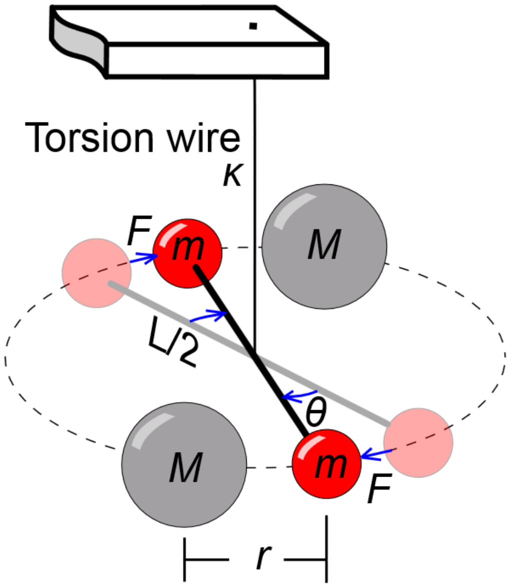

The Cavendish Experiment

Measuring *G* using a torsional pendulum

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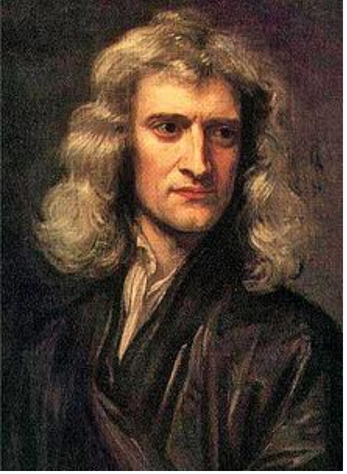
Department of Physics, Winchester College.
19/20 December 2016



Isaac Newton proposed a **Law of Universal Gravitation** to calculate the *attractive* force between point masses m and M, separated by a distance r

$$F = \frac{GMm}{r^2}$$

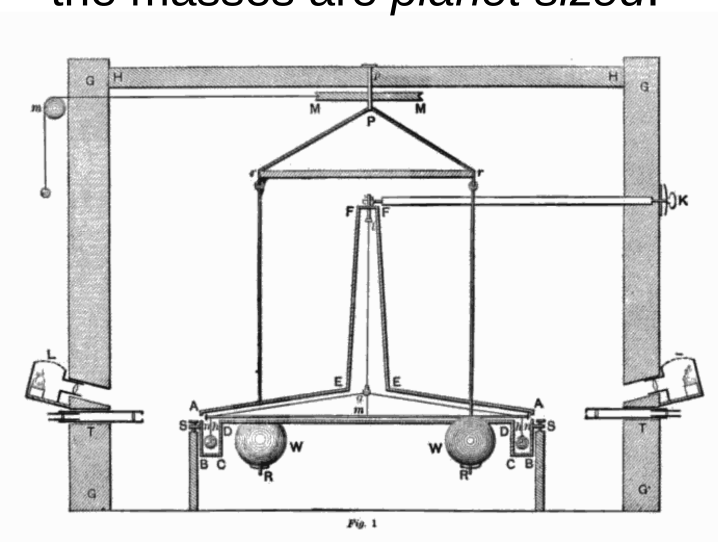
Isaac Newton
1642-1727

The constant G is:

$$G = 6.67384 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

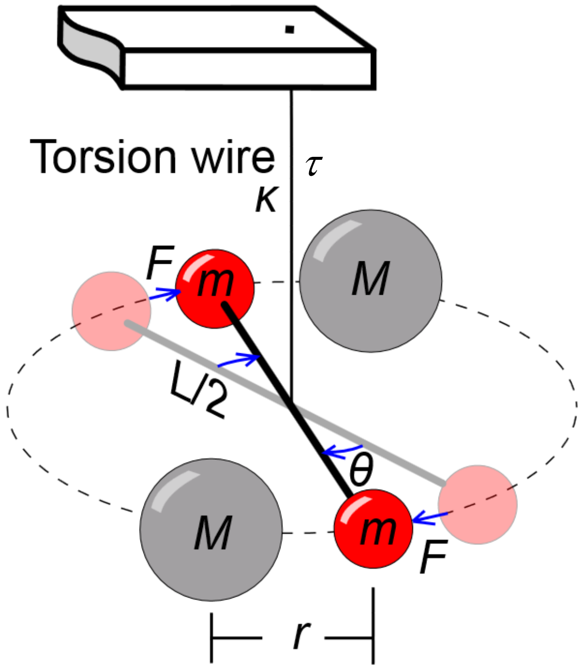
Henry Cavendish
1731-1810

To measure G experimentally on Earth is *difficult*, since gravitational forces only become important on a human scale when the masses are *planet sized*.



Henry Cavendish devised an ingenious experiment to overcome this problem. His equipment was based upon a **torsional pendulum**. Amazingly, the tiny forces resulting from the gravitational attraction of lead spheres could result in a *measurable* effect upon the oscillation of the pendulum.

Measuring G via the Cavendish experiment



Balance the torsion force (the twist) on the wire with the torque resulting from the gravitational attraction of masses M and m

$$\tau\theta = L \times \frac{GmM}{r^2}$$

$$\therefore G = \frac{\tau r^2 \theta}{LmM}$$

Angle in *radians*

The torsion constant τ can be found by measuring the period T of small oscillations of the pendulum.

$$T = 2\pi\sqrt{\frac{I}{\tau}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

For a mass on a spring of stiffness k

$$I = 2 \times m\left(\tfrac{1}{2}L\right)^2 = \tfrac{1}{2}mL^2$$

Moment of inertia I of the pendulum *about* the wire axis

$$\frac{T^2}{4\pi^2} = \frac{\frac{1}{2}mL^2}{\tau}$$

$$\therefore \tau = \frac{2\pi^2 mL^2}{T^2}$$

In the original experiment

$m = 0.73\text{kg}, \quad M = 158\text{kg}$

$L = 1.8\text{m}, \quad r = (230 - 4.1)\text{mm}$

$P = 875.3\text{s}, \quad \tfrac{1}{2}L\theta \approx 4.1\text{mm}$

We can now combine the expressions to find G

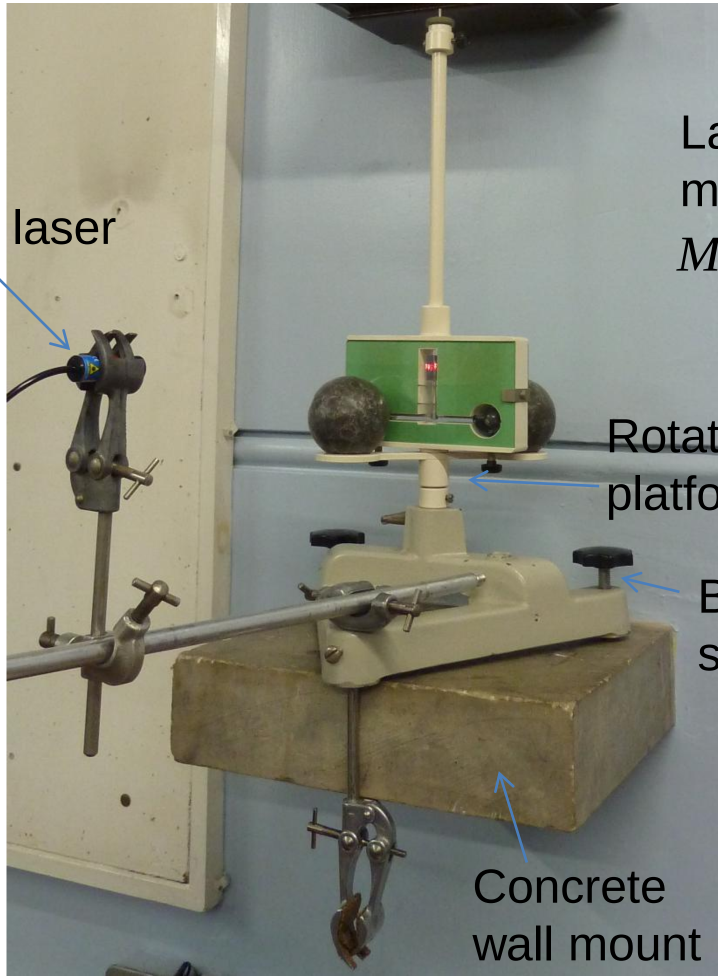
$$G = \frac{2\pi^2 mL^2}{T^2} \times \frac{r^2\theta}{LmM}$$

