**Solving equations using iteration** $x_{n+1} = F(x_n)$

In many situations it may *not* be possible to find the *exact* solution(s) to an equation *f(x) = 0* via algebraic means. However, we can *compute* a solution to the precision of our calculating device (e.g. 12.d.p for most modern calculators) using an *iterative* scheme.

In essence, the idea is to re-write *f(x) = 0* into the form *x = F(x)* and then consider the sequence $x_{n+1} = F(x_n)$ . If the initial value $x_1$ is suitably chosen, then the sequence *may* converge on the desired root of *f(x).* However, *f(x)* may have *many* roots and, depending on the gradient of *F(x)* near the root, the sequence may *diverge* or enter a *cycle* rather than converge. To understand what is going on, plot *y* vs *x* and *y* vs *F(x)* and trace the path of the iteration. This is called a *cobweb diagram* due to the cobweb-like manner the lines converge or diverge.

**Example**

$$f(x) = x^3 - 3x - 1$$
$$f(x) = 0$$
$$x_{n+1} = \frac{x_n^3 - 1}{3}$$
$$x_{n+1} = \sqrt[3]{3x_n + 1}$$

Two different iteration schemes of the $x_{n+1} = F(x_n)$ form

Note in the above case the first *F(x)* is the inverse of the other. In many cases $F^{-1}(x)$ will converge on a root when *F(x)* diverges, and vice-versa.

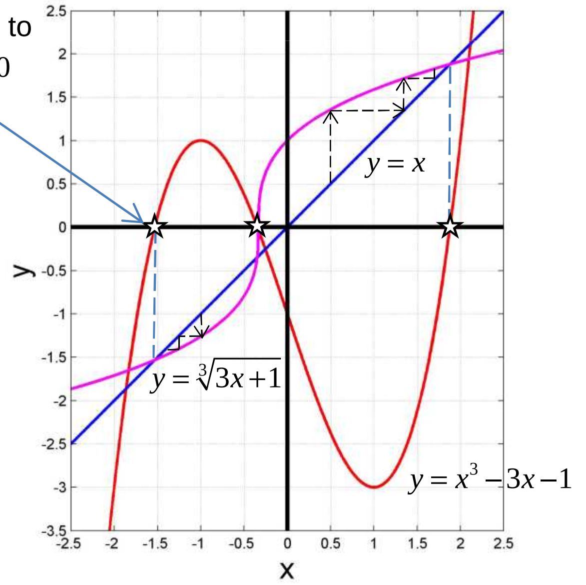
$$\begin{aligned} x_{n+1} &= \sqrt[3]{3x_n + 1} \\ x_1 &= -1 \\ x_2 &= -1.259921 \\ x_3 &= -1.406056 \\ x_4 &= -1.476396 \\ x_{33} &= -1.53208888624 \end{aligned}$$

$$\begin{aligned} x_{n+1} &= \sqrt[3]{3x_n + 1} \\ x_1 &= 0.5 \\ x_2 &= 1.357209 \\ x_3 &= 1.718103 \\ x_4 &= 1.932566 \\ x_{23} &= 1.87938524157 \end{aligned}$$

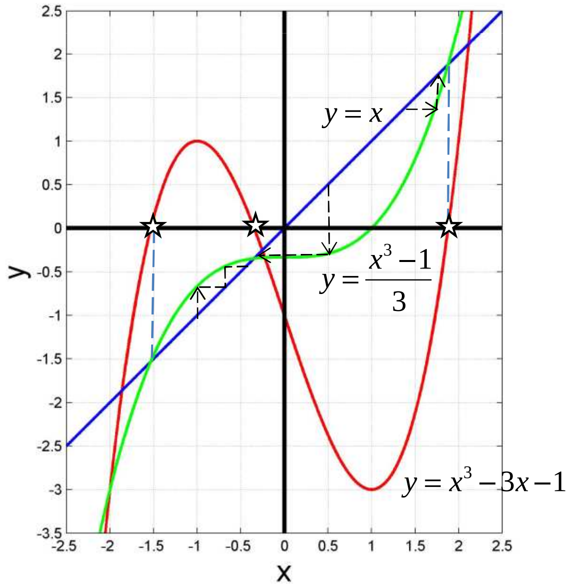
$$\begin{aligned} x_{n+1} &= \frac{x_n^3 - 1}{3} \\ x_1 &= 0.5 \\ x_2 &= -0.29167 \\ x_3 &= -0.34160 \\ x_4 &= -0.34662 \\ x_{16} &= -0.347296355334 \end{aligned}$$

