Calculus defines the **derivative** of a function *f(x)* to be its *gradient* at coordinate *(x,y)* along the curve *y = f(x).* Since we can often find a formula for the derivative of a function using calculus, this means we can work various features of the curve *y = f(x)* without having to plot the curve first:

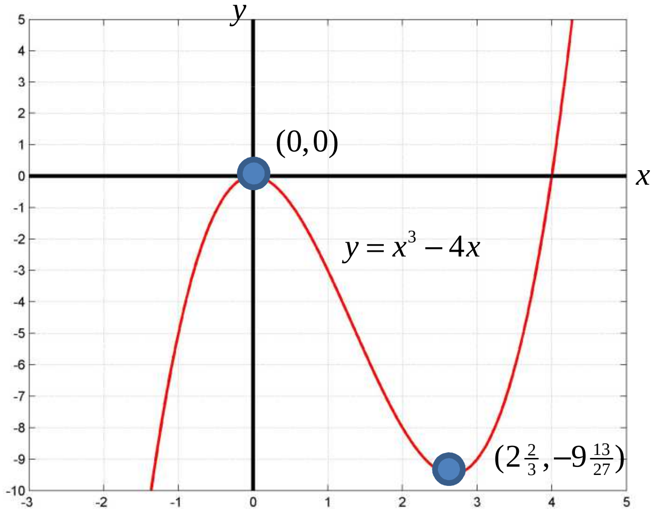
- The location and nature (maxima, minima, point of inflexion) of any stationary points (or 'turning points') i.e. when the gradient is zero.
- The domain of *y = f(x)* when f(x) is increasing (or decreasing)
- The equations of tangents and normals to *y = f(x)* which pass through *(x,y)*

Consider $y = x^3 - 4x^2$ $\quad y = x^2(x-4)$ so *y = 0* when *x= 0 or 4*

$$\frac{dy}{dx} = 3x^2 - 8x$$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow x(3x-8) = 0$$

$\therefore x = 0, 2\frac{2}{3}$ $\quad$ Therefore *stationary points* are $(0,0)$ & $(2\frac{2}{3}, -9\frac{13}{27})$



$$x = \frac{8}{3}$$

$$y = \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2$$

$$y = \frac{512}{27} - \frac{256}{9}$$

$$y = \frac{512}{27} - \frac{768}{27}$$

$$y = -9\frac{13}{27}$$

From the graph, (0,0) is clearly a maximum and the other stationary point is clearly a minimum. However, we don't *have* to sketch the curve to work this out.....

Note the function is *increasing* when the *gradient is positive* i.e.

$$\frac{dy}{dx} > 0 \Rightarrow x < 0, x > 2\frac{1}{3}$$

Use the stationary points and a sketch to work out the regions of the inequality

Consider the most basic quadratic:

$$y = ax^2$$

$$\frac{dy}{dx} = 2ax$$

$$\frac{d^2y}{dx^2} = 2a$$

If *a* is > 0 then the quadratic is a 'smile' and hence its vertex (i.e. the stationary point) is a *minimum*. The converse is true if *a* < 0 and the quadratic is an inverted parabola.

This is a general result – not just for quadratics!

$y = ax^2$	$y = ax^2$
$a < 0$	$a > 0$
Maxima	Minima
$\left.\frac{d^2y}{dx^2}\right\rvert_{\text{stationary point}} < 0$	$\left.\frac{d^2y}{dx^2}\right\rvert_{\text{stationary point}} > 0$

If the second derivative is zero it means the stationary point is *neither* a minimum or maximum. It is instead a ***point of inflexion***. The simplest example is the point (0,0) in $y = x^3$

Since, with suitable scaling of coordinates, **we can approximate *any* function in the vicinity of a stationary point by a *parabola***; *the sign of the second derivative evaluated at the stationary point is a general characteristic of the stationary point.*

In the example of $y = x^3 - 4x^2$

$$\frac{dy}{dx} = 3x^2 - 8x$$

$$\frac{d^2y}{dx^2} = 6x - 8$$

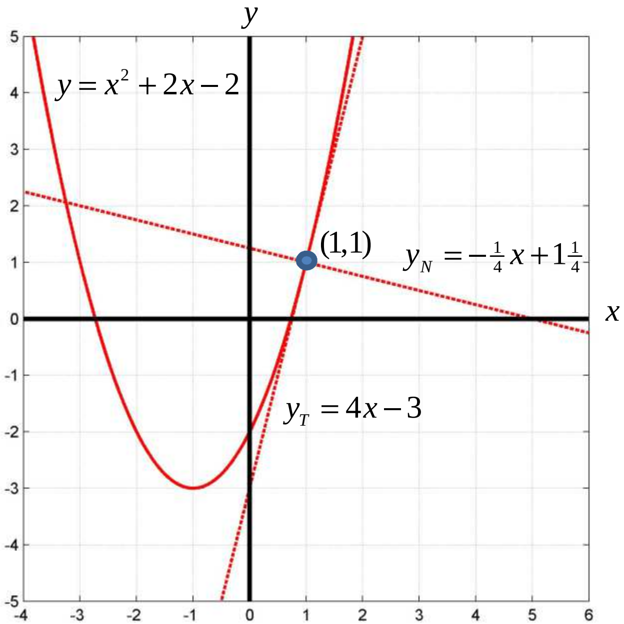
Evaluate the second-derivative *at the stationary point*

