Calculus of Variations

A large number of problems in applied mathematics concern finding the shape of a curve $x(t)$. such that an integral S of $x(t)$. (and possibly its derivative dx/dt and t also) is 'optimized'* with respect to *small variations* in $x(t)$.

To generalize, let the integrand L of be a function of a family of variables and associated derivatives:

$$S = \int_{t_a}^{t_b} L\left(x_1, x_2, ...x_N, \dot{x}_1, \dot{x}_2, ...\dot{x}_N, t\right) dt$$

$$\dot{x}_i = \frac{dx_i}{dt}$$

Consider a small perturbation ε such that: $X_i = x_i(t) + \varepsilon\eta_i(t)$

and $\eta_i(t_a) = \eta_i(t_b) = 0$

Define: $S_\varepsilon = \int_{t_a}^{t_b} L\left(X_1, X_2, ...X_N, \dot{X}_1, \dot{X}_2, ...\dot{X}_N, t\right) dt$

The goal is to find an equation for $x_i(t)$ such that $\left.\frac{dS_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = 0$

Now: $\frac{dS_\varepsilon}{d\varepsilon} = \int_{t_a}^{t_b} \frac{dL}{d\varepsilon} dt$

Since when $\varepsilon = 0$, $X_i = x_i$ criteria above is consistent with finding the **stationary** values of S. This means a minimum or maximum value, i.e. S is ***extremized***.

By the **chain rule**:

$$\frac{dL}{d\varepsilon} = \frac{\partial L}{\partial t}\frac{dt}{d\varepsilon} + \sum_{i=1}^{N}\left(\frac{\partial L}{\partial X_i}\frac{dX_i}{d\varepsilon} + \frac{\partial L}{\partial \dot{X}_i}\frac{d\dot{X}_i}{d\varepsilon}\right)$$

$X_i = x_i + \varepsilon\eta_i$ Hence: $\frac{dt}{d\varepsilon} = 0,\ \frac{dX_i}{d\varepsilon} = \eta_i,\ \frac{d\dot{X}_i}{d\varepsilon} = \dot{\eta}_i$

Therefore: $$\frac{dS_\varepsilon}{d\varepsilon} = \int_{t_a}^{t_b} \frac{dL}{d\varepsilon} dt = \int_{t_a}^{t_b} \sum_{i=1}^{N}\left(\frac{\partial L}{\partial X_i}\eta_i + \frac{\partial L}{\partial \dot{X}_i}\dot{\eta}_i\right) dt = \sum_{i=1}^{N}\left\{\int_{t_a}^{t_b} \frac{\partial L}{\partial X_i}\eta_i dt + \int_{t_a}^{t_b} \frac{\partial L}{\partial \dot{X}_i}\dot{\eta}_i dt\right\}$$

Using by-parts integration: $$\int_{t_a}^{t_b} \frac{\partial L}{\partial \dot{X}_i}\dot{\eta}_i dt = \left[\eta \frac{\partial L}{\partial \dot{X}_i}\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right)\eta_i dt = -\int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right)\eta_i dt$$

Note: $\eta_i(t_a) = \eta_i(t_b) = 0$

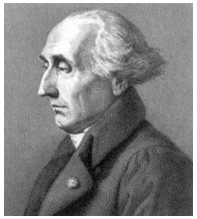
$$\therefore \frac{dS_\varepsilon}{d\varepsilon} = \sum_{i=1}^{N}\left\{\int_{t_a}^{t_b}\left\{\frac{\partial L}{\partial X_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right)\right\}\eta_i dt\right\}$$

Now: $\left.\frac{dS_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$

This yields the **Euler-Lagrange equation**, the solution of which will yield $x_i(t)$. Note there is a *distinct* equation for each variable $x_i(t)$. This feature enables a complicated multi-variable calculation to be *separated into component parts*, which will prove invaluable when we apply these ideas to mechanical systems.

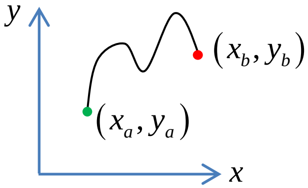
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$ **Euler-Lagrange equation**

Leonhard Euler
1707-1783



Joseph-Louis Lagrange
1736-1813

Example #1: The parametric curve $x(t)$, $y(t)$ which describes the shortest distance S between two points on a flat plane.