$$\boxed{G = \frac{2\pi^2 L r^2 \theta}{MT^2}}$$

Cavendish measured $\boxed{G = 6.74 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}}$

$[G_{\text{modern}} = 6.67 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}]$

At Winchester, we have a Cavendish Experiment kit supplied by **LD Didactic GmbH**



Red laser

Large lead masses

$M = 1.496.6\text{kg}$

Rotating mass platform

Balancing screws

Concrete wall mount

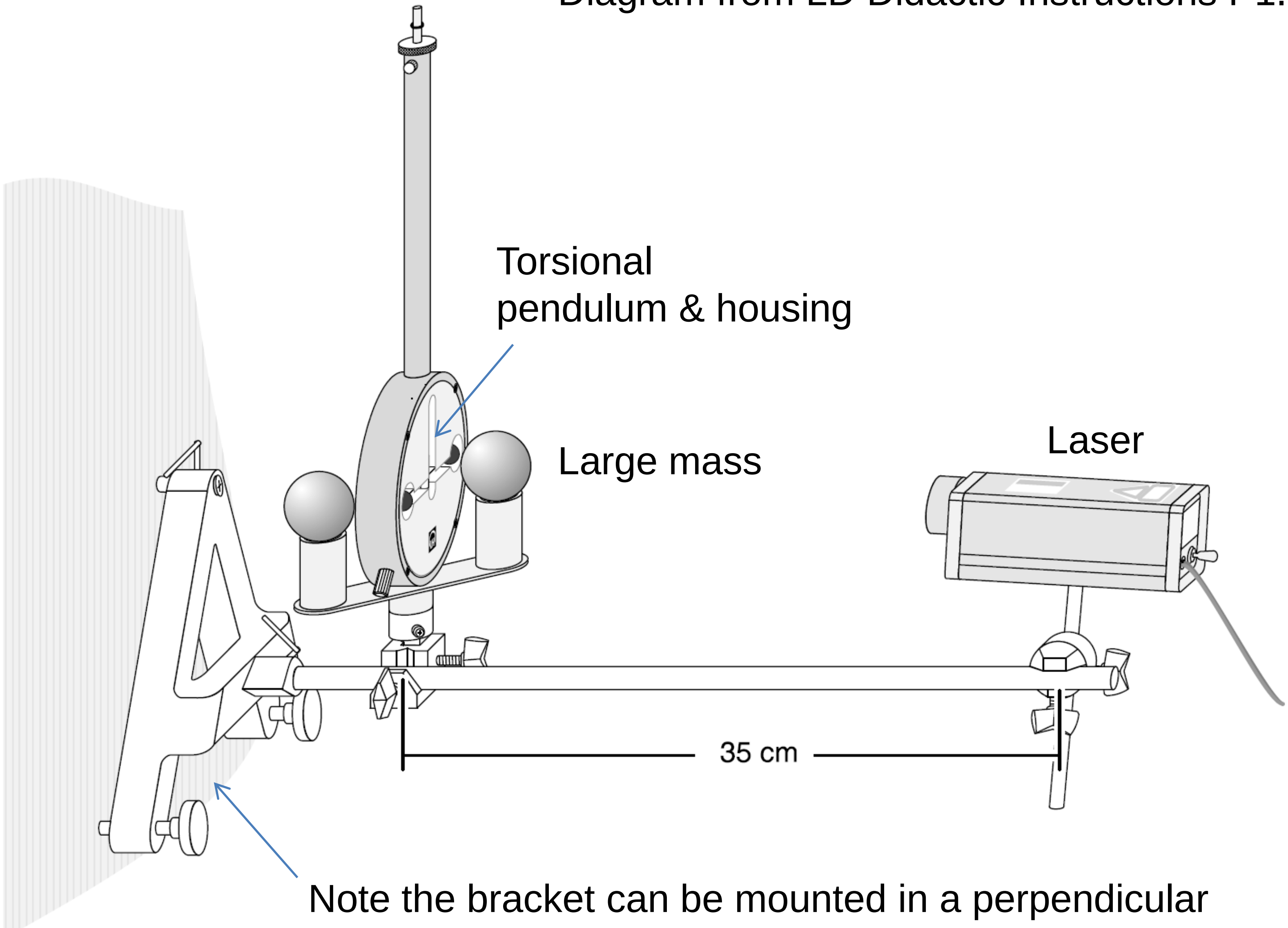
Torsion wire

Mirror

Small lead masses, fixed to torsional pendulum

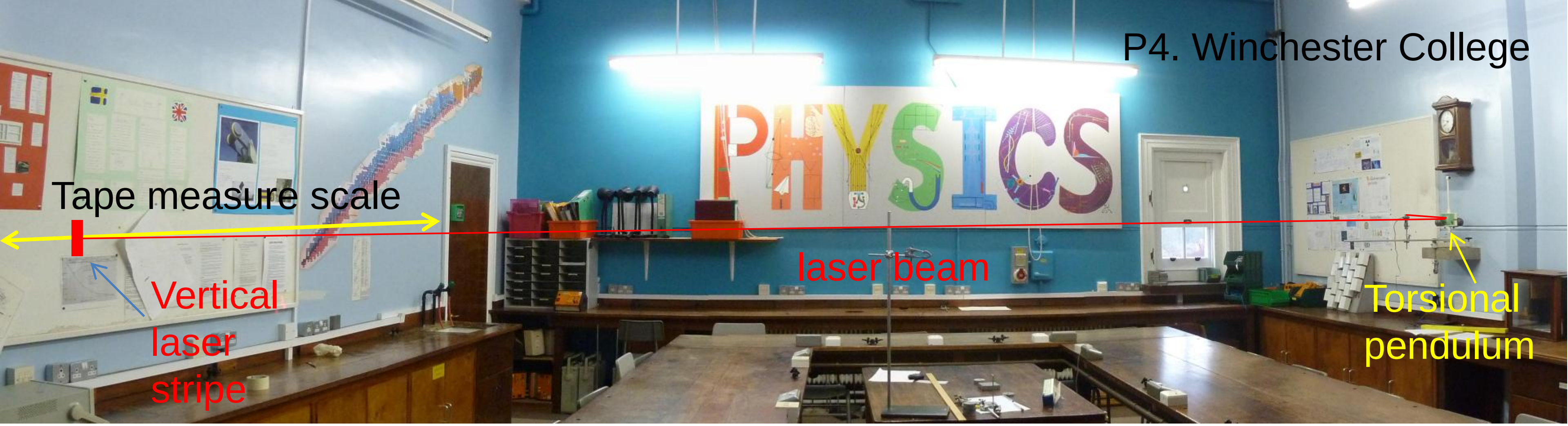
$m = 0.015\text{kg}$

Diagram from LD Didactic Instructions P1.1.3.1

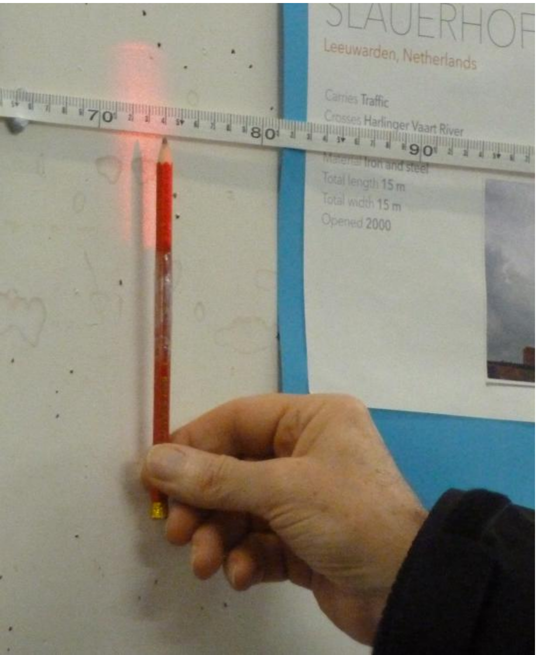


Note the bracket can be mounted in a perpendicular position – as in our case given we have the concrete platform installed in Laboratory P4.