$$\begin{aligned} x_{n+1} &= \frac{x_n^3 - 1}{3} \\ x_1 &= -1 \\ x_2 &= -0.666667 \\ x_3 &= -0.432099 \\ x_4 &= -0.360226 \\ x_{16} &= -0.347296355334 \end{aligned}$$

☆ Solutions to $x^3 - 3x - 1 = 0$




The iteration $x_{n+1} = \sqrt[3]{3x_n + 1}$ converges on one of the outer roots (-1.532 or 1.879) but *not* the middle root




The iteration $x_{n+1} = \frac{x_n^3 - 1}{3}$ converges on the inner root (-0.347), or diverges

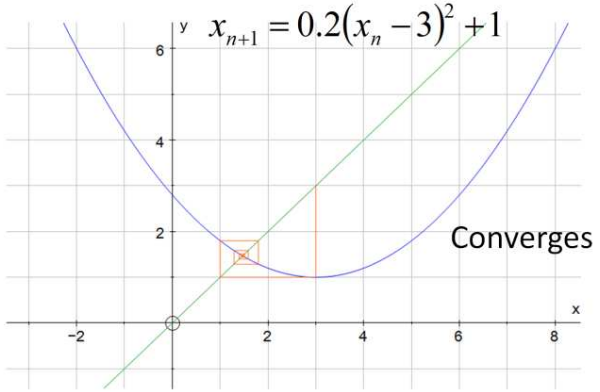
## Investigating convergence

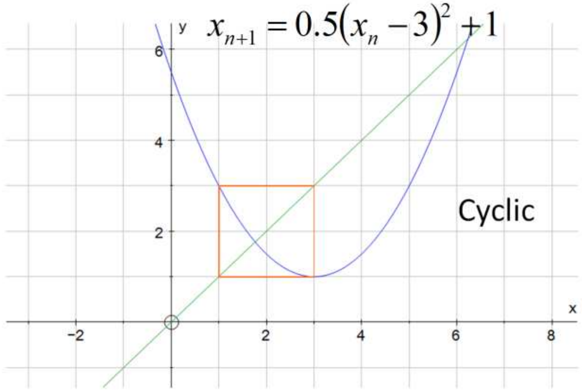
$x_{n+1} = a\left(x_n - 3\right)^2 + 1$

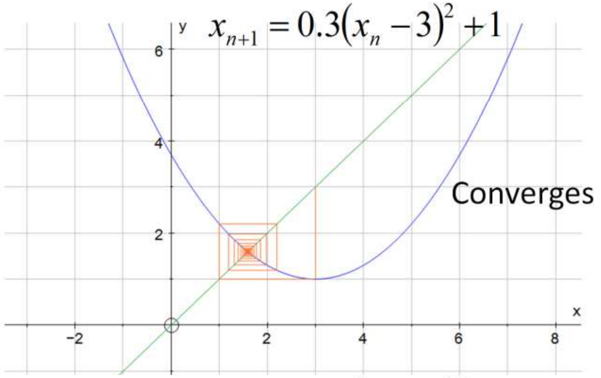
i.e. roots of

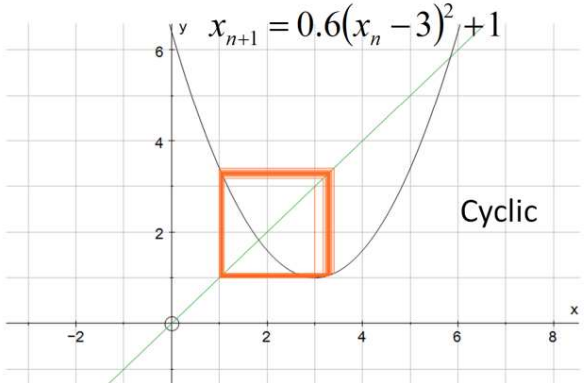
$f(x) = ax^2 - (6a+1)x + 1$

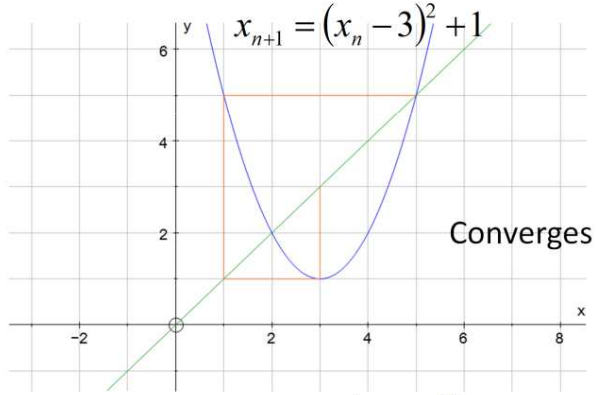
**Observations:**
If $f(x) = 0$ can be written as $x = F(x)$ and has a real solution:
$x_{n+1} = F(x_n)$ will typically converge toward the solution (the 'root') if $x_1$ is close to the root and $\left|\frac{dF}{dx}\right| < 1$ in the neighbourhood of the root.