$x(t=0) = x_a,\ x(t=\tau) = x_b$

$y(t=0) = y_a,\ y(t=\tau) = y_b$

$S = \int_{t=0}^{\tau}\sqrt{dx^2 + dy^2} = \int_{x_a}^{x_b}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ i.e. transform to a single set of variables y, dy/dx, x

$\therefore L = \sqrt{1 + \dot{y}^2}$

Hence assign: $\dot{y} = \frac{dy}{dx}$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$

$$\therefore \frac{d}{dx}\left(\tfrac{1}{2}2\dot{y}\left(1 + \dot{y}^2\right)^{-\frac{1}{2}}\right) = 0$$

$$\therefore \dot{y}\left(1 + \dot{y}^2\right)^{-\frac{1}{2}} = k$$

where k is a constant. This means:

$$\dot{y} = \frac{dy}{dx} = \text{constant}$$

i.e. a **straight line** →

$$y = y_a + \frac{x - x_a}{x_b - x_a}(y_b - y_a)$$

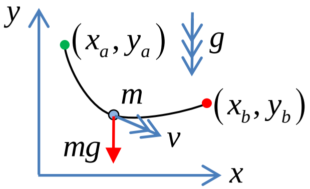
$$x(t) = x_a + \frac{t}{\tau}(x_b - x_a)$$

$$y(t) = y_a + \frac{t}{\tau}(y_b - y_a)$$

*i.e. either a maximum or minimum value

Example #2: The Brachitochrone problem

If a bead is allowed to slide frictionlessly along a wire, what shape should the wire be such that the bead takes the *least time* to slide between two points?



The time taken for the bead to slide between two points is:

$$\tau = \int_{x_a}^{x_b} \frac{\sqrt{dx^2 + dy^2}}{v}$$

$$\tau = \int_{x_a}^{x_b} dx \frac{1}{v}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Energy is conserved in the system since the bead is assumed to slide without friction:

Hence, if the bead starts from rest:

$$\tfrac{1}{2}mv^2 + mgy = mgy_a$$

$$\therefore v = \sqrt{2g(y_a - y)}$$

$$\therefore \tau = \tfrac{1}{\sqrt{2g}} \int_{x_a}^{x_b} dx \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y_a - y}}$$

Applying the Euler-Lagrange equation:

$$\dot{y} = \frac{dy}{dx}, \quad L = \frac{(1+\dot{y}^2)^{\frac{1}{2}}}{\sqrt{y_a - y}}, \quad \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$

Now it is worth noting that L does not explicitly include the integration variable x. This means we can use the **Beltrami Identity:**

If $\frac{\partial L}{\partial x} = 0$ this means $L - \dot{y}\frac{\partial L}{\partial \dot{y}} = \text{constant}$

$$\frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\dot{y} + \frac{\partial L}{\partial \dot{y}}\ddot{y}$$ Chain Rule

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$ Euler Lagrange equation

$$\therefore \frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right)\dot{y} + \frac{\partial L}{\partial \dot{y}}\ddot{y}$$

$$\therefore \frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{d}{dx}\left(\dot{y}\frac{\partial L}{\partial \dot{y}}\right)$$

$$\therefore \frac{d}{dx}\left\{L - \dot{y}\frac{\partial L}{\partial \dot{y}}\right\} = \frac{\partial L}{\partial x}$$

If $\frac{\partial L}{\partial x} = 0$ this means

$L - \dot{y}\frac{\partial L}{\partial \dot{y}} = \text{constant}$

We can apply the **Beltrami Identity** to the Brachitochrone problem

$$\frac{(1+\dot{y}^2)^{\frac{1}{2}}}{\sqrt{y_a - y}} - \dot{y} \times \tfrac{1}{2} 2\dot{y} \frac{(1+\dot{y}^2)^{-\frac{1}{2}}}{\sqrt{y_a - y}} = k$$

$$(1+\dot{y}^2)^{\frac{1}{2}} - \dot{y}^2 (1+\dot{y}^2)^{-\frac{1}{2}} = k\sqrt{y_a - y}$$

$$\frac{1+\dot{y}^2 - \dot{y}^2}{(1+\dot{y}^2)^{\frac{1}{2}}} = k\sqrt{y_a - y}$$

$$\therefore (y_a - y)(1+\dot{y}^2) = k^2$$

It turns out the solutions to this differential equation are **cylcloids**

$$y = y_a - \tfrac{1}{2}k^2(1 - \cos\theta)$$
$$x = x_a + \tfrac{1}{2}k^2(\theta - \sin\theta)$$

Parametric equations of a cycloid

$$\dot{y} = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-\frac{1}{2}k^2 \sin\theta}{\frac{1}{2}k^2(1-\cos\theta)} = -\frac{\sin\theta}{1-\cos\theta}$$

$$\therefore 1 + \dot{y}^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2}$$

$$\therefore 1 + \dot{y}^2 = \frac{2(1-\cos\theta)}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta}$$

$$y = y_a - \tfrac{1}{2}k^2(1-\cos\theta) \Rightarrow 1 - \cos\theta = \frac{2}{k^2}(y_a - y)$$

$$\therefore 1 + \dot{y}^2 = \frac{2}{\frac{2}{k^2}(y_a - y)} = \frac{k^2}{y_a - y} \quad *$$

