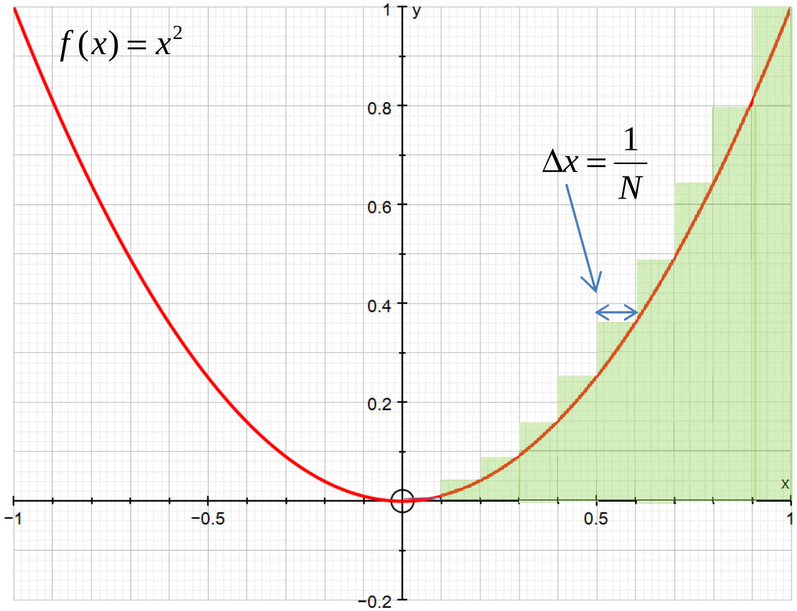
Introduction to integration



The area between a curve *f(x)* and the *x* axis is defined as the *integral* of the *integrand f(x)*

$$A=\lim_{\Delta x\to 0}\left\{\sum_{n=1}^{N} f(n\Delta x)\Delta x\right\}=\int_a^b f(x)dx$$

In the example above the limits would be *a = 0, b= 1*

The Old English 'S' means "sum rectangular strips of height *f(x)* and width Δ*x*, in the limit when Δ*x* tends to zero." Note the *dx* has the same meaning as *dx* in the **gradient** *df/dx*

Now, for polynomials: $y=ax^{n+1}$ $\quad \therefore \frac{dy}{dx}=(n+1)ax^n$ $\quad \therefore \frac{dy}{n+1}=ax^n dx$

$$\therefore \frac{1}{n+1}\int dy=\int ax^n dx$$

$$\frac{1}{n+1}y+c=\int ax^n dx$$

$$\therefore \boxed{\int ax^n dx=\frac{ax^{n+1}}{n+1}+c}$$

Note we have omitted *limits* here. This integral is said to be 'indefinite' (whereas an integral with limits is 'definite'). An indefinite integral must always have a constant of integration added to account for the fact the limits have not been stated.

What is the area *A* between a mathematically defined curve and the x axis? We can approximate this by summing a series of rectangular strips.

$$A\approx\sum_{n=1}^{N} f(n\Delta x)\Delta x$$

In the example on the left $f(x)=x^2$ and the limits of our area are $0\le x\le 1$

The (fixed) strip width is therefore $\Delta x=\frac{1}{N}$

Clearly, in order to get a better estimate, we must reduce the size of the strip width, and thus increase the number of strips to cover the interval [0,1]. The exact area is therefore:

$$A=\lim_{\Delta x\to 0}\left\{\sum_{n=1}^{N} f(n\Delta x)\Delta x\right\} \longleftarrow f(n\Delta x)=\left(\frac{n}{N}\right)^2$$

$$A=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(\frac{n}{N}\right)^2\frac{1}{N}\right\}=\lim_{N\to\infty}\left\{\frac{1}{N^3}\sum_{n=1}^{N}n^2\right\} \longleftarrow \sum_{n=1}^{N}n^2=\tfrac{1}{6}N(N+1)(2N+1)$$

$$A=\lim_{N\to\infty}\left\{\tfrac{1}{N^3}\tfrac{1}{6}N(N+1)(2N+1)\right\}=\lim_{N\to\infty}\left\{\tfrac{1}{N^3}\tfrac{1}{6}\left(2N^3+3N^2+N\right)\right\}$$

