Integration by parts

This is a calculus 'rule' which enables integrals of *products* to be evaluated. It looks rather complicated, but can be readily memorized via:

"First times the integral of the second, minus the integral *of that same integral* times the derivative of the first."

$$\int_a^b (uv)\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b \left\{\frac{du}{dx}\left(\int v\,dx\right)\right\}dx$$

Definite – i.e. an area between limits

$$\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\left(\int v\,dx\right)dx\right\}$$

Indefinite or 'anti-derivative' of $y = uv$

Example 1

$$\int_0^1 xe^x dx = \left[x\int e^x dx\right]_0^1 - \int_0^1\left\{\left(\frac{d}{dx}x\right)\left(\int e^x dx\right)\right\}dx$$

$$\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 1\times e^x dx$$

$$\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \left[e^x\right]_0^1$$

$$\int_0^1 xe^x dx = e - 0 - \{e-1\}$$

$$\boxed{\int_0^1 xe^x dx = 1}$$

Example 2

$$\boxed{\int \ln x\,dx = x\ln x - x + c}$$

Here we prove how to integrate logarithms

$$\int_1^e \ln x\,dx = \int_1^e (\ln x \times 1)\,dx$$

Another useful multiply by one trick!

$$\int_1^e (\ln x\times 1)\,dx = \left[\ln x \times \int 1 dx\right]_1^e - \int_1^e\left\{\left(\frac{d}{dx}\ln x\right)\times\int 1dx\right\}dx$$

$$\int_1^e (\ln x\times 1)\,dx = \left[x\ln x\right]_1^e - \int_1^e\left\{\frac{1}{x}\times x\right\}dx$$

$$\int_1^e (\ln x\times 1)\,dx = \left[x\ln x\right]_1^e - \int_1^e dx$$

$$\int_1^e (\ln x\times 1)\,dx = \left[x\ln x - x\right]_1^e$$

$$\int_1^e (\ln x\times 1)\,dx = (e\ln e - e) - (1\times\ln 1 - 1) = e - e - 0 + 1$$

$$\boxed{\int_1^e (\ln x\times 1)\,dx = 1}$$

Proof

$$y = uw$$

$$\frac{dy}{dx} = u\frac{dw}{dx} + w\frac{du}{dx}$$

Product rule

$$\int_a^b \frac{dy}{dx}dx = \int_a^b u\frac{dw}{dx}dx + \int_a^b w\frac{du}{dx}dx$$

Integrate both sides of the above

$$\int_a^b dy = \int_a^b d(uw) = \int_a^b u\frac{dw}{dx}dx + \int_a^b w\frac{du}{dx}dx$$

$$\boxed{[uw]_a^b - \int_a^b w\frac{du}{dx}dx = \int_a^b u\frac{dw}{dx}dx}$$

This is how by-parts integration is usually quoted in books – but I think it is helpful to one step further

$$\text{Let } v = \frac{dw}{dx} \quad \therefore w = \int v\,dx$$

$$\therefore \boxed{\int_a^b (uv)\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b\left\{\frac{du}{dx}\left(\int v\,dx\right)\right\}dx}$$

Example 3 – Integrate by-parts until you end up with the integral you want

$$I = \int e^{-x}\sin x\,dx$$

$$I = e^{-x}\left(\int \sin x\,dx\right) - \int\left(-e^{-x}\right)\left(\int\sin x\,dx\right)dx + c$$

$$I = -e^{-x}\cos x - \int e^{-x}\cos x\,dx + c$$

$$I = -e^{-x}\cos x - \left\{e^{-x}\left(\int\cos x\,dx\right) - \int\left(-e^{-x}\right)\left(\int\cos x\,dx\right)dx\right\} + c$$

$$I = -e^{-x}\cos x - \left\{e^{-x}\sin x + \int e^{-x}\sin x\,dx\right\} + c$$

$$I = -e^{-x}\cos x - e^{-x}\sin x - I + c$$

$$2I = -e^{-x}(\cos x + \sin x) + c$$

$$\boxed{I = -\tfrac{1}{2}e^{-x}(\cos x + \sin x) + k}$$