Putting this together:

$$(y_a - y)(1+\dot{y}^2) = k^2$$ From application of Beltrami Identity

$$1 + \dot{y}^2 = \frac{k^2}{y_a - y}$$ Using cycloid result * above

$$\therefore (y_a - y)(1+\dot{y}^2) = (y_a - y)\frac{k^2}{y_a - y} = k^2$$

Note the Brachitochrone curve is *independent* of not only the beam mass, but also the strength of gravity g. The constant k and the final value of θ depend on the start and end x,y coordinates. Note $\theta = 0$ yields the initial coordinates.

$$y_b = y_a - \tfrac{1}{2}k^2(1-\cos\theta_b) \quad \therefore \tfrac{1}{2}k^2 = \frac{y_a - y_b}{1 - \cos\theta_b}$$

$$x_b = x_a + \tfrac{1}{2}k^2(\theta_b - \sin\theta_b) \quad \therefore \tfrac{1}{2}k^2 = \frac{x_b - x_a}{\theta_b - \sin\theta_b}$$

$$\therefore \frac{y_a - y_b}{1-\cos\theta_b} = \frac{x_b - x_a}{\theta_b - \sin\theta_b} \qquad \alpha = \frac{y_a - y_b}{x_b - x_a}$$

$$\therefore (\theta_b - \sin\theta_b)\alpha = 1 - \cos\theta_b$$

$$\therefore f(\theta_b) = \frac{1-\cos\theta_b}{\alpha} + \sin\theta_b - \theta_b = 0$$

This can be solved using an iterative scheme e.g. **Newton-Raphson** (see next page)

$$y = y_a - \tfrac{1}{2}k^2\left(1-\cos\theta\right)$$
$$x = x_a + \tfrac{1}{2}k^2\left(\theta - \sin\theta\right)$$

$$c = \tfrac{1}{2}k^2 = \frac{y_a - y_b}{1-\cos\theta_b}$$

$$\alpha = \frac{y_a - y_b}{x_b - x_a}$$

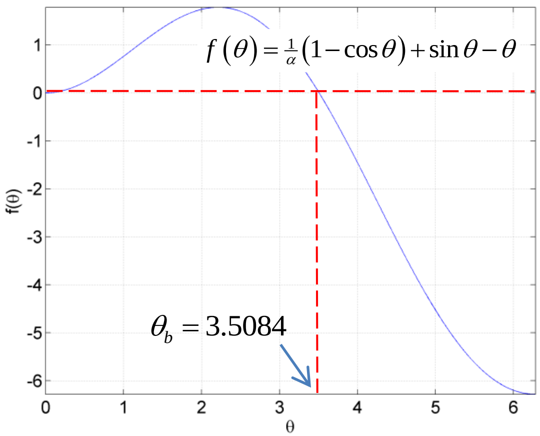
$$f(\theta) = \tfrac{1}{\alpha}\left(1-\cos\theta\right) + \sin\theta - \theta = 0$$

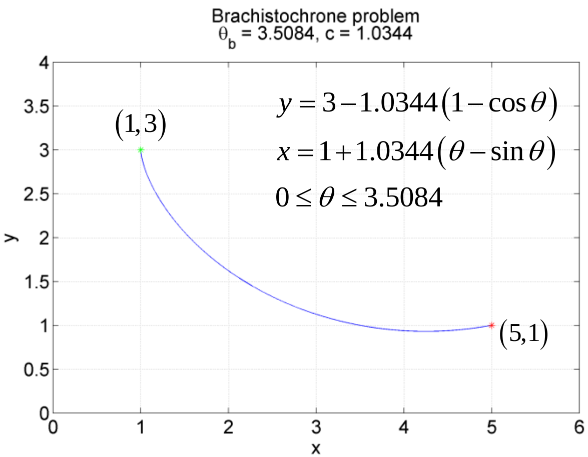
$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)}$$ Newton-Raphson method

$$\therefore \theta_{n+1} = \theta_n - \frac{\tfrac{1}{\alpha}\left(1-\cos\theta_n\right) + \sin\theta_n - \theta_n}{\tfrac{1}{\alpha}\sin\theta_n + \cos\theta_n - 1}$$

$$x_a = 1,\ x_b = 5,\ y_a = 3,\ y_b = 1$$

$$\theta_1 = 4,\ \theta_b = 3.5084,\ c = 1.0344$$



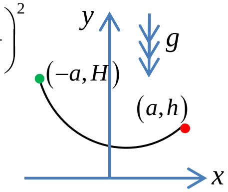


Example #3: The Catenary:

Incorporating constraints using Lagrange Multipliers

If a uniform, inextensible chain of length l is connected between two points *(-a,H)* and *(a,h)* separated by distance $d < l$, what curve does the chain take if gravitational potential energy is to be minimized? The total GPE V of the chain, which is of uniform mass per unit length ρ is:

$$V = \rho g \int_{x=-a}^{a} y\sqrt{dx^2 + dy^2} = \int_{-a}^{a} \rho g y dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



The curve of the chain is also *constrained* by its actual length

$$l = \int_{-a}^{a} \sqrt{dx^2 + dy^2} = \int_{-a}^{a} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

