Taylor & Maclaurin expansions

A *Taylor series* is a *polynomial expansion* of a function $f(x)$ in the vicinity of a particular point $(x_0, f(x_0))$

The *Macluarin expansion* is a special case when $x_0 = 0$

Taylor and Maclaurin expansions are useful as *approximations* to functions. They can enable numeric evaluations of functions such as sine, cosine etc and also integrals that cannot be analytically evaluated.

Taylor series

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + ...$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

$$f^{(n)}(x_0) = \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}$$

Maclaurin series

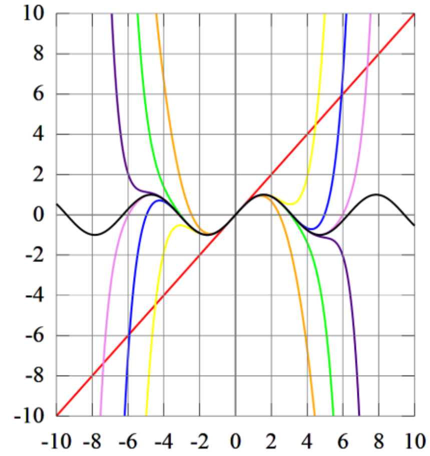
$$f(x) = f(0) + x f'(x_0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^{(n)}(0) + ...$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

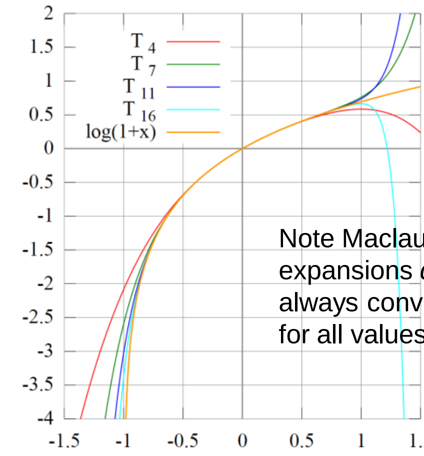
$$f^{(n)}(0) = \left. \frac{d^n f(x)}{dx^n} \right|_{x=0}$$

Brook Taylor (1685-1731)

Colin Maclaurin (1698-1746)



Maclaurin series for $f(x) = \sin x$
Increased number of terms improves the approximation further away from $x = 0$



The Maclaurin series for $f(x) = \ln(1+x)$ *diverges* outside the range [-1,1] *regardless of the number of terms in the series*. Indeed, adding more terms makes the series approximation *worse* outside [-1,1]

Maclaurin expansion for trigonometric functions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ... \quad |x| < \tfrac{1}{2}\pi$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + ... \quad |x| < \tfrac{1}{2}\pi$$

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + ... \quad |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ... \quad |x| < \pi$$

Maclaurin expansion for exponential, logarithmic and hyperbolic functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad -1 < x \le 1$$

$$\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ... \quad |x| < 1$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

$$\sinh x = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + ... \quad |x| < \tfrac{1}{2}\pi$$

Maclaurin series of functions defined as integrals

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$$

Generalized Binomial Expansion

$$(1+x)^n = 1+nx+\tfrac{1}{2!}n(n-1)x^2+\tfrac{1}{3!}n(n-1)(n-2)x^3+...$$

$$\left(1+t^2\right)^{-1} = 1+(-1)(t^2)+(-1)(-2)\frac{(t^2)^2}{2!}+(-1)(-2)(-3)\frac{(t^2)^3}{3!}+... \quad \left|t^2\right|<1 \Rightarrow |t|<1$$

$$\left(1+t^2\right)^{-1} = 1-t^2+t^4-t^6+... \quad |t|<1$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \left[t-\tfrac{1}{3}t^3+\tfrac{1}{5}t^5-\tfrac{1}{7}t^7+...\right]_0^x \quad |x|<1$$

$$\boxed{\tan^{-1}x = x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+... \quad |x|<1}$$

$$\sin^{-1}x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\left(1-t^2\right)^{-\frac{1}{2}} = 1+(-\tfrac{1}{2})(-t^2)+(-\tfrac{1}{2})(-\tfrac{3}{2})\frac{(-t^2)^2}{2!}+(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})\frac{(-t^2)^3}{3!}+... \quad \left|t^2\right|<1 \Rightarrow |t|<1$$

$$\left(1-t^2\right)^{-\frac{1}{2}} = 1+\tfrac{1}{2}t^2+\tfrac{3}{2\times 4}t^4+\tfrac{3\times 5}{2\times 4\times 6}t^6+... \quad |t|<1$$

