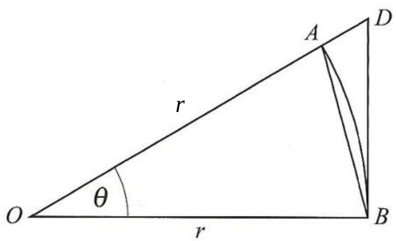
Calculus with trigonometric functions

Consider the right angled triangle *ODB*, the sector *OAB* and the isosceles triangle *OAB*. (i.e. *OA* = *OB* = *r*). Areas are:



$$A_{ODB} = \tfrac{1}{2} r \times r \tan\theta$$
$$A_{OAB} = \frac{\theta}{360^\circ}\pi r^2$$
$$A_{\Delta OAB} = \tfrac{1}{2} r \times r \sin\theta$$

If we use *radians* for our angular measure (π radians = 180°) the areas become:

$$A_{ODB} = \tfrac{1}{2} r^2 \tan\theta$$
$$A_{OAB} = \tfrac{1}{2} r^2\theta$$
$$A_{\Delta OAB} = \tfrac{1}{2} r^2 \sin\theta$$

Now from the diagram:

$$A_{ODB} > A_{OAB} > A_{\Delta OAB}$$
$$\tfrac{1}{2} r^2 \tan\theta > \tfrac{1}{2} r^2\theta > \tfrac{1}{2} r^2 \sin\theta$$

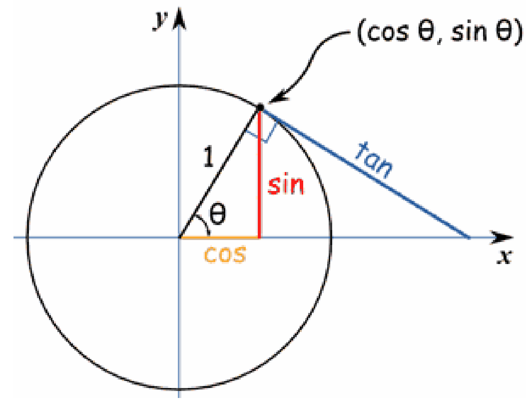
$$\therefore \quad \tan\theta > \theta > \sin\theta$$

As θ becomes small, the diagram justifies the result:

$$\lim_{\theta\to 0}\left(\tan\theta\right) = \lim_{\theta\to 0}\left(\sin\theta\right) = \theta$$

where θ is in radians

Now consider the unit circle definitions of sine and cosine



$$x = \cos\theta$$
$$y = \sin\theta$$

Hence: $x + \delta x = \cos(\theta + \delta\theta)$

$y + \delta y = \sin(\theta + \delta\theta)$

Expanding the latter using addition formula for sine and cosine we find:

$$\delta x = \cos(\theta + \delta\theta) - x$$
$$\delta x = \cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta - \cos\theta$$
$$\frac{\delta x}{\delta\theta} = \frac{\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta - \cos\theta}{\delta\theta}$$

$$y + \delta y = \sin(\theta + \delta\theta)$$
$$\delta y = \sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta - \sin\theta$$
$$\frac{\delta y}{\delta\theta} = \frac{\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta - \sin\theta}{\delta\theta}$$

The gradient of sine and cosine functions are defined as:

$$\frac{d}{d\theta}\sin\theta = \lim_{\delta\theta\to 0}\left(\frac{\delta y}{\delta\theta}\right)$$
$$\frac{d}{d\theta}\cos\theta = \lim_{\delta\theta\to 0}\left(\frac{\delta x}{\delta\theta}\right)$$

Now we can make use of the results

$$\lim_{\delta\theta\to 0}\left(\sin\delta\theta\right) = \delta\theta$$
$$\lim_{\delta\theta\to 0}\left(\cos\delta\theta\right) = 1$$

Hence: $$\frac{d}{d\theta}\sin\theta = \lim_{\delta\theta\to 0}\left(\frac{\sin\theta\times 1 + \delta\theta\cos\theta - \sin\theta}{\delta\theta}\right)$$

$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

$$\frac{d}{d\theta}\cos\theta = \lim_{\delta\theta\to 0}\left(\frac{\cos\theta\times 1 - \delta\theta\sin\theta - \cos\theta}{\delta\theta}\right)$$

$$\boxed{\frac{d}{d\theta}\cos\theta = -\sin\theta}$$

We can use identities plus the quotient rule to compute derivatives (and hence integrals) of the other trigonometric functions.