We can use the Euler-Lagrange equation to solve this by incorporating the **integrand of the constraint** into the L function using an addition via a **Lagrange Multiplier** λ

$$\dot{y} = \frac{dy}{dx},\quad L = \rho g y\left(1+\dot{y}^2\right)^{\frac{1}{2}} + \lambda\left(1+\dot{y}^2\right)^{\frac{1}{2}} = \left(1+\dot{y}^2\right)^{\frac{1}{2}}\left(\rho g y + \lambda\right)$$

$$\frac{\partial L}{\partial x} = 0 \quad \therefore L - \dot{y}\frac{\partial L}{\partial \dot{y}} = k$$ Beltrami Identity

$$\therefore \left(1+\dot{y}^2\right)^{\frac{1}{2}}\left(\rho g y + \lambda\right) - \dot{y} \times \tfrac{1}{2}2\dot{y}\left(1+\dot{y}^2\right)^{-\frac{1}{2}}\left(\rho g y + \lambda\right) = k$$

$$\therefore \frac{1+\dot{y}^2 - \dot{y}^2}{\left(1+\dot{y}^2\right)^{\frac{1}{2}}} = \frac{k}{\rho g y + \lambda} \quad \Rightarrow \tfrac{1}{k^2}\left(\rho g y + \lambda\right)^2 = 1 + \dot{y}^2$$

$$\therefore \frac{dy}{dx} = \sqrt{\tfrac{1}{k^2}\left(\rho g y + \lambda\right)^2 - 1}$$

$$\frac{\rho g y + \lambda}{k} = \cosh z \quad \therefore \frac{dy}{dx} = \frac{k}{\rho g}\sinh z \frac{dz}{dx}$$ Choose this substitution*

$$\therefore \frac{dy}{dx} = \sqrt{\cosh^2 z - 1} = \sinh z$$

$$\therefore \frac{k \sinh z}{\rho g}\frac{dz}{dx} = \sinh z \Rightarrow \frac{k}{\rho g}\frac{dz}{dx} = 1$$

$$\therefore \frac{kz}{\rho g} = x + c \Rightarrow \cosh^{-1}\left(\frac{\rho g y + \lambda}{k}\right) = \frac{\rho g}{k}(x+c)$$ Integration constant c

$$\therefore y = \frac{k}{\rho g}\cosh\left\{\frac{\rho g}{k}(x+c)\right\} - \frac{\lambda}{\rho g}$$

So the catenary is a cosh curve. Now to find the three constants!

To match the start and end points:

$$H = \frac{k}{\rho g}\cosh\left\{\frac{\rho g}{k}(-a+c)\right\} - \frac{\lambda}{\rho g}$$
$$h = \frac{k}{\rho g}\cosh\left\{\frac{\rho g}{k}(a+c)\right\} - \frac{\lambda}{\rho g}$$

Using the chain length constraint we can generate a third equation :

$$l = \int_{-a}^{a}\sqrt{dx^2 + dy^2} = \int_{-a}^{a} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{dy}{dx} = \sinh\left\{\frac{\rho g}{k}(x+c)\right\}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left\{\frac{\rho g}{k}(x+c)\right\}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2\left\{\frac{\rho g}{k}(x+c)\right\}$$

$$\therefore l = \int_{-a}^{a}\cosh\left\{\frac{\rho g}{k}(x+c)\right\}dx$$

$$\therefore l = \tfrac{k}{\rho g}\sinh\left\{\frac{\rho g}{k}(a+c)\right\} - \tfrac{k}{\rho g}\sinh\left\{\frac{\rho g}{k}(-a+c)\right\}$$

Note hyperbolic identities:

$$\cosh 2A - \cosh 2B = 2\sinh(A+B)\sinh(A-B)$$
$$\sinh 2A - \sinh 2B = 2\cosh(A+B)\sinh(A-B)$$

Hence: $$H - h = -\tfrac{2k}{\rho g}\sinh\left(\frac{\rho g c}{k}\right)\sinh\left(\frac{\rho g a}{k}\right)$$

$$l = \tfrac{2k}{\rho g}\cosh\left(\frac{\rho g c}{k}\right)\sinh\left(\frac{\rho g a}{k}\right)$$

$$\therefore \frac{H-h}{l} = -\tanh\left(\frac{\rho g c}{k}\right)$$

$$\therefore \frac{\rho g c}{k} = -\tanh^{-1}\left(\frac{H-h}{l}\right)$$

*This part of the derivation follows *Mathematical Methods for Physics & Engineering* (3rd Edition) pp786

$$H-h=-\tfrac{2k}{\rho g}\sinh\left(\frac{\rho gc}{k}\right)\sinh\left(\frac{\rho ga}{k}\right)$$

$$\frac{\rho gc}{k}=-\tanh^{-1}\left(\frac{H-h}{l}\right)$$

$$\therefore H-h=-\tfrac{2k}{\rho g}\sinh\left(-\tanh^{-1}\left(\frac{H-h}{l}\right)\right)\sinh\left(\frac{\rho ga}{k}\right)$$

