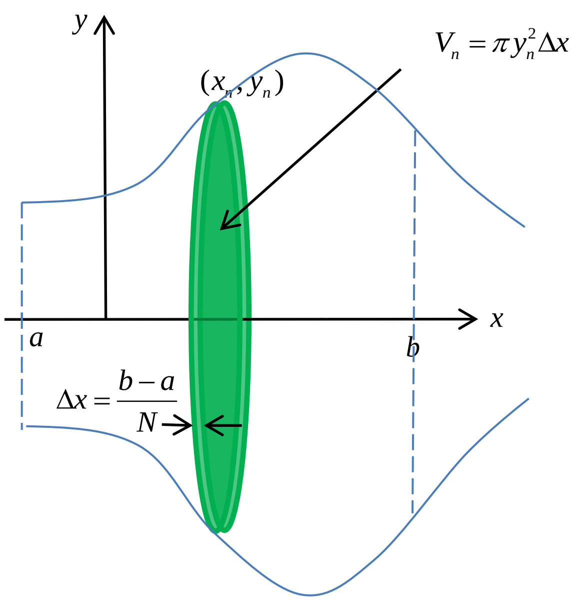
Volumes of revolution

The *volume of revolution* of a curve about an axis (e.g. the *x* or *y* axis) can be computed from the sum of volumes of *infinitesimally thin discs*, whose radii vary according to the curve. The limit of *N* such discs, of width $\Delta x = (b-a)/N$ such that *N* tends to infinity can be written as (for volume of revolution about the *x* axis)*

$$V = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\pi y_n^2\left(\frac{b-a}{N}\right)\right\} = \lim_{\Delta x\to 0}\left\{\sum_{n=1}^{N}\pi y_n^2\Delta x\right\} = \boxed{\int_a^b \pi y^2\,dx}$$



i.e. the volume can be written as an *integral* over the *square* of the function *y(x)*

Example1 : Find the volume of a solid formed by revolving $y = e^{-x}$ about the *x* axis between *x* limits of [0,4]

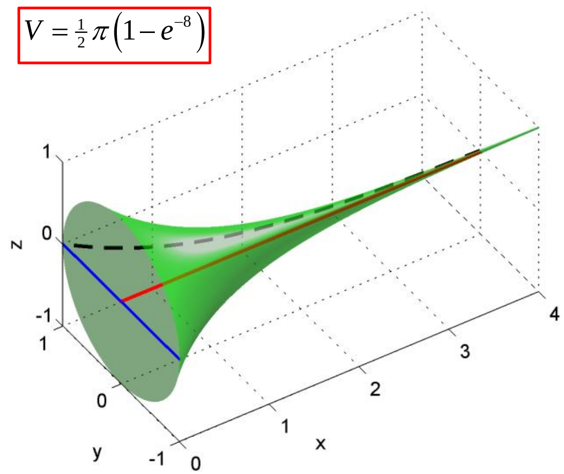
$$V = \int_0^4 \pi y^2\,dx$$

$$V = \int_0^4 \pi e^{-2x}\,dx$$

$$V = \left[\frac{\pi e^{-2x}}{-2}\right]_0^4$$

$$V = \left(\frac{\pi e^{-8}}{-2}\right) - \left(\frac{\pi}{-2}\right)$$

$$\boxed{V = \tfrac{1}{2}\pi\left(1-e^{-8}\right)}$$



Example2 : Find the volume of a solid formed by revolving $y = 1 + \sin x$ about the *x* axis between *x* limits of [0,2π]

$$V = \int_0^{2\pi} \pi y^2\,dx$$

$$\therefore V = \pi\int_0^{2\pi} (1+\sin x)^2\,dx$$

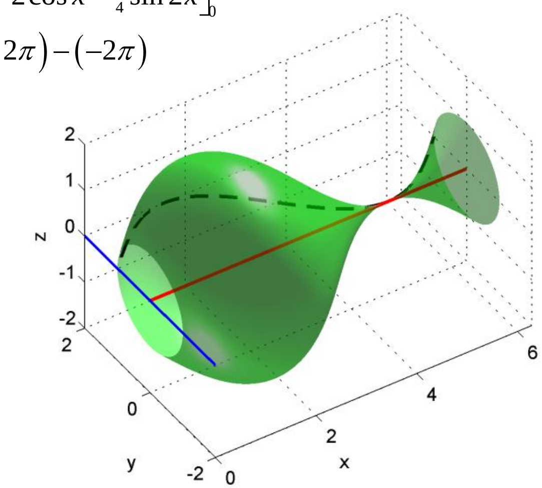
$$V = \pi\int_0^{2\pi} \left(1 + 2\sin x + \sin^2 x\right)dx$$

$$V = \pi\int_0^{2\pi} \left(1 + 2\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)dx$$

$$V = \pi\left[\tfrac{3}{2}x - 2\cos x - \tfrac{1}{4}\sin 2x\right]_0^{2\pi}$$

$$V = \left(3\pi^2 - 2\pi\right) - \left(-2\pi\right)$$

$$\boxed{V = 3\pi^2}$$



*The equivalent expression for a volume of revolution about the *y* axis from *y* limits of [*a*,*b*] is

$$V_y = \int_{y=a}^{b} \pi x^2\,dy$$

To evaluate this you will need to rearrange *x* in terms of *y*. i.e. find the *inverse* function of *y = f(x)*. Then hope the square of this can be integrated!