Implicit differentiation is when all terms of an equation are differentiated, yielding an expression (called a *differential equation*) which includes a mixture of variables *and their derivatives*. Sometimes this is a more elegant method of finding the derivative *dy/dx* of a curve defined by *f(x,y) = 0*, than firstly rearranging to form *y = g(x)* and then working out *dy/dx = g'(x)*

Essentially we are exploiting the *chain rule*: $\frac{d}{dx} f(y) = \frac{df}{dy} \times \frac{dy}{dx}$ e.g. $\frac{d}{dx}\left(y^4 + \tan y\right) = \left(4y^3 + \sec^2 y\right)\frac{dy}{dx}$

Example #1: Find the derivative and hence the gradient of a tangent to a circle

$$(x-a)^2 + (y-b)^2 = r^2$$

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x-a}{y-b}$$

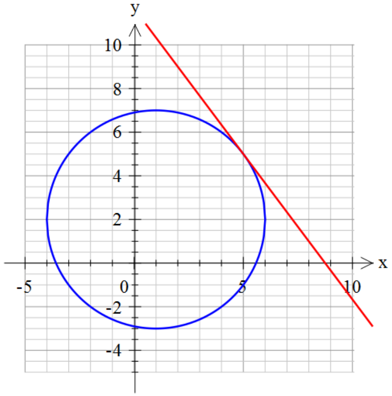
$$(x-1)^2 + (y-2)^2 = 25$$

$$\left.\frac{dy}{dx}\right|_{x=5, y=5} = -\frac{5-1}{5-2} = -\tfrac{4}{3}$$

$$y_T = -\tfrac{4}{3}x + c$$

$$5 = -\tfrac{4}{3}(5) + c \Rightarrow c = 11\tfrac{2}{3}$$

$$y_T = -\tfrac{4}{3}x + 11\tfrac{2}{3}$$



Example #2: Find *dy/dx* in terms of *x* and *y*

$$y^2 + 2x + e^{-x} - \cos y = 0$$

$$2y\frac{dy}{dx} + 2 - e^{-x} + \sin y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(2y + \sin y\right) = e^{-x} - 2$$

$$\frac{dy}{dx} = \frac{e^{-x} - 2}{2y + \sin y}$$

Example #3: Find the acceleration *dv/dt* using the following *energy equation of a harmonic oscillator*

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$\frac{dE}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt}$$

$$\frac{dE}{dt} = 0$$ If total energy E is a constant

$$v = \frac{dx}{dt}$$

$$0 = mv\frac{dv}{dt} + kxv$$

$$0 = v\left(m\frac{dv}{dt} + kx\right)$$

$$\therefore m\frac{dv}{dt} = -kx$$

i.e. Newton's Second law
mass x acceleration = sum of forces
(in this case a Hookean Spring)

Example #4: If $y = x^x$, find *dy/dx*

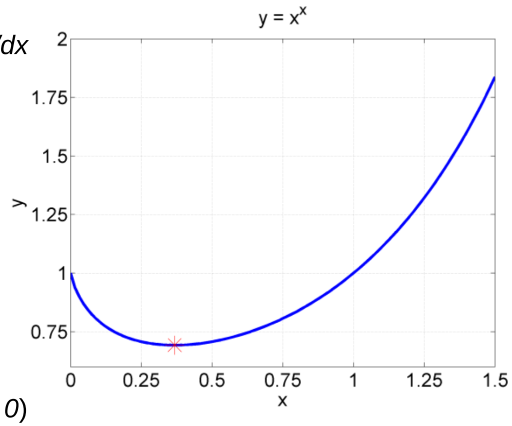
$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{d}{dx}\ln y = x\frac{1}{x} + \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = x^x\left(1 + \ln x\right)$$



So there is a *minima* (*dy/dx = 0*) when *lnx = -1* i.e. *x = 1/e*

A **parametric** *(x,y)* curve is defined in terms of (typically one) variable – in other words there are separate equations for *x(t)* and *y(t)* where *t* is a parameter. In mechanics, time is the obvious parameter, hence the typical choice of *t* as the symbol for a parametric variable.