$$\tanh^{-1}z=\ln\sqrt{\frac{1+z}{1-z}}$$

$$\therefore \sinh\left(\tanh^{-1}z\right)=\tfrac{1}{2}\left(e^{\ln\sqrt{\frac{1+z}{1-z}}}-e^{-\ln\sqrt{\frac{1+z}{1-z}}}\right)=\tfrac{1}{2}\left(\sqrt{\frac{1+z}{1-z}}-\sqrt{\frac{1-z}{1+z}}\right)$$

$$\therefore \sinh\left(\tanh^{-1}z\right)=\tfrac{1}{2}\left(\frac{1+z-(1-z)}{\sqrt{(1-z)(1+z)}}\right)=\frac{z}{\sqrt{1-z^2}}$$

$$\therefore H-h=\tfrac{2k}{\rho g}\frac{\frac{H-h}{l}}{\sqrt{1-\left(\frac{H-h}{l}\right)^2}}\sinh\left(\frac{\rho ga}{k}\right)$$

$$\therefore l\sqrt{1-\left(\frac{H-h}{l}\right)^2}=\frac{2k}{\rho g}\sinh\left(\frac{\rho ga}{k}\right)$$

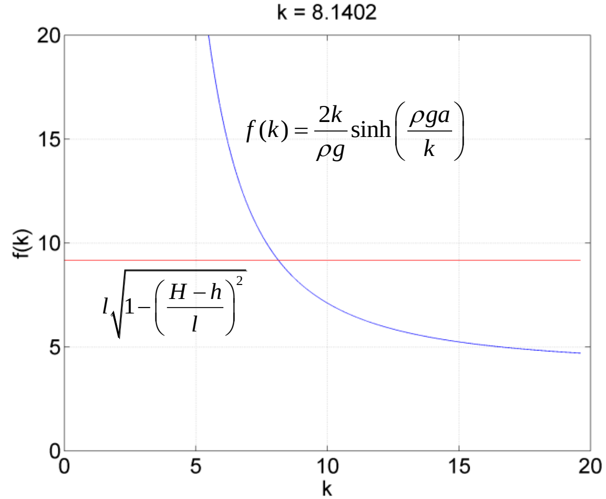
Solve this via a numeric root finding scheme

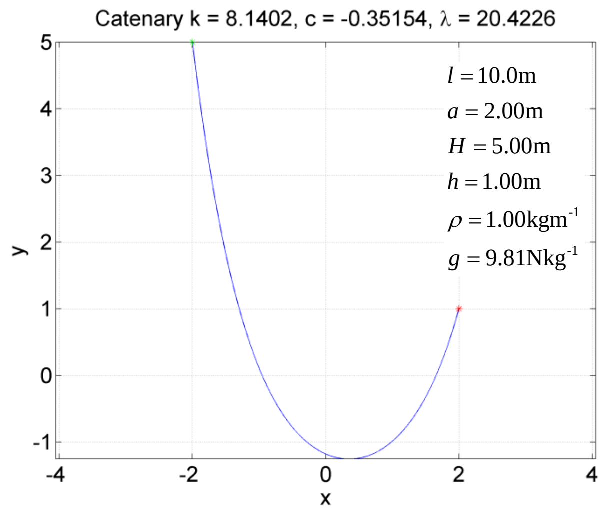
So if k is known we can use the above results to define work out the other parameters and hence define the catenary

$$c=-\frac{k}{\rho g}\tanh^{-1}\left(\frac{H-h}{l}\right)$$

$$\lambda=k\cosh\left\{\frac{\rho g}{k}(a+c)\right\}-h\rho g$$

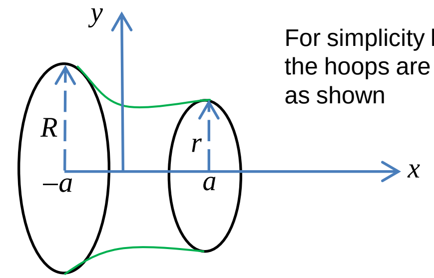
$$y=\frac{k}{\rho g}\cosh\left\{\frac{\rho g}{k}(x+c)\right\}-\frac{\lambda}{\rho g}$$





Example #4: Minimal surface between two circular hoops

What is the shape of a soap bubble film between two circular hoops? Since creating a surface requires energy, it is going to be the surface with minimum surface area. Let us ignore the effect of gravity on the energy of this surface (i.e. we expect surface tension to be the dominant effect).



