Integration by substitution

In order to evaluate many integrals, we may need to change the variable using a suitable **substitution**. A substitution *may* be sufficient to transform the integral into one you already know how to do. Note this aspect of Mathematics is not entirely prescriptive! A substitution *may not always* make the integral easier to evaluate. Sometimes *more than one* substitution may be required.

Remember **to change the limits** of a definite integral if you are using a substitution.

Integral includes	Try substitution
$(ax+b)^n$	$u = ax+b$
$\sqrt[n]{ax+b}$	$u^n = ax+b$
$a - bx^2$	$a\sin^2 u = bx^2$
$a + bx^2$	$a\tan^2 u = bx^2$
$bx^2 - a$	$a\sec^2 u = bx^2$
e^x	$u = e^x$
$\ln(ax+b)$	$e^u = ax+b$

Example 1

$$I = \int \frac{1}{1+x^2}dx$$

$$x = \tan u \quad \therefore u = \tan^{-1} x$$

$$\frac{dx}{du} = \sec^2 u \quad \therefore \quad dx = \sec^2 u du$$

$$1+x^2 = 1+\tan^2 u = \sec^2 u$$

$$I = \int \frac{1}{1+x^2}dx = \int \frac{\sec^2 u}{\sec^2 u}du$$

$$I = \int du = u + c$$

$$\boxed{I = \tan^{-1} x + c}$$

Example 2

$$I = \int \sin^6 x \cos x dx$$

$u = \sin x$ ← Define substitution

$\frac{du}{dx} = \cos x \quad \therefore \quad dx = \frac{du}{\cos x}$ Differentiate substitution and find out what dx turns into

$$I = \int u^6 \cos x \frac{du}{\cos x}$$

Replace terms in integral and hope any functions in x cancel (leaving only functions in u that can then be integrated).

$$I = \int u^6 du = \tfrac{1}{7}u^7 + c$$

$\boxed{I = \tfrac{1}{7}\sin^7 x + c}$ ← Replace u by its function of x to yield the final answer.

Example 3

$$I = \int_0^1 \sqrt[3]{1+2x}dx$$

$$u = 1+2x$$

$$\frac{du}{dx} = 2 \quad \therefore dx = \tfrac{1}{2}du$$

$$I = \tfrac{1}{2}\int_{u=1}^{3} u^{\frac{1}{3}}du$$

$$I = \tfrac{1}{2}\left[\tfrac{3}{4}u^{\frac{4}{3}}\right]_1^3$$

$$\boxed{I = \tfrac{3}{8}\left(3^{\frac{4}{3}} - 1\right)}$$

Change the limits to be those of the new variable

Example 4

$$I = \int_{x=0}^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}dx$$

$$3x = \sin u \quad \therefore \quad u = \sin^{-1}(3x)$$

$$3\frac{dx}{du} = \cos u \quad \therefore \quad dx = \tfrac{1}{3}\cos u du$$

$$I = \int_{u=\sin^{-1} 0}^{\sin^{-1}\frac{1}{2}} \frac{1}{\sqrt{1-\sin^2 u}} \tfrac{1}{3}\cos u du$$

$$I = \tfrac{1}{3}\int_0^{\frac{1}{6}\pi} \frac{1}{\cos u}\cos u du$$

$$I = \tfrac{1}{3}\int_0^{\frac{1}{6}\pi} du$$

$$I = \tfrac{1}{3}[u]_0^{\frac{1}{6}\pi}$$

$$\boxed{I = \tfrac{1}{18}\pi}$$

Example 5

$$I = \int \frac{1}{1-\sqrt{x}}dx$$

$$u = 1-\sqrt{x} \quad \therefore \sqrt{x} = 1-u$$

$$\frac{du}{dx} = -\frac{1}{2\sqrt{x}} \quad \therefore dx = -2\sqrt{x}du = 2(u-1)du$$

$$I = \int \frac{1}{u}2(u-1)du$$

$$I = 2\int du - 2\int \frac{1}{u}du$$

$$I = 2\left(u - \ln|u|\right) + c$$

$$\boxed{I = 2\left(1-\sqrt{x} - \ln\left|1-\sqrt{x}\right|\right) + c}$$

CSM's Challenge!