P4. Winchester College

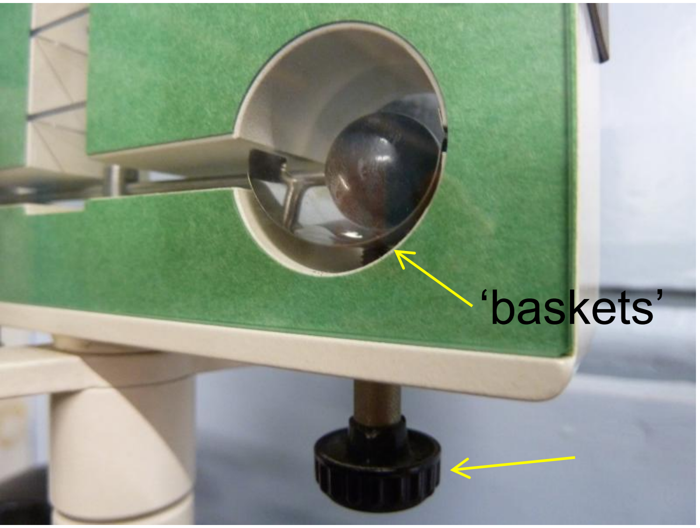


The laser is reflected off a mirror, which rotates with the pendulum. This causes a vertical stripe to slowly track back and forth across the opposing wall of the laboratory. The dynamics of these oscillations are used to determine G.



Once the pendulum was aligned, measurements were taken, every 30s, from the *centre* of the stripe along the tape measure scale.

A measuring tape was affixed to the wall such that the stripe tracks along it.



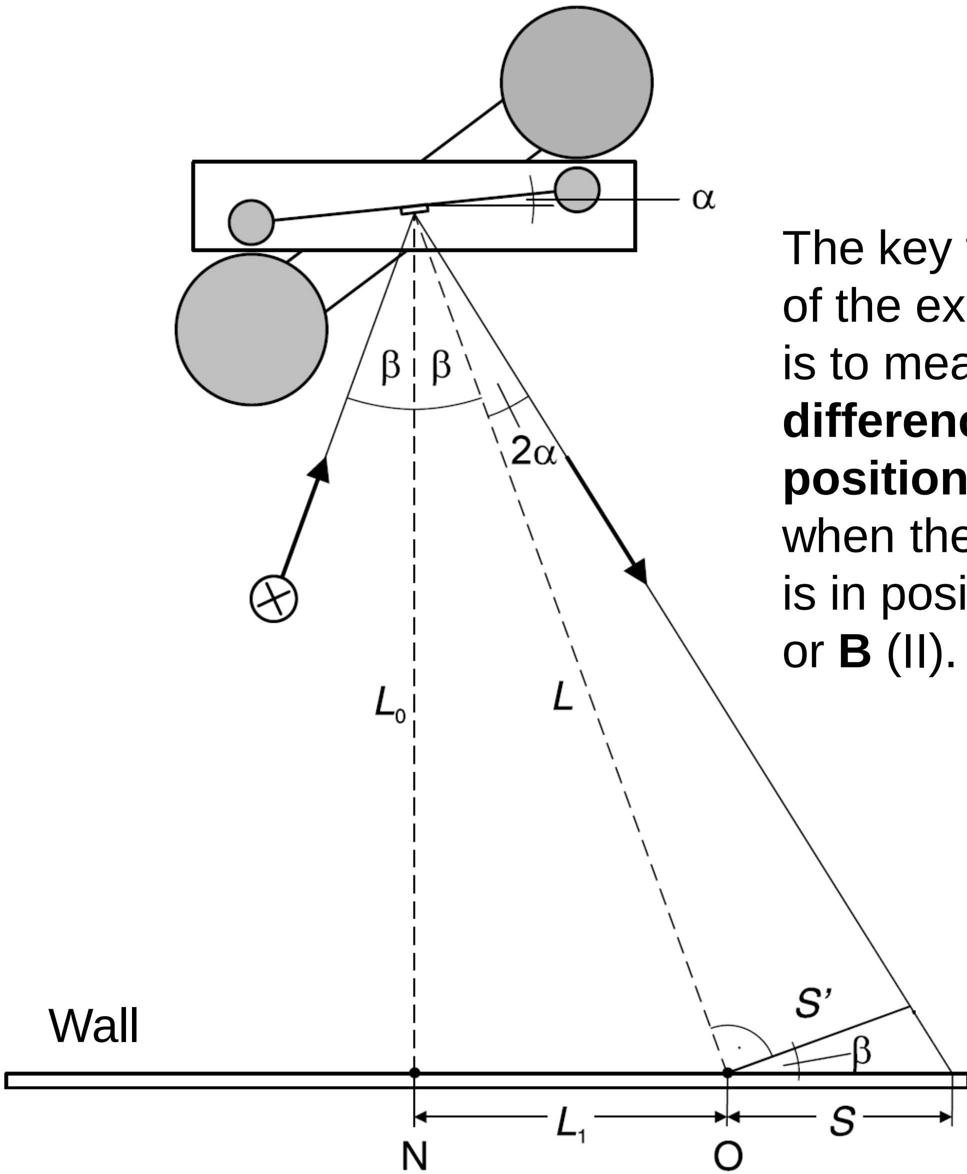
These screws are used to lower metal ‘baskets’ which lock the pendulum in place. The baskets *must* be raised when the equipment is transported.