For simplicity let us assume the hoops are on axis as shown

Surface area S is a volume of revolution:

$$S=\int_{x=-a}^{a}2\pi y\sqrt{dx^2+dy^2}=\int_{-a}^{a}2\pi y\,dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Applying the Euler-Lagrange equation and Beltrami identity as in previous examples:

$$L=2\pi y\sqrt{1+\dot{y}^2},\quad \dot{y}=\frac{dy}{dx},\quad L-\dot{y}\frac{\partial L}{\partial \dot{y}}=k$$

$$\therefore 2\pi y\sqrt{1+\dot{y}^2}-\dot{y}\times\tfrac{1}{2}\frac{2\pi y}{\sqrt{1+\dot{y}^2}}(2\dot{y})=k$$

$$\therefore 2\pi y\frac{1+\dot{y}^2-\dot{y}^2}{\sqrt{1+\dot{y}^2}}=k$$

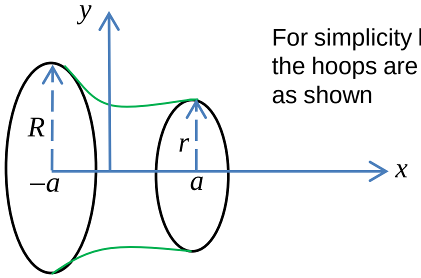
$$\therefore \tfrac{1}{\alpha^2}y^2=1+\dot{y}^2,\qquad \alpha=\frac{k}{2\pi}$$

$$\therefore \frac{dy}{dx}=\sqrt{\tfrac{1}{\alpha^2}y^2-1}$$

$$\cosh z=\tfrac{1}{\alpha}y\quad \therefore \alpha\sinh z\frac{dz}{dx}=\frac{dy}{dx}$$

$$\therefore \alpha\sinh z\frac{dz}{dx}=\sqrt{\cosh^2 z-1}=\sinh z\quad \therefore\ z=\tfrac{1}{\alpha}x+c$$

$$\therefore y=\alpha\cosh\left(\tfrac{1}{\alpha}x+c\right)$$



For simplicity let us assume the hoops are on axis as shown

$$y = \alpha\cosh\left(\tfrac{1}{\alpha}x + c\right)$$

$$R = \alpha\cosh\left(-\tfrac{1}{\alpha}a + c\right)$$

$$r = \alpha\cosh\left(\tfrac{1}{\alpha}a + c\right)$$

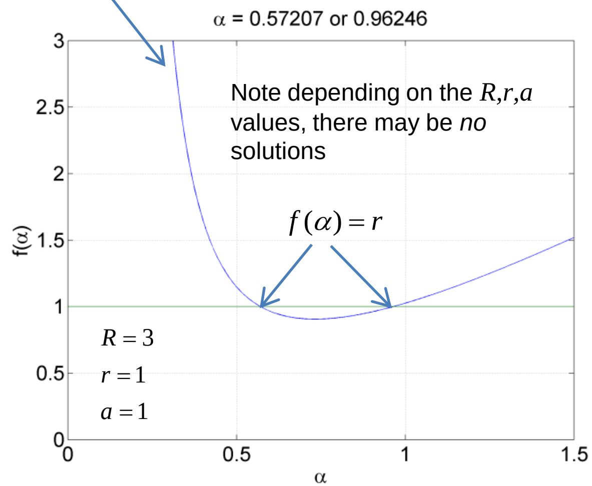
$$\frac{R-r}{\alpha} = \cosh\left(-\tfrac{1}{\alpha}a + c\right) - \cosh\left(\tfrac{1}{\alpha}a + c\right)$$

$$\frac{R-r}{\alpha} = -2\sinh c \sinh\left(\tfrac{1}{\alpha}a\right)$$

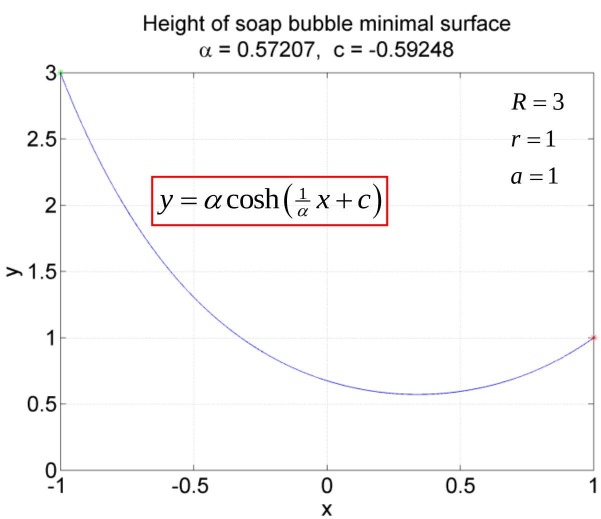
$$\therefore c = -\sinh^{-1}\left(\frac{R-r}{2\alpha\sinh\left(\tfrac{1}{\alpha}a\right)}\right)$$