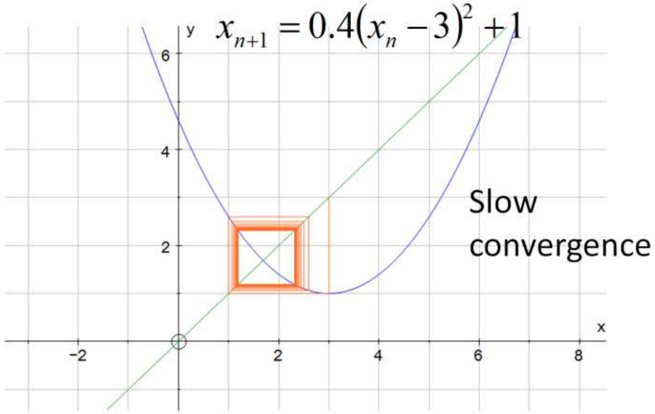


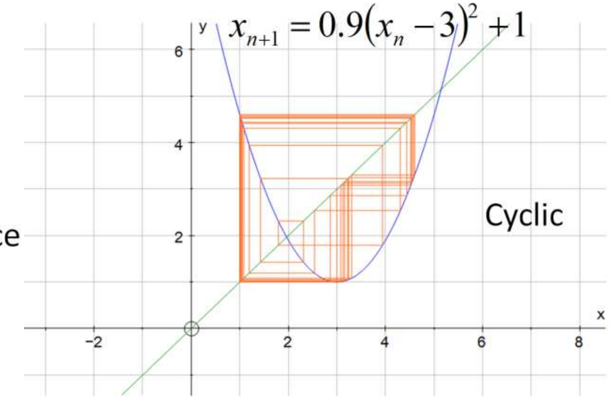



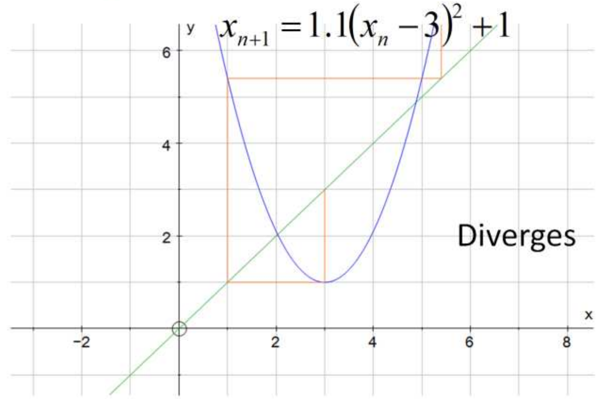



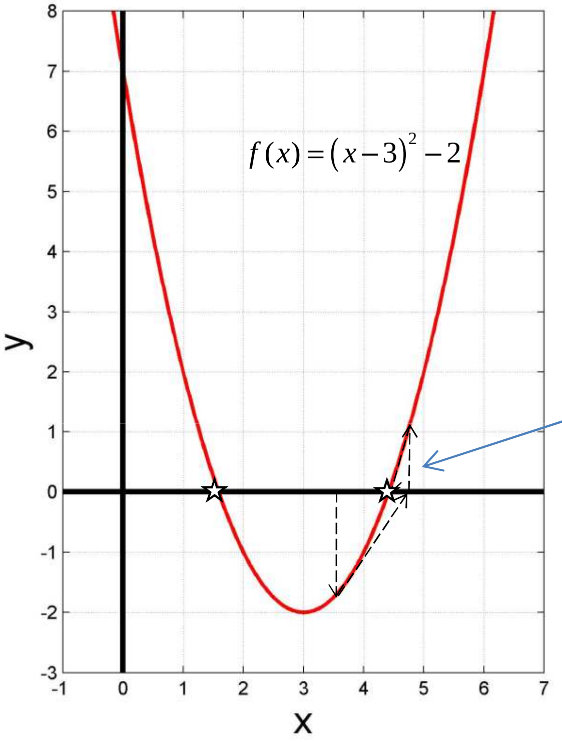
## The Newton-Raphson method

Rather than educated guesswork at what iteration scheme of the form $x_{n+1} = F(x_n)$ one should use, the *Newton-Raphson* method offers a more sophisticated (and faster converging) iterative scheme. It works very well unless the root is close to a *stationary point* of the equation $x = F(x)$ (in which case it diverges).

$$\boxed{\begin{array}{l} f(x)=0 \\ x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)} \\ f'(x)=\dfrac{df}{dx} \end{array}}$$

The idea is as follows:

(1) Guess an initial value $x=x_1$ for the root
(2) Find where the tangent at $f(x)$ crosses the $x$ axis
(3) Let this crossing point be the next $x$ value in the sequence




This method can result in very rapid convergence

$$f(x)=(x-3)^2-2$$
$$f'(x)=2(x-3)$$

Of course the Newton-Raphson method also *assumes* the function *f(x) can be differentiated* at all points near the root of *f(x).* This *may not be true* for all functions. e.g. tan*(x) and 1/x* have *undefined* gradients at, respectively, $x = \pi/2$ and 0.