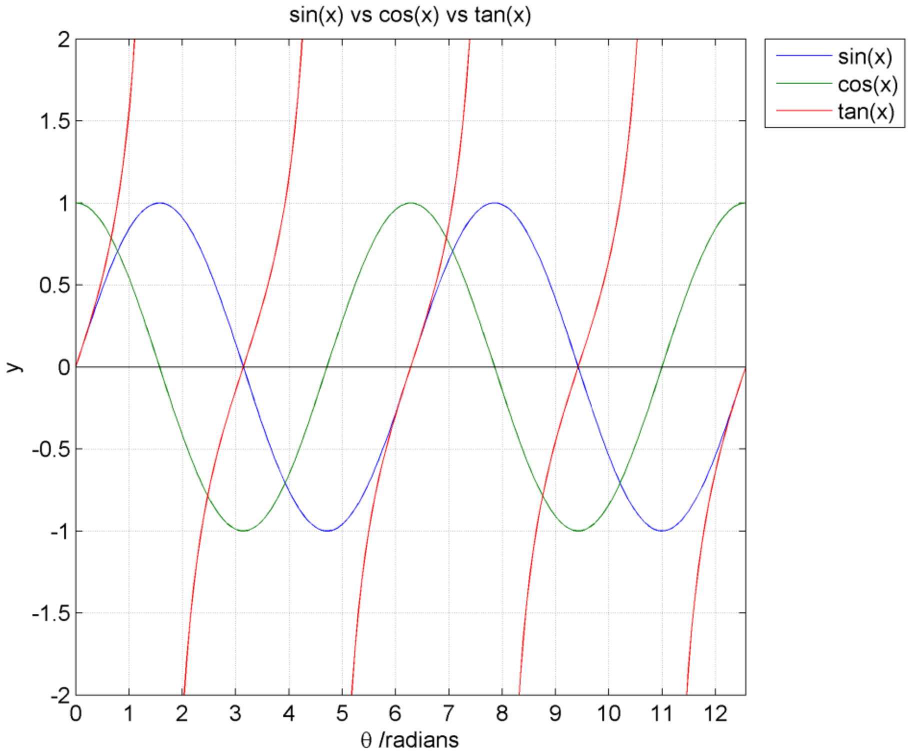
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\frac{d}{d\theta}\tan\theta = \frac{\cos\theta\frac{d}{d\theta}\sin\theta - \sin\theta\frac{d}{d\theta}\cos\theta}{\cos^2\theta}$$

$$\frac{d}{d\theta}\tan\theta = \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta}$$

$$\frac{d}{d\theta}\tan\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$\boxed{\frac{d}{d\theta}\tan\theta = \sec^2\theta}$$



$$\frac{d}{d\theta}\cos\theta = -\sin\theta$$

$$-\int d\cos\theta = \int \sin\theta d\theta$$

$$\boxed{\int \sin\theta d\theta = -\cos\theta + c}$$

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

$$\int d\sin\theta = \int \cos\theta d\theta$$

$$\boxed{\int \cos\theta d\theta = \sin\theta + c}$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta$$

$$\int d\tan\theta = \int \sec^2\theta d\theta$$

$$\boxed{\int \sec^2\theta d\theta = \tan\theta + c}$$

Using a simple substitution and the chain rule we can derive the following more general results:

$$\frac{d}{d\theta}\sin(a\theta) = a\cos(a\theta)$$

$$\frac{d}{d\theta}\cos(a\theta) = -a\sin(a\theta)$$

$$\frac{d}{d\theta}\tan(a\theta) = a\sec^2(a\theta)$$

$$\int \sin(a\theta)d\theta = -\tfrac{1}{a}\cos(a\theta) + c$$

$$\int \cos(a\theta)d\theta = \tfrac{1}{a}\sin(a\theta) + c$$

$$\int \tan(a\theta)d\theta = \tfrac{1}{a}\sec^2(a\theta) + c$$