$$A=\lim_{N\to\infty}\left\{\tfrac{1}{6}\left(2+\frac{3}{N}+\frac{1}{N^2}\right)\right\}=\boxed{\frac{1}{3}}$$

$$\int_0^1 x^2dx=\int_{-\infty}^1 x^2dx-\int_{-\infty}^0 x^2dx$$

i.e. we work out *definite integrals* by finding the difference in curve areas between the limits, relative to minus infinity. The constant of integration is therefore the same for each, and hence cancels in the difference.

Hence, using this rule for integrating polynomials: $\int_0^1 x^2dx=\left[\tfrac{1}{3}x^3\right]_0^1=\left(\tfrac{1}{3}\right)-(0) = \boxed{\tfrac{1}{3}}$

Integration is *anti-differentiation*

$$\int \frac{dy}{dx}dx = \int dy = y + c$$

Differentiate

$$y \rightarrow \frac{dy}{dx}$$

Integrate

$$\int \frac{dy}{dx}dx$$

The caveat is that integration *creates information* (i.e. the constant of integration)

i.e. $y = x^3 + 2$

$$\frac{dy}{dx} = 3x^2$$

$$\int 3x^2 dx = x^3 + c$$

Regardless of what the constant added to x^3 is, differentiating it will yield the *same answer.*

Integrating the result will restore x^3, but *any constant is possible.*

More information e.g $y = 3$ when $x = 1$ is needed to define what the constant is.

Since integration is essentially a summation of terms:

$$\sum \{ f(n) + g(n) \} = \sum f(n) + \sum g(n)$$

$$\sum af(n) = a \sum f(n)$$

$$\int \{ f(x) + g(x) \} dx = \int f(x)dx + \int g(x)dx$$

$$\int af(x)dx = a \int f(x)dx$$

e.g. $$\int \{ 2x^2 + \sqrt{x} \} dx \quad = \quad 2\int x^2 dx + \int x^{\frac{1}{2}} dx \quad = \quad \tfrac{2}{3}x^3 + \tfrac{2}{3}x^{\frac{3}{2}} + c$$

We can use the idea of integration as anti-differentiation to work out the integrals of many functions, using what we know about their derivatives

Polynomials

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

Exponentials

$$y = e^{ax}$$

$$\frac{dy}{dx} = ae^{ax} \quad \therefore \tfrac{1}{a}dy = e^{ax}dx$$

$$\therefore \int e^{ax}dx = \tfrac{1}{a}\int dy = \tfrac{1}{a}y + c$$

$$\therefore \int e^{ax}dx = \tfrac{1}{a}e^{ax} + c$$

Trigonometric

$$y = \sin ax$$

$$\frac{dy}{dx} = a\cos ax \quad \therefore \tfrac{1}{a}dy = \cos axdx$$

$$\therefore \int \cos axdx = \tfrac{1}{a}\int dy = \tfrac{1}{a}y + c$$

$$\therefore \int \cos axdx = \tfrac{1}{a}\sin ax + c$$

$$y = \cos ax$$

$$\frac{dy}{dx} = -a\sin ax \quad \therefore -\tfrac{1}{a}dy = \sin axdx$$

$$\therefore \int \sin axdx = -\tfrac{1}{a}\int dy = -\tfrac{1}{a}y + c$$

$$\therefore \int \sin axdx = -\tfrac{1}{a}\cos ax + c$$

$$y = \tan ax$$

$$\frac{dy}{dx} = a\sec^2 ax \quad \therefore \tfrac{1}{a}dy = \sec^2 axdx$$

$$\therefore \int \sec^2 axdx = \tfrac{1}{a}\int dy = \tfrac{1}{a}y + c$$

$$\therefore \int \sec^2 axdx = \tfrac{1}{a}\tan ax + c$$

Note this process is somewhat restrictive, for example it does not tell us what the integral of tanx is, or indeed logarithms of x. To compute these results we need *other techniques* such as *integration by parts* and *integration by substitution*.