$$x_{n+1}=x_n-\frac{(x_n-3)^2-2}{2(x_n-3)}$$

$x_1 = 1$
$x_2 = 1.5$
$x_3 = 1.583333$
$x_4 = 1.585784$
$x_6 = 1.58578643763$

$$x_{n+1}=x_n-\frac{(x_n-3)^2-2}{2(x_n-3)}$$

$x_1 = 3.5$
$x_2 = 5.25$
$x_3 = 4.56944$
$x_4 = 4.42189$
$x_5 = 4.41423$
$x_6 = 4.41214$
$x_7 = 4.1421356237$

### Derivation

Tangent to f(x) at $x_n$ is

$$y_T = xf'(x_n)+c$$
$$f(x_n)=x_n f'(x_n)+c$$
$$c=f(x_n)-x_n f'(x_n)$$
$$\therefore y_T=(x-x_n)f'(x_n)+f(x_n)$$

The tangent crosses the x axis when

$$y_T=0$$
$$x=x_n-\frac{f(x_n)}{f'(x_n)}$$

Hence is this x axis crossing point is the next x value in the sequence

$$\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}}$$

The Newton-Raphson method is possibly divergent when a root is near a stationary point i.e. where *f′(x) = 0*

At this point the ratio *f(x) / f′(x)* will tend to 0/0 which may be undefined.

'Possible divergence' depends on how 'fast' *f(x)* and *f′(x)* tend to zero near the stationary point.