Horizontal alignment screws

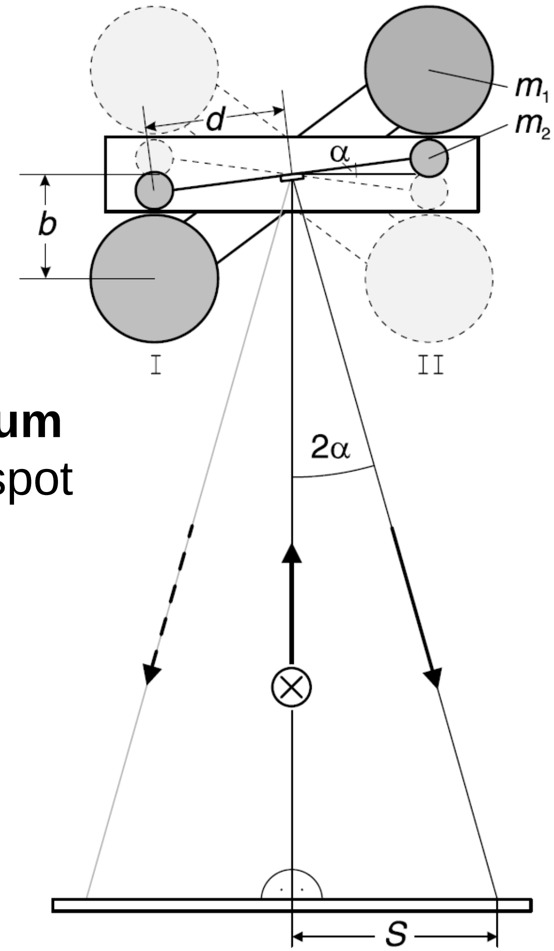
This screw seems to relate to how the wire is fixed in place. Turning it has broken the kit in the past....

Once the lowered baskets have released the pendulum, it is often discovered that the pendulum will ‘crash’ into the glass windows. i.e. it has too much angular momentum and the equilibrium position is not central. The top thumb screw enables the wire and mirror to be manually rotated to correct for this.

This process of alignment can be time consuming and turns of the screw should be subtle!

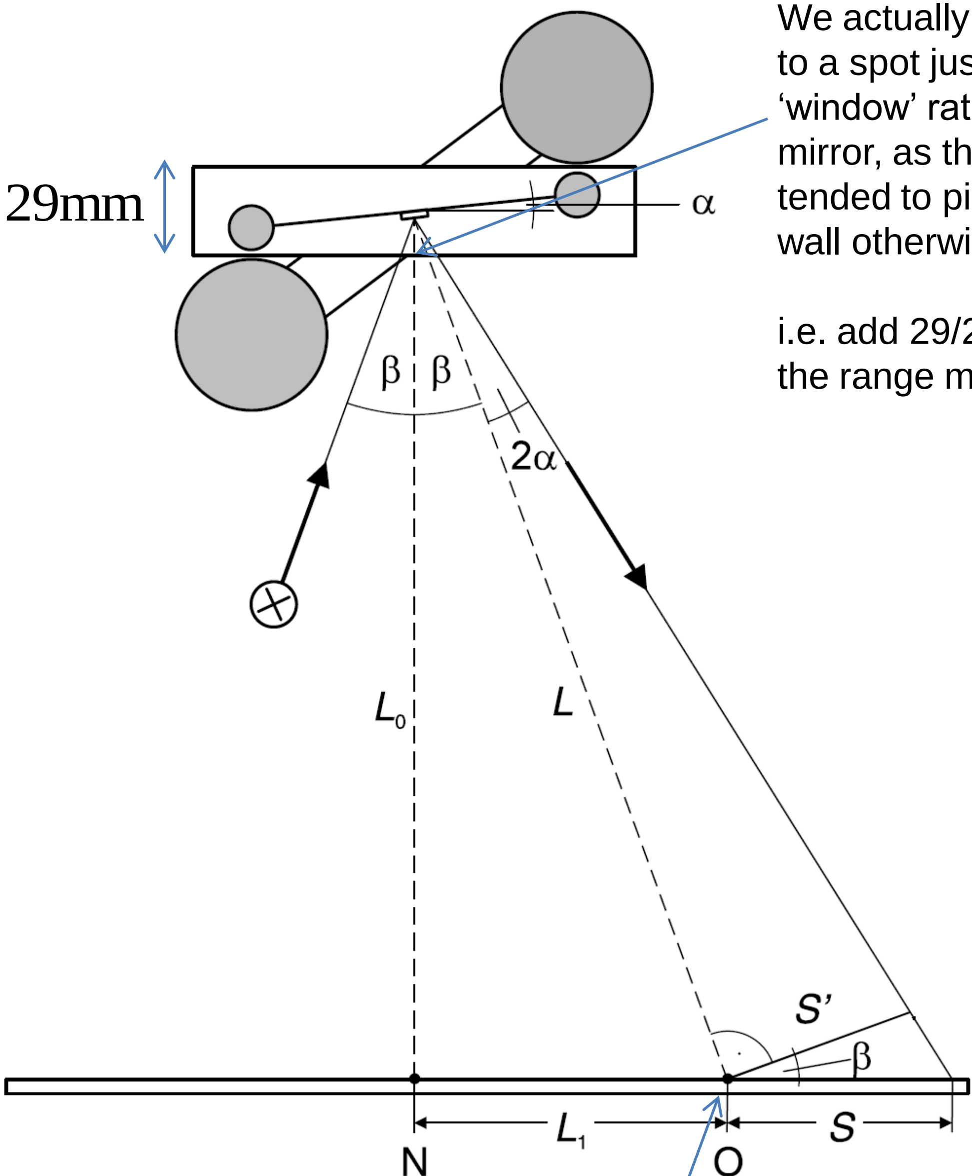


The key feature
of the experiment
is to measure the
difference in equilibrium
positions of the laser spot
when the large mass
is in positions **A** (I)
or **B** (II).



The large mass
is rotated till it *just* touches
the glass panel

Diagrams from LD Didactic Instructions P1.1.3.1



A **laser range-finder** was used to determine the distances between the equipment and the wall.