$\therefore \left.\frac{d^2y}{dx^2}\right\rvert_{x=0} = -8$ $\quad$ i.e. –ve therefore (0,0) is a *maxima*

$\therefore \left.\frac{d^2y}{dx^2}\right\rvert_{x=\frac{8}{3}} = 8$ $\quad$ i.e. +ve therefore this point is a *minima*

Evaluating the derivative of *f(x)* at a particular *x* value enables us to work out the *tangent* through the point *(x,y)* on the curve *y = f(x).*

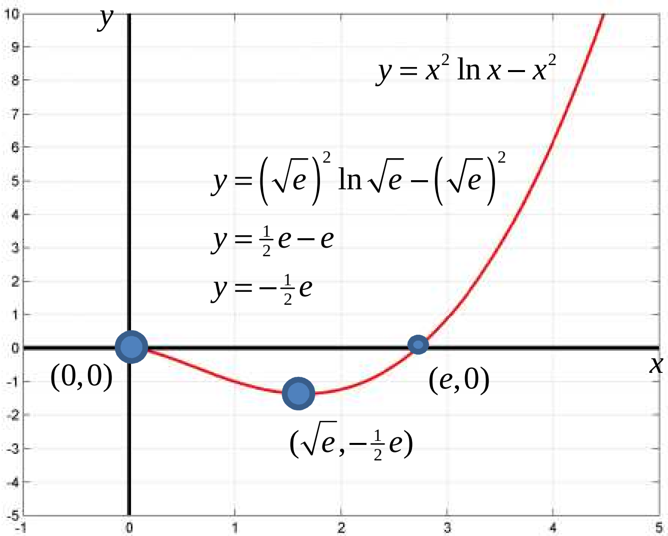
$$y=(x+1)^2-3$$
$$y=x^2+2x-2$$
$$\frac{dy}{dx}=2x+2$$
$$\left.\frac{dy}{dx}\right|_{x=1}=4$$
$$\therefore y_T=4x+c$$
This must pass through (1,1)
$$1=4+c$$
$$\therefore \boxed{y_T=4x-3}$$



Since ***the negative-reciprocal of the gradient gives the gradient of the normal***, we can also find the equation of the normal.

$$y=x^2+2x-2$$
$$\frac{dy}{dx}=2x+2$$
$$\left.\frac{dy}{dx}\right|_{x=1}=4$$
$$\therefore y_N=-\tfrac{1}{4}x+c$$
This must pass through (1,1)
$$1=-\tfrac{1}{4}+c$$
$$\therefore \boxed{y_N=-\tfrac{1}{4}x+1\tfrac{1}{4}}$$

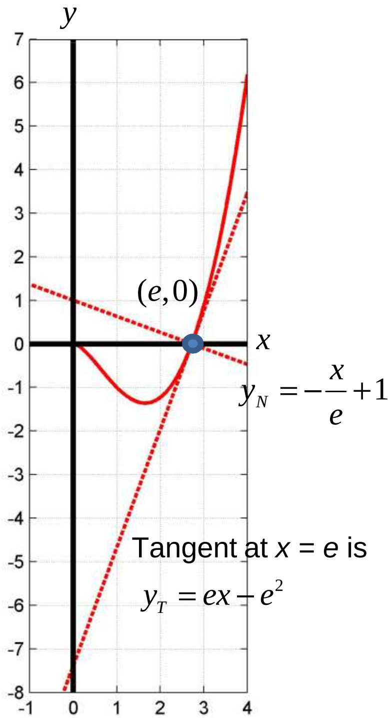
Example not involving polynomials (may require chain rule, product rule etc).



$$y=x^2\ln x-x^2$$
$$\frac{dy}{dx}=x^2\frac{1}{x}+2x\ln x-2x$$
$$\frac{dy}{dx}=2x\ln x-x$$
$$\frac{dy}{dx}=x(2\ln x-1)$$
$$\frac{dy}{dx}=0\Rightarrow x=0,\sqrt{e}$$

Stationary points are:

$$\boxed{(0,0)\ \&\ (\sqrt{e},-\tfrac{1}{2}e)}$$



$$\frac{d^2y}{dx^2}=\frac{d}{dx}(2x\ln x-x)$$
$$\frac{d^2y}{dx^2}=2x\frac{1}{x}+2\ln x-1$$
$$\frac{d^2y}{dx^2}=1+2\ln x$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=-\infty$$
i.e. –ve therefore (0,0) is a *maxima*
$$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{e}}=2$$
i.e. +ve therefore this point is a *minima*