$$\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \left[t+\tfrac{1}{2}\tfrac{1}{3}t^3+\tfrac{3}{2\times 4}\tfrac{1}{5}t^5+\tfrac{3\times 5}{2\times 4\times 6}\tfrac{1}{7}t^7+...\right]_0^x \quad |x|<1$$

$$\boxed{\sin^{-1}x = x+\tfrac{1}{2}\tfrac{1}{3}x^3+\tfrac{1\times 3}{2\times 4}\tfrac{1}{5}x^5+\tfrac{1\times 3\times 5}{2\times 4\times 6}\tfrac{1}{7}x^7+... \quad |x|<1}$$

Summary of Maclaurin expansions of inverse trigonometric functions

$$\sin^{-1}x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+...|x|<1$$

$$\tan^{-1}x = \begin{cases} x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+... & |x|\le 1 \\ \tfrac{1}{2}\pi-\dfrac{1}{x}+\dfrac{1}{3x^3}-\dfrac{1}{5x^5}+... & x>1 \\ -\tfrac{1}{2}\pi-\dfrac{1}{x}+\dfrac{1}{3x^3}-\dfrac{1}{5x^5}+.. & x<-1 \end{cases}$$

Note:

$$\boxed{\begin{aligned}\cos^{-1}x &= \tfrac{1}{2}\pi-\sin^{-1}x \\ \cot^{-1}x &= \tfrac{1}{2}\pi-\tan^{-1}x\end{aligned}}$$

Approximations to integrals that can't be evaluated analytically

$$f(x) = \int_0^x e^{-\frac{1}{2}t^2}\,dt \quad \leftarrow \quad e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

$$e^{-\frac{1}{2}t^2} = 1+\left(-\tfrac{1}{2}t^2\right)+\tfrac{1}{2}\left(-\tfrac{1}{2}t^2\right)^2+\tfrac{1}{6}\left(-\tfrac{1}{2}t^2\right)^3+...$$

$$e^{-\frac{1}{2}t^2} = 1-\tfrac{1}{2}t^2+\tfrac{1}{8}t^4-\tfrac{1}{48}t^6+...$$

$$\boxed{\int_0^x e^{-\frac{1}{2}t^2}\,dt = x-\tfrac{1}{2}\tfrac{1}{3}x^3+\tfrac{1}{8}\tfrac{1}{5}x^5-\tfrac{1}{48}\tfrac{1}{7}x^7+...}$$

This result has an application in *probability theory.* The *Normal Distribution* has a probability density function *p(z)* i.e. where the probability of 'normalized' random variable *z* being between *z* and *z+dz* is *p(z)dz*

$$\boxed{p(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}} \qquad z = \frac{x-\mu}{\sigma}$$

For more general random variable *x*, mean μ and standard deviation σ

The probability of *z* being less than *z** is therefore:

$$P(z<z^*) = \int_{-\infty}^{z^*} p(z)\,dz$$

$$\boxed{P(z<z^*) = \begin{cases} \tfrac{1}{2}+\displaystyle\int_0^{z^*} p(z)\,dz & z^*>0 \\ \tfrac{1}{2}-\displaystyle\int_0^{|z^*|} p(z)\,dz & z^*<0 \end{cases}}$$

Since $\int_{-\infty}^{\infty} p(z)\,dz = 1$ and *p(z)* is an even function

$$\boxed{\int_0^{|z^*|} p(z)\,dz = |z^*|-\tfrac{1}{2}\tfrac{1}{3}|z^*|^3+\tfrac{1}{8}\tfrac{1}{5}|z^*|^5-\tfrac{1}{48}\tfrac{1}{7}|z^*|^7+...}$$

This is how tables of the *cumulative Normal distribution* are worked out numerically

Proof of the Taylor expansion

$$R_n(x) = \int_{x_0}^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt \quad \text{Define this function}$$

$$R_n(x) = \left[\frac{(x-t)^n}{n!}f^{(n)}(t)\right]_{x_0}^x - \int_{x_0}^x f^{(n)}(t)\times\frac{n(x-t)^{n-1}(-1)}{n!}\,dt \quad \text{Integration by parts}$$

$$R_n(x) = -\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+nR_{n-1}$$

Expect $\lim_{n\to\infty} R_n = 0$

$$R_n(x) = -\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)-\frac{(x-x_0)^{n-1}}{(n-1)!}f^{(n)}(x_0)-....+f(x)$$

$$\boxed{f(x) = f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+....+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)}+R_n$$