$$I=\int\frac{1}{1+\cos a\cos x}dx$$

$$u=\frac{\cos\left(\frac{a-x}{2}\right)}{\cos\left(\frac{a+x}{2}\right)}$$

This substitution is perhaps what you would call an inspired guess!*

$$\frac{du}{dx}=\frac{\cos\left(\frac{a+x}{2}\right)\tfrac{1}{2}\sin\left(\frac{a-x}{2}\right)+\cos\left(\frac{a-x}{2}\right)\tfrac{1}{2}\sin\left(\frac{a+x}{2}\right)}{\cos^2\left(\frac{a+x}{2}\right)}$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$A=\frac{a-x}{2},\ \ B=\frac{a+x}{2},\ \ \therefore A+B=a$$

$$\therefore\sin\left(\frac{a-x}{2}\right)\cos\left(\frac{a+x}{2}\right)+\cos\left(\frac{a-x}{2}\right)\sin\left(\frac{a+x}{2}\right)=\sin a$$

$$\therefore\frac{du}{dx}=\frac{\tfrac{1}{2}\sin a}{\cos^2\left(\frac{a+x}{2}\right)}$$

$$\therefore dx=\frac{2}{\sin a}\cos^2\left(\frac{a+x}{2}\right)du$$

$$\therefore I=\frac{2}{\sin a}\int\frac{1}{1+\cos a\cos x}\cos^2\left(\frac{a+x}{2}\right)du$$

$$u^2=\frac{\cos^2\left(\frac{a-x}{2}\right)}{\cos^2\left(\frac{a+x}{2}\right)}=\frac{1+\cos(a-x)}{1+\cos(a+x)}$$

$$\sin^2\tfrac{1}{2}\theta=\tfrac{1}{2}(1-\cos\theta)$$
$$\cos^2\tfrac{1}{2}\theta=\tfrac{1}{2}(1+\cos\theta)$$
$$\tan^2\tfrac{1}{2}\theta=\frac{\sin\theta}{1+\cos\theta}$$

$$\therefore 1+u^2=\frac{1+\cos(a+x)+1+\cos(a-x)}{1+\cos(a+x)}$$

$$\therefore 1+u^2=\frac{2+\cos a\cos x-\sin a\sin x+\cos a\cos x+\sin a\sin x}{1+\cos(a+x)}$$

$$\therefore 1+u^2=\frac{2+2\cos a\cos x}{1+\cos(a+x)}$$

$$1+\cos(a+x)=2\cos^2\left(\frac{a+x}{2}\right)$$

$$\cos^2\tfrac{1}{2}\theta=\tfrac{1}{2}(1+\cos\theta)$$

$$\therefore 1+u^2=\frac{1+\cos a\cos x}{\cos^2\left(\frac{a+x}{2}\right)}$$

$$\therefore\frac{\cos^2\left(\frac{a+x}{2}\right)}{1+\cos a\cos x}=\frac{1}{1+u^2}$$

$$\therefore I=\frac{2}{\sin a}\int\frac{1}{1+\cos a\cos x}\cos^2\left(\frac{a+x}{2}\right)du$$

$$\therefore I=\frac{2}{\sin a}\int\frac{1}{1+u^2}du$$

$$\therefore I=\frac{2}{\sin a}\tan^{-1}u+c$$

$$\therefore I=\frac{2}{\sin a}\tan^{-1}\left(\frac{\cos\left(\frac{a-x}{2}\right)}{\cos\left(\frac{a+x}{2}\right)}\right)+c$$

* or a visit to http://www.wolframalpha.com/calculators/integral-calculator/