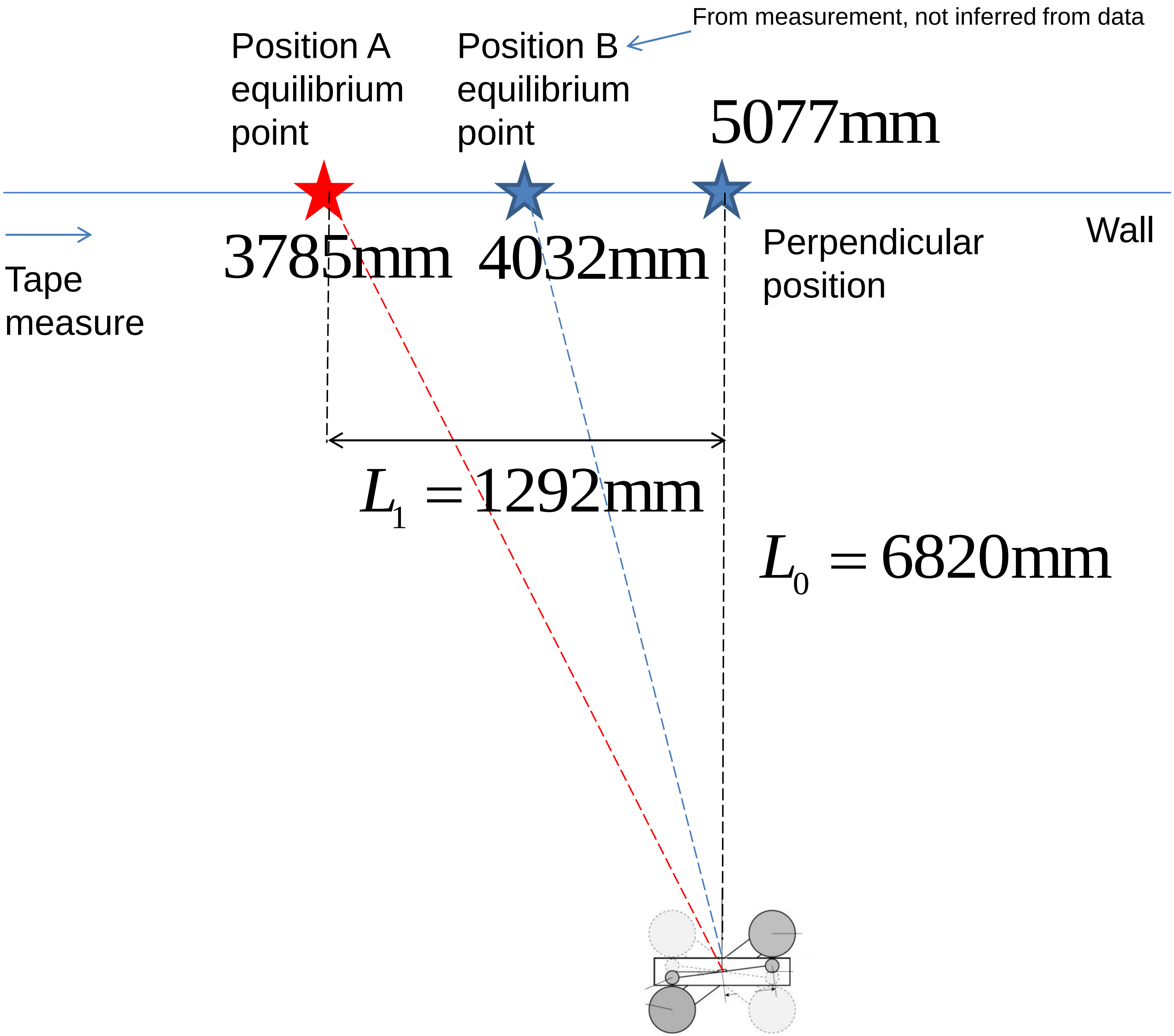
$$L_0 = 6820\text{mm}$$

The distance from the perpendicular point N to the **equilibrium of Position A** was measured to be

$$L_1 = 1292\text{mm}$$

The offset angle of the laser will mean the equilibrium positions are offset from N

Errors of about 10mm (0.8%) due to wobble of the laser when button pressed, misalignment of walls etc

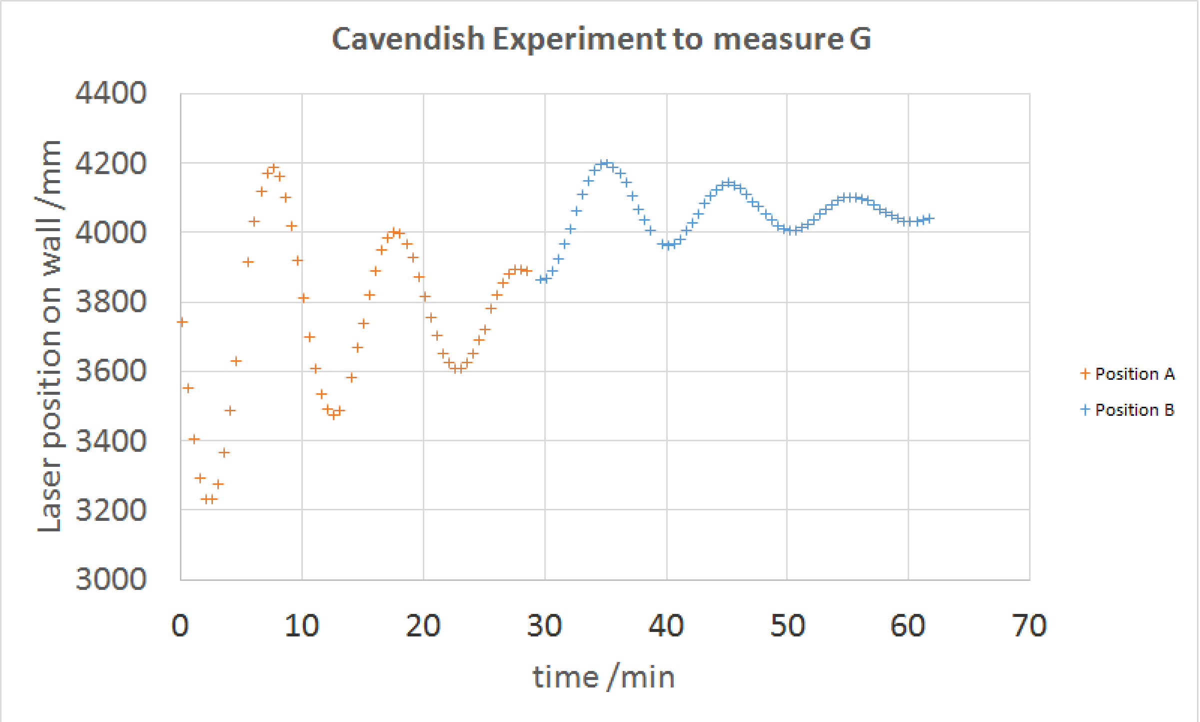

From measurement, not inferred from data
Position A equilibrium point
Position B equilibrium point
5077mm
3785mm
4032mm
Perpendicular position
Wall
Tape measure
$L_1 = 1292\text{mm}$
$L_0 = 6820\text{mm}$

Once the system was aligned, and it was confirmed that oscillations were contained within the pendulum housing, (i.e. no crashing against the interior windows) laser stripe positions along the tape were recorded every 30s.

After four oscillations, Position A was changed to Position B, and another three complete oscillations were recorded.

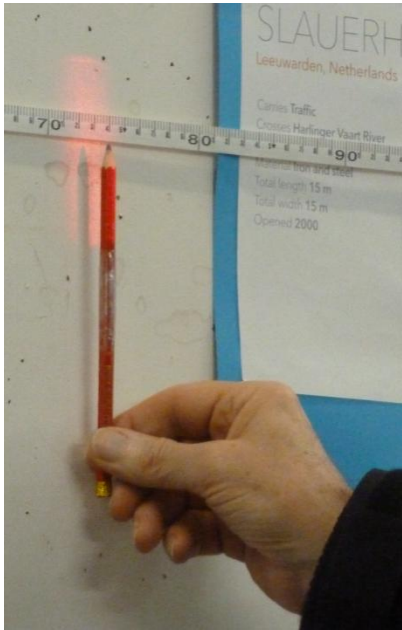
This amounted to an experimental run time of about 60 minutes.

