Second order 'ordinary' differential equations (ODEs) are of the general form:

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The goal is to find a *closed form* expression for $y(x)$

Second order ODEs contain *up to the second derivative* of the variable y so one expects *two arbitrary constants*, which will result from integration, and which must occur in two independent instances to remove the derivative(s). Hence to solve the ODE, we will need to know one point on the (x,y) curve and the corresponding derivative, or two points on the curve, or two derivatives etc....

Linear second order differential equations
This is a special case, which in many scenarios will have a closed form solution.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx}p(x) + yr(x) = q(x)$$

Linearity means all derivatives in the equation are raised to a single power

The solution is the solution when $q(x) = 0$ "The **Complimentary Function**" (CF) plus a "**Particular Integral**" (PI which is typically something which has the same form as $q(x)$. If this doesn't work try $xq(x)$, $x^2q(x)$ etc...

$$\frac{d^2y}{dx^2} + \frac{dy}{dx}p(x) + yr(x) = 0$$
$$y_{CF}(x) = Ag(x) + Bh(x)$$

Alas, there is often no simple method beyond 'inspired guesswork' to find the Complimentary Function in the *general case* where p and r are functions of x

However, you *can* solve the scenario when p and r are *constants*....

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
$$y = Ae^{\lambda x}$$
$$\therefore aA\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$\therefore a\lambda^2 + b\lambda + c = 0$$
$$\therefore \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This quadratic polynomial is called the **Auxiliary Equation**

The two roots of the Auxiliary Equation yield the two *independent functions* of the Complimentary Function

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$y = Ae^{\lambda_+ x} + Be^{\lambda_- x}$$

A and B are arbitrary constants to be set by the initial conditions e.g. y and dy/dx values when $x = 0$

Special case: repeated roots
We must have the possibility of two arbitrary constants for a second order equation, so if the *discriminant* of the auxiliary equation is zero:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
$$b^2 - 4ac = 0$$
$$y = (Ax + B)e^{-\frac{b}{2a}x}$$

i.e. an extra linear term

Check:

$$y = (Ax + B)e^{\lambda x}$$
$$\frac{dy}{dx} = \lambda(Ax + B)e^{\lambda x} + Ae^{\lambda x} = (\lambda Ax + \lambda B + A)e^{\lambda x}$$
$$\therefore b\frac{dy}{dx} = (\lambda Abx + \lambda bB + Ab)e^{\lambda x}$$
$$\frac{d^2y}{dx^2} = (\lambda^2 Ax + \lambda^2 B + \lambda A)e^{\lambda x} + \lambda Ae^{\lambda x}$$
$$\therefore a\frac{d^2y}{dx^2} = (\lambda^2 Aax + \lambda^2 aB + 2\lambda aA)e^{\lambda x}$$
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$$
$$= (\lambda^2 Aax + \lambda^2 aB + 2\lambda aA + \lambda Abx + \lambda bB + Ab + cAx + Bc)e^{\lambda x}$$
$$= (\lambda^2 Aa + \lambda Ab + cA)xe^{\lambda x} + (\lambda^2 aB + 2\lambda aA + \lambda bB + Ab + Bc)e^{\lambda x}$$

i.e. the specific relationships for this repeated root scenario

$$\lambda = -\tfrac{b}{2a}, \quad b^2 - 4ac = 0 \Rightarrow c = \tfrac{b^2}{4a}$$
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$$
$$= \left(\tfrac{b^2}{4a^2}a - \tfrac{b}{2a}b + \tfrac{b^2}{4a}\right)Axe^{\lambda x} + \left(\tfrac{b^2}{4a^2}aB - \tfrac{b}{a}aA - \tfrac{b}{2a}bB + Ab + B\tfrac{b^2}{4a}\right)e^{\lambda x}$$
$$= \left(\tfrac{b^2}{4a} - \tfrac{b^2}{2a} + \tfrac{b^2}{4a}\right)Axe^{\lambda x} + \left(\tfrac{b^2}{4a}B - bA - \tfrac{b^2}{2a}B + Ab + B\tfrac{b^2}{4a}\right)e^{\lambda x}$$
$$= (0)Axe^{\lambda x} + (0)e^{\lambda x} = 0 \quad \checkmark$$