$$\dot{x}=\frac{dx}{dt} \qquad \dot{y}=\frac{dy}{dt}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$ Chain rule

$$\frac{dt}{dx}=\left(\frac{dx}{dt}\right)^{-1}$$ Reciprocal rule

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)$$

$$\frac{d^2y}{dx^2}=\frac{\dot{x}\frac{d\dot{y}}{dx}-\dot{y}\frac{d\dot{x}}{dx}}{\dot{x}^2}$$

$$\ddot{y}=\frac{d\dot{y}}{dt}=\frac{d^2y}{dt^2} \qquad \ddot{x}=\frac{d\dot{x}}{dt}=\frac{d^2x}{dt^2}$$

$$\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}\frac{dt}{dx}-\dot{y}\ddot{x}\frac{dt}{dx}}{\dot{x}^2}$$

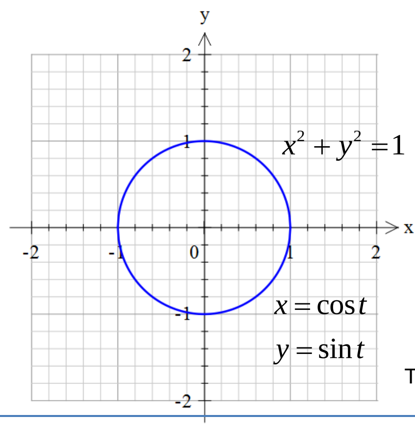
$$\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}\frac{1}{\dot{x}}-\dot{y}\ddot{x}\frac{1}{\dot{x}}}{\dot{x}^2}$$

$$\frac{d^2y}{dx^2}=\frac{\ddot{y}-\frac{\dot{y}\ddot{x}}{\dot{x}}}{\dot{x}^2}$$

$$\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}$$

Example #1: Find first and second derivatives of a unit circle defined parametrically

$$x=\cos t$$
$$y=\sin t$$
$$\dot{x}=-\sin t$$
$$\dot{y}=\cos t$$
$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{\cos t}{-\sin t}$$
$$\frac{dy}{dx}=-\frac{x}{y}$$
$$\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}$$
$$\frac{d^2y}{dx^2}=\frac{-\sin t(-\sin t)-\cos t(-\cos t)}{-\sin^3 t}$$
$$\frac{d^2y}{dx^2}=\frac{\sin^2 t+\cos^2 t}{-\sin^3 t}$$
$$\frac{d^2y}{dx^2}=-\frac{1}{y^3}$$



The unit circle

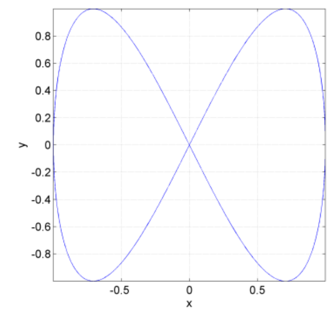
Check using implicit differentiation

$$x^2+y^2=1$$
$$2x+2y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{x}{y}$$
$$\frac{d^2y}{dx^2}=\frac{y(-1)-(-x)\frac{dy}{dx}}{y^2}$$
$$\frac{d^2y}{dx^2}=\frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$$
$$\frac{d^2y}{dx^2}=\frac{-y^2-x^2}{y^3}$$
$$\frac{d^2y}{dx^2}=-\frac{1}{y^3}$$

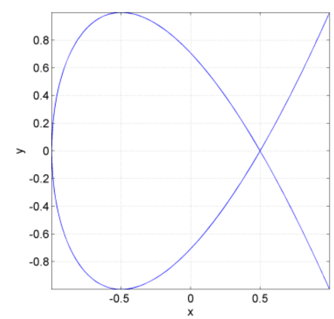
Example #2: *Lissajous* figures

$$x=\sin\omega t$$
$$y=\sin(\Omega t+\phi)$$

$\omega=1,\ \Omega=2,\ \phi=0$



$\omega=2,\ \Omega=3,\ \phi=-\frac{\pi}{4}$



$\omega=3,\ \Omega=7,\ \phi=\frac{\pi}{2}$