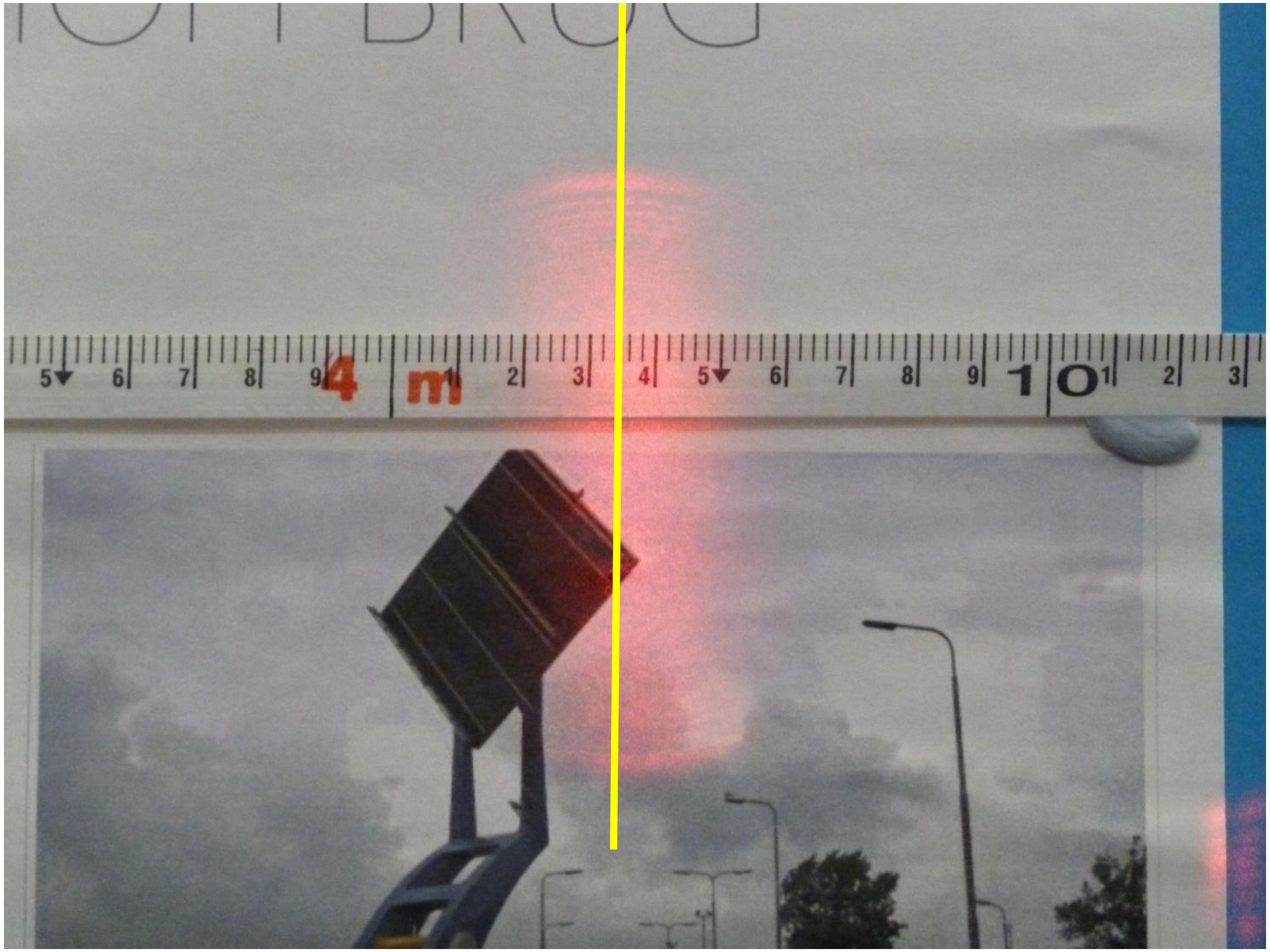
Data was recorded directly into **Microsoft Excel** and plotted as the data was collected.



Note the equilibrium stripe location for position A will be **inferred** via curve fitting (rather than wait many hours!)

This could be a source of error in both **period** and **AB equilibrium offset**





After several hours following the cessation of measurements recorded in **Position B**, the following measurement was photographed.
This is assumed to be the **Position B equilibrium reading**:

4032mm

The data was then imported into **MATLAB** for further analysis.

1. **Cubic spline interpolation** to smooth between the data points.

2. **Peak (and trough) finding routine** to determine the peak and trough times and positions for each A and B oscillation.

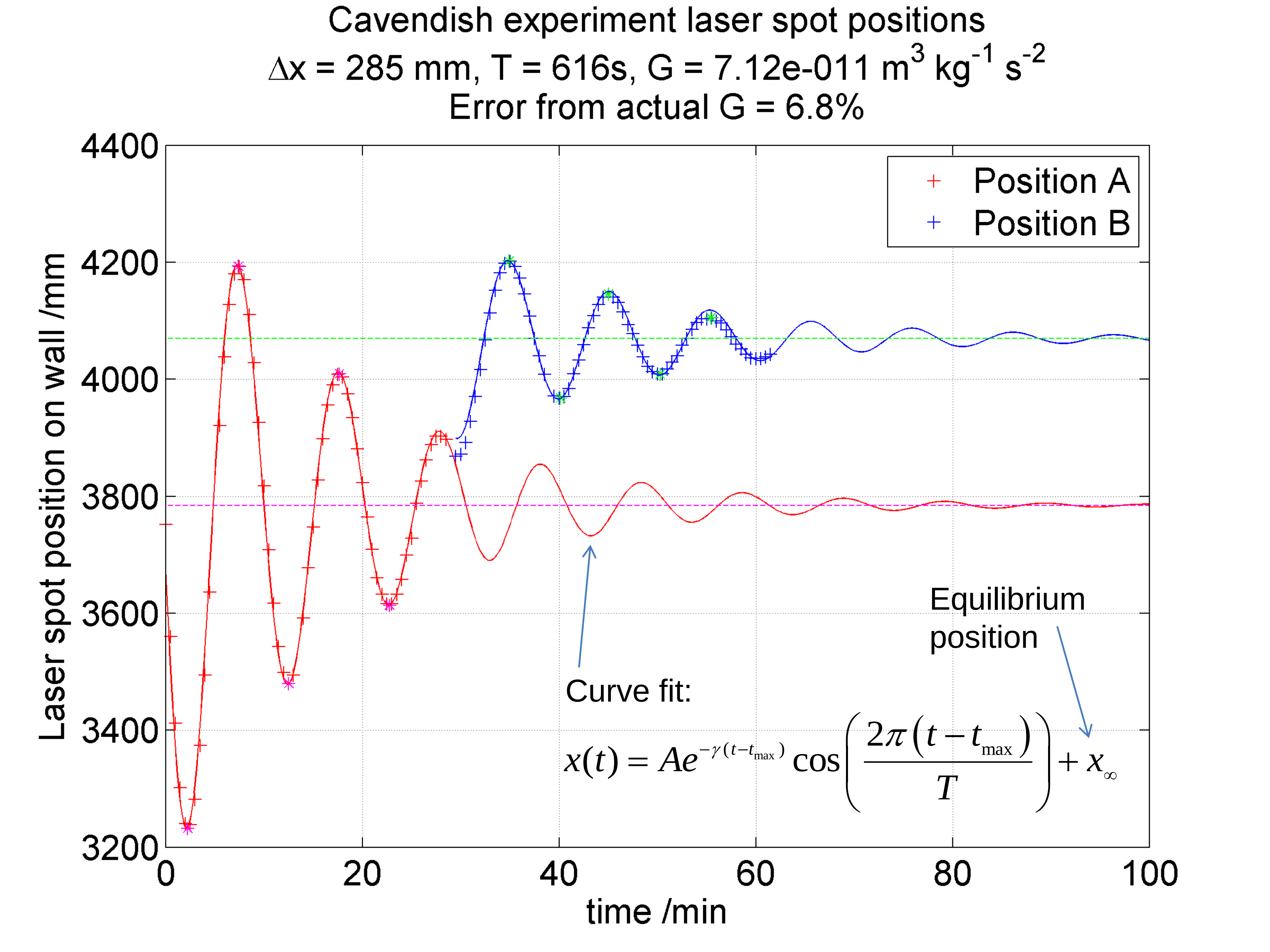
3. The difference between peak times were also used to calculate the **period** of torsional oscillations, which was calculated to be **616s.**

