A **radiodrome** is the *curve of pursuit* $y(t)$ *vs* $x(t)$ of an object moving at constant speed u, starting from $(a,0)$, which aims to intercept another object moving upwards from (0,0) at constant speed v.

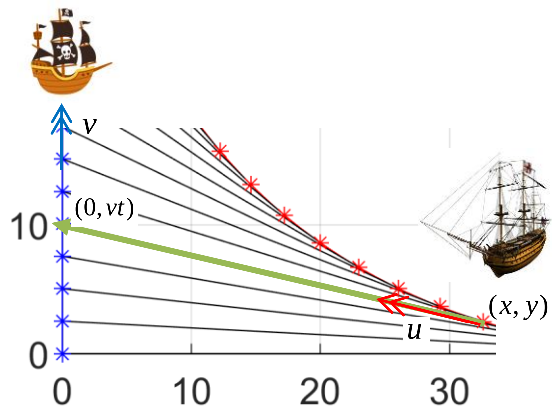
$$\frac{dy}{dx} = -\frac{vt-y}{x}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x\left(v\frac{dt}{dx} - \frac{dy}{dx}\right) - (vt-y)(1)}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x}\left(v\frac{dt}{dx} - \frac{dy}{dx} - \frac{vt-y}{x}\right)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x}\left(v\frac{dt}{dx} - \frac{dy}{dx} + \frac{dy}{dx}\right)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{v}{x}\frac{dt}{dx}$$



$$\frac{dx}{dt} = -u\cos\theta$$

$$\tan\theta = -\frac{dy}{dx}$$

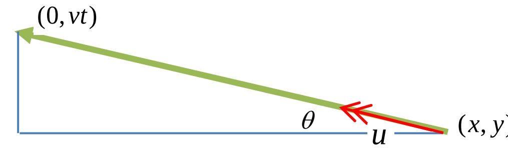
$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\therefore \frac{dx}{dt} = -\frac{u}{\sqrt{1+(dy/dx)^2}}$$

$$\therefore \frac{dt}{dx} = -\tfrac{1}{u}\sqrt{1+(dy/dx)^2}$$



The ship-of-the-line at always points to the current position of the pirate ship.

Pierre Bouguer (1698-1758)

$$\frac{d^2y}{dx^2} = -\frac{v}{x}\frac{dt}{dx}, \quad \frac{dt}{dx} = -\tfrac{1}{u}\sqrt{1+(dy/dx)^2} \quad \therefore \frac{d^2y}{dx^2} = \frac{v/u}{x}\sqrt{1+(dy/dx)^2}$$

$$z = dy/dx \quad \therefore \frac{dz}{dx} = \frac{v/u}{x}\sqrt{1+z^2} \quad \therefore \int \frac{dz}{\sqrt{1+z^2}} = \tfrac{v}{u}\int \frac{1}{x}\,dx \Rightarrow \boxed{\sinh^{-1} z = \tfrac{v}{u}\ln x + c}$$

$$\sinh^{-1} z = \ln\left(z + \sqrt{1+z^2}\right)$$

$$\ln\left(z+\sqrt{1+z^2}\right) = \tfrac{v}{u}\ln x + c$$

$$\ln\left(z+\sqrt{1+z^2}\right) = \ln x^{\frac{v}{u}} + \ln b = \ln\left(bx^{\frac{v}{u}}\right)$$

$$\therefore z + \sqrt{1+z^2} = bx^{\frac{v}{u}}$$

$$z = \left.\frac{dy}{dx}\right|_{x=a} = 0 \quad \therefore 1 = ba^{\frac{v}{u}} \Rightarrow b = a^{-\frac{v}{u}}$$

$$\sinh z = \tfrac{1}{2}\left(e^z - e^{-z}\right)$$

$$\therefore z = \sinh\left(\ln\left(\left(\frac{x}{a}\right)^{\frac{v}{u}}\right)\right) = \tfrac{1}{2}\left\{\left(\frac{x}{a}\right)^{\frac{v}{u}} - \left(\frac{x}{a}\right)^{-\frac{v}{u}}\right\}$$

$$\therefore y = \tfrac{1}{2}a\left\{\frac{\left(\frac{x}{a}\right)^{\frac{v}{u}+1}}{\frac{v}{u}+1} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{u}}}{1-\frac{v}{u}}\right\} + d$$

$x = a, y = 0$

$$\therefore 0 = \tfrac{1}{2}a\left\{\frac{1}{\frac{v}{u}+1} - \frac{1}{1-\frac{v}{u}}\right\} + d = \tfrac{u}{2}a\left\{\frac{1}{v+u} - \frac{1}{u-v}\right\} + d$$

$$\therefore 0 = \tfrac{u}{2}a\left\{\frac{u-v-v-u}{u^2-v^2}\right\} + d \quad \therefore d = \frac{uva}{u^2-v^2}$$

So $u > v$

$$\boxed{y = \tfrac{1}{2}a\left\{\frac{\left(\frac{x}{a}\right)^{\frac{v}{u}+1}}{\frac{v}{u}+1} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{u}}}{1-\frac{v}{u}}\right\} + \frac{uva}{u^2-v^2}}$$

$$\frac{dy}{dx} = -\frac{vt-y}{x}$$

$$\frac{dy}{dx} = \tfrac{1}{2}\left\{\left(\frac{x}{a}\right)^{\frac{v}{u}} - \left(\frac{x}{a}\right)^{-\frac{v}{u}}\right\}$$

$$h = \frac{uva}{u^2-v^2} \leftarrow x = 0, y = h$$

$$\boxed{t = \tfrac{1}{v}\left(-x\frac{dy}{dx} + y\right)}$$

$$u^2 - v^2 - uv\frac{a}{h} = 0$$

$$\boxed{x = 0,\ t = \frac{ua}{u^2-v^2}}$$

$$\left(u - v\frac{a}{2h}\right)^2 - v^2 - v^2\frac{a^2}{4h^2} = 0$$

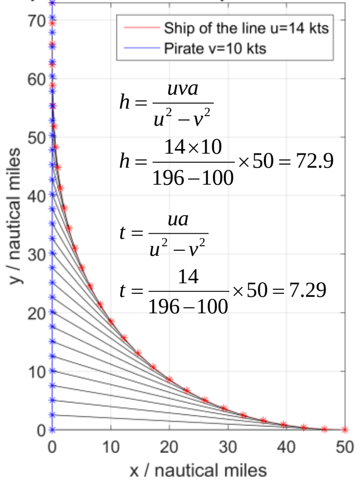
$u > v$ so take +ve root

$$\boxed{\frac{u}{v} = \frac{a}{2h} + \sqrt{1+\frac{a^2}{4h^2}}}$$

$$h = a \Rightarrow \frac{u}{v} = \tfrac{1+\sqrt{5}}{2}$$

Golden ratio

Radiodrome pursuit curve. Intercept in T = 7.29 hours at (0,72.9)



Radiodrome pursuit curve. Intercept in T = 5 hours at (0,50)