Special case: complex roots

If the discriminant is *negative*, then our auxilliary equation has *no real roots.*
This means the Complimentary Function has exponential solutions with *imaginary powers* i.e. oscillatory (sine and cosine) form.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$
$$b^2-4ac<0$$
$$\lambda_\pm=\frac{-b\pm i\sqrt{4ac-b^2}}{2a}$$
$$\gamma=\frac{b}{2a},\quad k=\frac{\sqrt{4ac-b^2}}{2a}$$
$$\therefore y=Ae^{-\gamma x}e^{ikx}+Be^{-\gamma x}e^{-ikx}$$
$$B=Ae^{2i\phi}$$
Can assign this without loss of generality
$$\therefore y=Ae^{-\gamma x}\left(e^{ikx}+e^{-ikx+2i\phi}\right)$$
$$\therefore y=Ae^{-\gamma x}e^{i\phi}\left(e^{ikx-i\phi}+e^{-(ikx-\phi)}\right)$$
$$\therefore y=2Ae^{-\gamma x}e^{i\phi}\cos(kx-\phi)$$

So if y is real, then:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$
$$b^2-4ac<0$$
$$\gamma=\frac{b}{2a},\quad k=\frac{\sqrt{4ac-b^2}}{2a}$$
$$y=\alpha e^{-\gamma x}\cos(kx-\phi)$$

where α is a real arbitrary constant and ϕ is an arbitrary phase constant.

Example: Driven Simple Harmonic Motion (SHM)

A linear second order differential equation with constant coefficients and a sinusoidal q describes many *oscillatory* systems in Physics, and results from Newton's Second Law (springs, pendulums etc) or Ohm's law (applied to a *reactive* circuit with an inductor, capacitor and resistor).

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=A_0\sin\omega t$$

Assume $\gamma,\omega_0>0$

Auxilliary equation, considering solutions to the CF of the form $x(t)=ae^{\lambda t}$

$$\lambda^2+2\gamma\lambda+\omega_0^2=0$$
$$\therefore \lambda=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$$

Three special cases:

$\gamma>\omega_0$ "**Overdamped**" i.e. not a single oscillation

$\gamma=\omega_0$ "**Critically damped**" i.e. *nearly* a single oscillation

See next page for general solutions...

$\gamma<\omega_0$ "**Underdamped**" i.e. *oscillatory* solutions

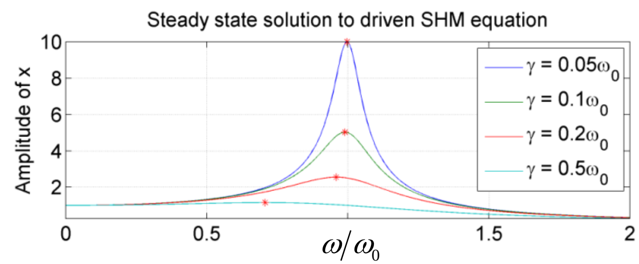
The positive γ will always mean an *exponentially decaying* complementary function, or "*transient*" since in SHM, x is typically the amplitude of an oscillation as a function of time.

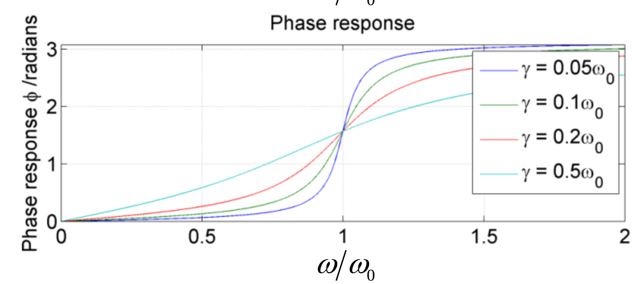
The **steady state solution** is therefore the **Particular Integral**. Let this take the form: $x(t)=B\sin(\omega t-\phi)$
It is easier to consider a **complex** SHM version (!), and then extract the imaginary part....

$$x(t)=B\cos(\omega t-\phi)+iB\sin(\omega t-\phi)=Be^{i(\omega t-\phi)}$$
$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=A_0e^{i\omega t}$$
Complex SHM equation
$$\therefore -\omega^2Be^{i(\omega t-\phi)}+2i\gamma\omega Be^{i(\omega t-\phi)}+\omega_0^2Be^{i(\omega t-\phi)}=A_0e^{i\omega t}$$
$$\therefore\left(-\omega^2+2i\gamma+\omega_0^2\right)Be^{-i\phi}=A_0$$
$$\therefore B\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}\,e^{i\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)}e^{-i\phi}=A_0$$
$$\therefore B=\frac{A_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}$$
$$\therefore \phi=\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