4. Curve fitting to estimate the longer term oscillation, and predict the final equilibrium positions.

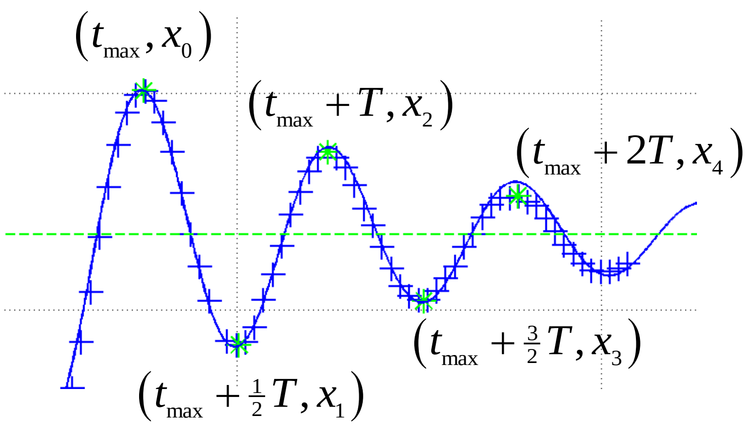
5. Calculation of G (see later slide for details)

6. Automatic graph plot and .png file creation.

MATLAB
The Language of Technical Computing


Cavendish experiment laser spot positions
Δx = 285 mm, T = 616s, G = 7.12e-011 m^3 kg^-1 s^-2
Error from actual G = 6.8%
Position A
Position B
Laser spot position on wall /mm
4400
4200
4000
3800
3600
3400
3200
0
20
40
60
80
100
time /min
Equilibrium position
Curve fit:
$x(t) = Ae^{-\gamma(t-t_{max})}\cos\left(\frac{2\pi(t-t_{max})}{T}\right) + x_\infty$

Curve fitting to determine the equilibrium position



$$x(t) = Ae^{-\gamma(t-t_{max})}\cos\left(\frac{2\pi(t-t_{max})}{T}\right) + x_\infty$$

$$x_0 = A + x_\infty$$

$$x_1 = -Ae^{-\frac{1}{2}\gamma T} + x_\infty$$

$$x_2 = Ae^{-\gamma T} + x_\infty$$

$$x_3 = -Ae^{-\frac{3}{2}\gamma T} + x_\infty$$

Use peak and trough coordinates

$$\frac{x_0 - x_1}{x_2 - x_3} = \frac{A + Ae^{-\frac{1}{2}\gamma T}}{Ae^{-\gamma T} + Ae^{-\frac{3}{2}\gamma T}}$$

$$\frac{x_0 - x_1}{x_2 - x_3} = \frac{1 + e^{-\frac{1}{2}\gamma T}}{e^{-\gamma T}\left(1 + e^{-\frac{1}{2}\gamma T}\right)} = e^{\gamma T}$$

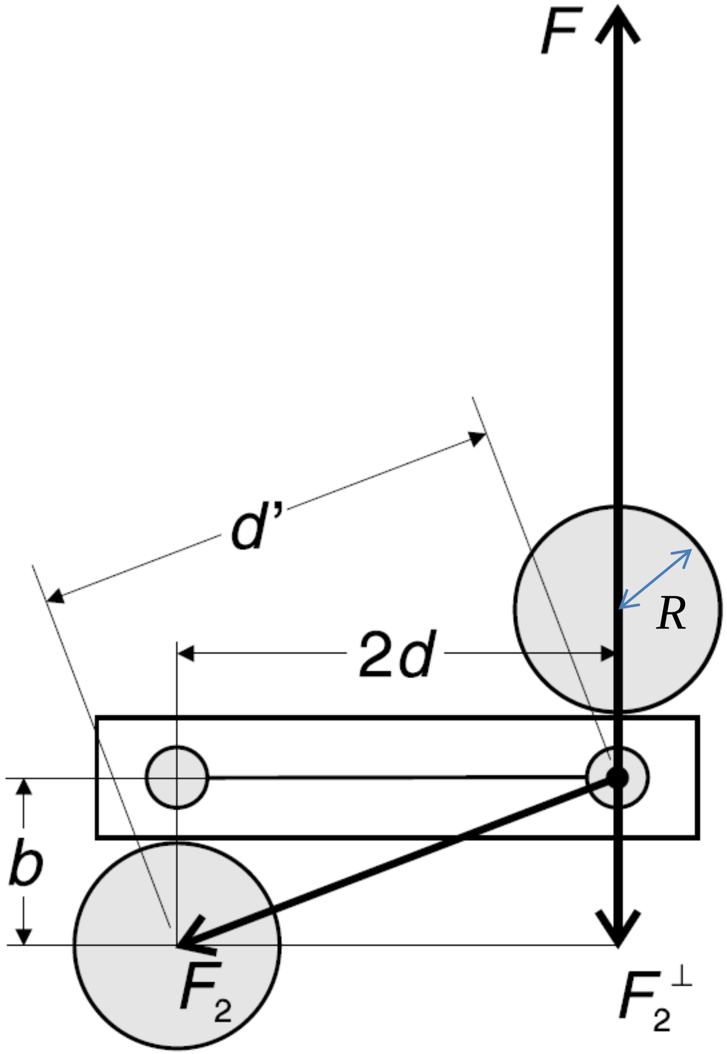
$$\therefore \gamma = \frac{1}{T}\ln\left(\frac{x_0 - x_1}{x_2 - x_3}\right)$$

$$\frac{x_0 - x_1}{x_2 - x_3} = \frac{A + Ae^{-\frac{1}{2}\gamma T}}{Ae^{-\gamma T} + Ae^{-\frac{3}{2}\gamma T}}$$

$$x_0 - x_1 = A\left(1 + e^{-\frac{1}{2}\gamma T}\right)$$

$$\therefore A = \frac{x_0 - x_1}{1 + e^{-\frac{1}{2}\gamma T}}$$

$$\therefore x_\infty = x_0 - A$$



We measured this

$M = 1.4966\text{kg}$

$R = 32\text{mm}$

Large sphere

$m = 0.015\text{kg}$

$r = 6.9\text{mm}$

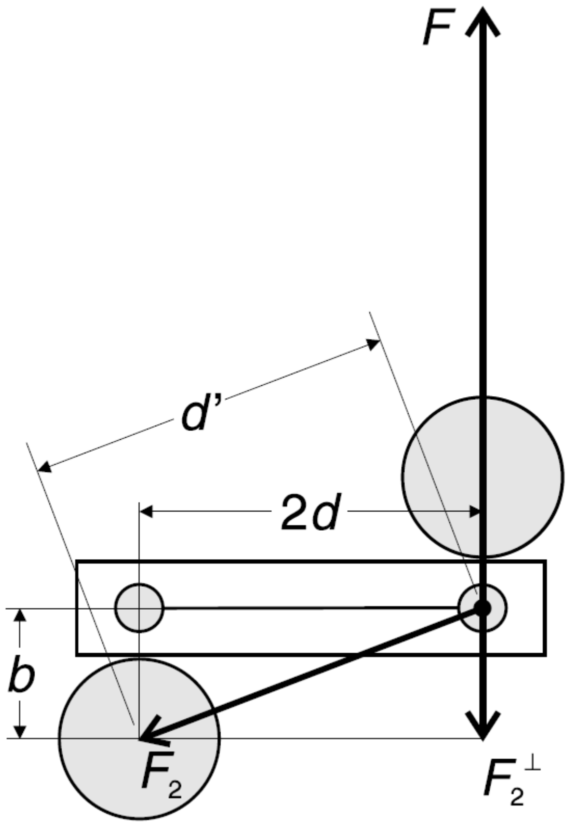
Small sphere

$d = 50\text{mm}$

$b = 46.5\text{mm}$

The French language version (supplied with the kit!) states 46.5mm whereas the English language version states 50mm....