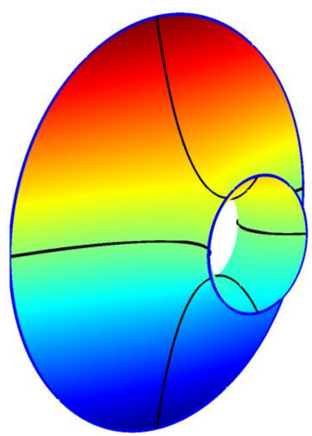
$$\therefore r = \alpha\cosh\left(\tfrac{1}{\alpha}a - \sinh^{-1}\left(\frac{R-r}{2\alpha\sinh\left(\tfrac{1}{\alpha}a\right)}\right)\right)$$

$$f(\alpha) = \alpha\cosh\left(\tfrac{1}{\alpha}a - \sinh^{-1}\left(\frac{R-r}{2\alpha\sinh\left(\tfrac{1}{\alpha}a\right)}\right)\right)$$



There are clearly *two* possible values of α in this example. However, since the surface area scales with *y* and *y* is proportional to α, we should choose the *smallest* α to yield the **minimal surface** defined by *y(x)*





MATLAB rendering of soap bubble surface using the parameters above

Emmy Noether 1882-1935

Application of Calculus of Variations in Mechanics and Optics

The Euler-Lagrange equation can be used most profitably in providing an efficient solution mechanism for mechanical systems, especially those with multiple variables.

In **Classical Mechanics**, the *Lagrangian* L is defined as:

$$L = T - V$$

Kinetic energy (T) — Potential energy (V)

See ***Lagrangian Mechanics*** Eclecticon note for a discussion of this, and the related ideas of *Hamiltonian Dynamics.*

Fermat's principle of Optics can also be applied using the E-L equation. Fermat's principle states that the path of light is such which minimizes the amount of time taken to travel through a medium. i.e. similar to the Brachisochrone problem. This can be used to show that (i) **light travels in straight lines in a medium of constant refractive index**; (ii) **Snell's Law of Refraction at a boundary of two optical media**; (iii) why sound waves refract in curved arcs if sound speed changes linearly. The **Feynman path integral** formulation of **Quantum Mechanics** puts Fermat's Principle on a stronger footing. The *probability* of a particular path is proportional to the complex exponential of the **phase** of the path. The path of least time will tend to be the path of highest probability.

Noether's Theorem . Classical Mechanics s fundamentally based upon conservation laws. Noether's Theorem states that for every **symmetry** in one of the parameters of L, there is a **corresponding conservation law.**

$$x_i \to x_i + \varepsilon K_i(x_i)$$ Define a translational symmetry

$$\frac{dL}{d\varepsilon} = \sum_i \left(\frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \varepsilon}\right) = \sum_i \left(\frac{\partial L}{\partial x_i}K_i + \frac{\partial L}{\partial \dot{x}_i}\dot{K}_i\right)$$ Chain rule

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$ Euler-Lagrange equation

$$\therefore \frac{dL}{d\varepsilon} = \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)K_i + \frac{\partial L}{\partial \dot{x}_i}\dot{K}_i\right)$$ Using product rule of differentiation

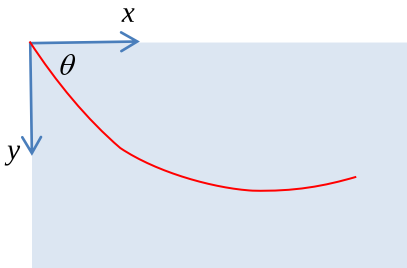
$$\therefore \frac{dL}{d\varepsilon} = \frac{d}{dt}\sum_i \left(\frac{\partial L}{\partial \dot{x}_i}K_i\right)$$

$$\therefore \frac{dL}{d\varepsilon} = 0 \Rightarrow \sum_i \left(\frac{\partial L}{\partial \dot{x}_i}K_i\right) = \text{constant}$$

So if Lagrangian doesn't change to first order in ε, there is a **conserved quantity associated with the symmetry**.

Example #5: Calculating the path of rays in a medium with a linearly changing wave speed

Fermat's Principle of Optics states that the path of a ray is associated with the *path of least time*.



Assume a ray emerges at angle θ from the x axis at the origin. The time to get to position (x,y) is:

$$t = \int_0^x \frac{\sqrt{dx^2 + dy^2}}{v(y)}$$

Assume the wave speed varies linearly with y

$$\boxed{v = \alpha + \beta y}$$

$$\therefore t = \int_0^x \frac{1}{\alpha + \beta y}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\dot{y} = \frac{dy}{dx}$$

$$\therefore L = \frac{1}{\alpha + \beta y}\left(1 + \dot{y}^2\right)^{\frac{1}{2}}$$

$$\frac{\partial L}{\partial x} = 0 \quad \therefore L - \dot{y}\frac{\partial L}{\partial \dot{y}} = k$$

$$\therefore \frac{1}{\alpha + \beta y}\left(1 + \dot{y}^2\right)^{\frac{1}{2}} - \ldots$$

$$\ldots \dot{y}\frac{1}{\alpha + \beta y}\tfrac{1}{2}2\dot{y}\left(1 + \dot{y}^2\right)^{-\frac{1}{2}} = k$$

$$\therefore \frac{1 + \dot{y}^2 - \dot{y}^2}{\left(1 + \dot{y}^2\right)^{\frac{1}{2}}} = k\left(c_o + \alpha y\right)$$