Since A_0 and B are *real* constants the phases must cancel

These equations model **resonance** effects





de Moivre's theorem $e^{ikx}=\cos kx+i\sin kx$

SHM in summary

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=A_0\sin\omega t \qquad x(0)=x_0,\quad \left.\frac{dx}{dt}\right|_{t=0}=\dot{x}_0$$

'Transient'

Under-damped - oscillatory

$$\gamma<\omega_0$$

$$x(t)=Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2}-\Phi\right)+B\sin\left(\omega t-\phi\right)$$

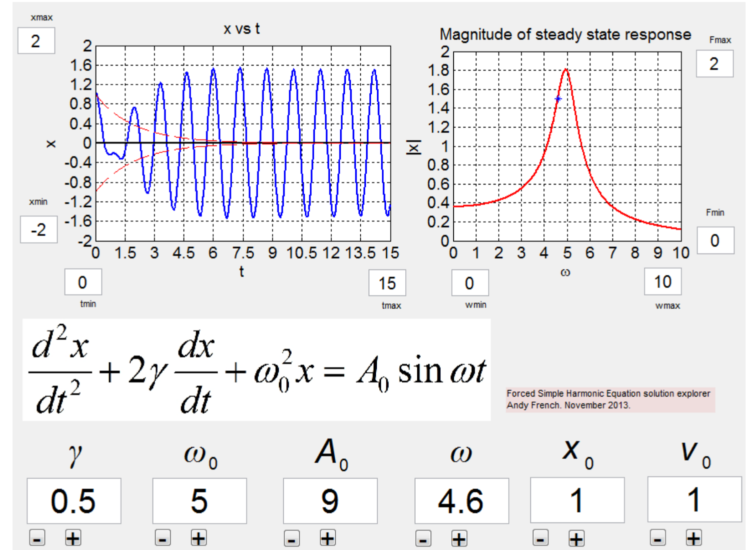
$$\Phi=\tan^{-1}\left(\frac{\dot{x}_0+\gamma\left(x_0+B\sin\phi\right)-B\omega\cos\phi}{\left(x_0+B\sin\phi\right)\sqrt{\omega_0^2-\gamma^2}}\right)$$

$$A=\frac{x_0+B\sin\phi}{\cos\Phi}$$

$$B=\frac{A_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}$$

$$\phi=\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

'Steady state' solution amplitude and phase



Critically damped

$$\gamma=\omega_0$$

$$x(t)=e^{-\gamma t}\left(A_1+A_2t\right)+B\sin\left(\omega t-\phi\right)$$

$$A_1=x_0+B\sin\phi$$

$$A_2=\dot{x}_0+\gamma\left(x_0+B\sin\phi\right)-B\omega\cos\phi$$

$$B_{max}\text{ when }\quad \omega=\sqrt{\omega_0^2-2\gamma^2}$$

$$B_{max}=\frac{A_0}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

Note this *resonance frequency* will always be less than or equal to the *natural frequency*

Note we only get a **resonance peak** when

$$\omega_0^2-2\gamma^2\geq0$$

$$\gamma\leq\frac{\omega_0}{\sqrt{2}}$$

$$\gamma\leq0.7011\omega_0$$

Over-damped

$$\gamma>\omega_0$$

$$x(t)=e^{-\gamma t}\left(A_1e^{t\sqrt{\gamma^2-\omega_0^2}}+A_2e^{-t\sqrt{\gamma^2-\omega_0^2}}\right)+B\sin\left(\omega t-\phi\right)$$

$$A_1=\tfrac{1}{2}\left(x_0+B\sin\phi\right)+\frac{\dot{x}_0+\gamma\left(x_0+B\sin\phi\right)-B\omega\cos\phi}{2\sqrt{\gamma^2-\omega_0^2}}$$

$$A_2=\tfrac{1}{2}\left(x_0+B\sin\phi\right)-\frac{\dot{x}_0+\gamma\left(x_0+B\sin\phi\right)-B\omega\cos\phi}{2\sqrt{\gamma^2-\omega_0^2}}$$