$$\therefore 1 + \dot{y}^2 = \tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2}$$

$$\therefore \frac{dy}{dx} = \sqrt{\tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2} - 1}$$

$$\therefore x + c = \int\left(\tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2} - 1\right)^{-\frac{1}{2}} dy$$

$$z = \tfrac{1}{k^2} - \left(\alpha + \beta y\right)^2 \quad \therefore \left(\alpha + \beta y\right)^2 = \tfrac{1}{k^2} - z$$

$$dz = -2\left(\alpha + \beta y\right)\beta\, dy$$

$$\therefore dy = \frac{dz}{-2\left(\alpha + \beta y\right)\beta} = \frac{dz}{-2\beta\sqrt{\tfrac{1}{k^2} - z}}$$

$$\therefore \int\left(\tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2} - 1\right)^{-\frac{1}{2}} dy = -\tfrac{1}{2\beta}\int\left(\tfrac{1}{k^2}\frac{1}{\tfrac{1}{k^2} - z} - 1\right)^{-\frac{1}{2}}\left(\tfrac{1}{k^2} - z\right)^{-\frac{1}{2}} dz$$

$$= -\tfrac{1}{2\beta}\int\left(\tfrac{1}{k^2} - \tfrac{1}{k^2} + z\right)^{-\frac{1}{2}} dz = -\tfrac{1}{2\beta}\int z^{-\frac{1}{2}} dz = -\tfrac{1}{\beta}\sqrt{z} + \text{constant}$$

$$\therefore \int\left(\tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2} - 1\right)^{-\frac{1}{2}} dy = -\tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \left(\alpha + \beta y\right)^2} + \text{constant}$$

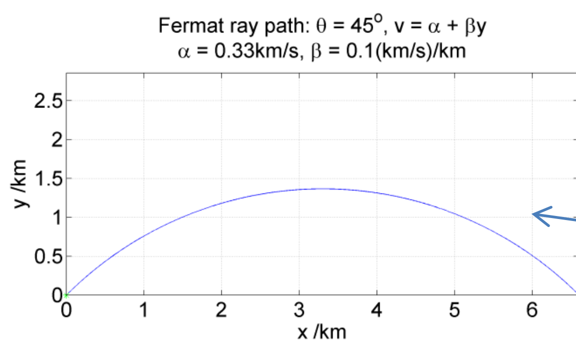
Hence:

$$x + c = -\tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \left(\alpha + \beta y\right)^2}$$

$$\therefore \beta^2\left(x + c\right)^2 = \tfrac{1}{k^2} - \left(\alpha + \beta y\right)^2$$

$$\therefore \alpha + \beta y = \sqrt{\tfrac{1}{k^2} - \beta^2\left(x + c\right)^2}$$

$$\therefore y(x) = \tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \beta^2\left(x + c\right)^2} - \frac{\alpha}{\beta}$$



If wave speed *increases* with depth y, rays are *bent upwards towards the surface.* This helps to explain 'anomalously far' propagation of radio waves in the atmosphere, or sound waves in the Earth or ocean.

$$\left.\frac{dy}{dx}\right|_{x=0,y=0} = \tan\theta$$

$$\frac{dy}{dx} = \sqrt{\tfrac{1}{k^2}\frac{1}{\left(\alpha + \beta y\right)^2} - 1}$$

$$\therefore \tan\theta = \sqrt{\tfrac{1}{k^2}\frac{1}{\alpha^2} - 1}$$

$$\therefore \tfrac{1}{k^2} = \alpha^2\left(\tan\theta^2 + 1\right)$$

$$x = 0, y = 0, \quad x + c = -\tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \left(\alpha + \beta y\right)^2}$$

$$\therefore c = -\tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \alpha^2}$$

$$\therefore c = -\tfrac{1}{\beta}\sqrt{\alpha^2\left(\tan\theta^2 + 1\right) - \alpha^2}$$

$$\therefore c = -\tfrac{1}{\beta}\alpha\tan\theta$$

$$y(x) = \tfrac{1}{\beta}\sqrt{\tfrac{1}{k^2} - \beta^2\left(x + c\right)^2} - \frac{\alpha}{\beta}$$

$$y(x) = \tfrac{1}{\beta}\sqrt{\alpha^2\left(\tan^2\theta + 1\right) - \beta^2\left(x - \tfrac{1}{\beta}\alpha\tan\theta\right)^2} - \frac{\alpha}{\beta}$$

$$y(x) = \tfrac{1}{\beta}\sqrt{\alpha^2\left(\tan^2\theta + 1\right) - \beta^2 x^2 + 2\beta x\alpha\tan\theta - \alpha^2\tan^2\theta} - \frac{\alpha}{\beta}$$

$$\boxed{\therefore y(x) = \tfrac{1}{\beta}\sqrt{\alpha^2 - \beta^2 x^2 + 2\beta x\alpha\tan\theta} - \frac{\alpha}{\beta}}$$