Diagram and parameters from LD Didactic literature



From LD Didactic Instructions P1.1.3.1 pp3

Antitorque moment of the "second" lead sphere:

In addition to the moment of momentum caused by the attractive force F of the respective opposing large lead spheres (distance b), the attractive force F_2 of the respectively more distant sphere (distance d') gives rise to an antitorque moment (see Fig. 3). Thus, to formulate it more precisely than equation (II), the following applies for the moment of momentum M_l:

$$M_\text{l} = 2\,(F + F_2{}^\perp) \cdot d,$$

where

$$F_2{}^\perp = -F_2 \cdot \frac{b}{d'}$$

is the component of force

$$F_2 = F \cdot \frac{b^2}{d'^2}$$

which is acting perpendicular to the transverse beam (see (I)). The gravitational constant G is thus greater than calculated in (X) by the correction factor

$$K = \frac{F}{F + F_2{}^\perp} = \frac{1}{1 - \frac{b^3}{d'^3}} \qquad \text{(XI)}$$

With

$$d' = \sqrt{(2d)^2 + b^2}$$

we can calculate the numerical value K = 1.083.

Calculating *G*

Mirror rotation is twice pendulum rotation, and between positions A and B we *double θ*

$$r = b,\ \ L = 2d,\ \ \theta = \frac{\theta_A - \theta_B}{4}$$

$$\therefore G = \frac{\pi^2 b^2 d}{MT^2}(\theta_A - \theta_B)$$

$$\theta_A = \tan^{-1}\left(\frac{1293}{6820}\right)$$

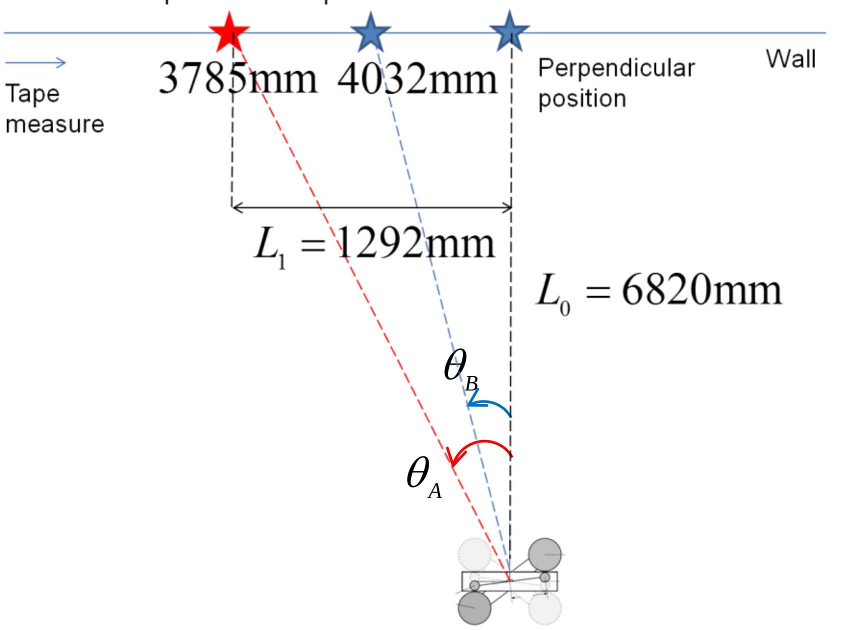
$$\theta_B = \tan^{-1}\left(\frac{1045}{6820}\right)$$

$$M = 1.4966\text{kg}$$

$$d = 50\text{mm}$$

$$b = 46.5\text{mm}$$

$$K = 1.083$$



$$G = K \times \frac{\pi^2 (46.5\times10^{-3})^2 \times 50\times10^{-3}}{1.4966\times616^2}\left(\tan^{-1}\left(\frac{1293}{6820}\right) - \tan^{-1}\left(\frac{1045}{6820}\right)\right)$$

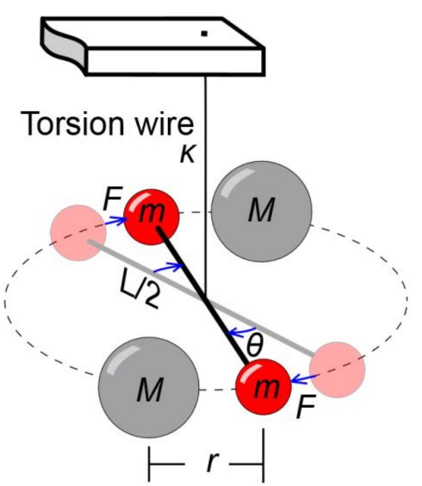
$$G = 7.1\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

So 6.8% out from the correct value

$$G_{\text{Cavendish}} = 6.74\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

$$G_{\text{modern}} = 6.67\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

$$G = \frac{2\pi^2 L r^2 \theta}{MT^2}$$



Error analysis

$$G = \frac{\pi^2 b^2 d}{MT^2}\left(\theta_A - \theta_B\right)$$

$$1.49655 \le M < 1.49665\text{kg}$$

$$49.5 \le d < 50.5\text{mm}$$

$$46.45 \le b < 46.55\text{mm}$$

$$K = 1.083$$

$$615.5\text{s} \le T < 616.5\text{s}$$

We'll ignore changes in K

There is probably also a systematic error in the larger value due to the inferred nature of the equilibrium position of A

$$\tan^{-1}\left(\frac{1292}{6821}\right) - \tan^{-1}\left(\frac{1046}{6819}\right) \le \theta_A - \theta_B < \tan^{-1}\left(\frac{1294}{6819}\right) - \tan^{-1}\left(\frac{1044}{6821}\right)$$

$$G_{\max} = K \times \frac{\pi^2\left(46.55\times10^{-3}\right)^2 \times 50.5\times10^{-3}}{1.49655\times615.5^2}\left(\tan^{-1}\left(\frac{1294}{6819}\right) - \tan^{-1}\left(\frac{1044}{6821}\right)\right)$$

$$G_{\max} = 7.4\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

$$G_{\min} = K \times \frac{\pi^2\left(46.45\times10^{-3}\right)^2 \times 49.5\times10^{-3}}{1.49665\times616.5^2}\left(\tan^{-1}\left(\frac{1292}{6821}\right) - \tan^{-1}\left(\frac{1046}{6819}\right)\right)$$

$$G_{\min} = 7.0\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

We shall assume a +/- precision error of 1mm for all length measurements

So we have about a 5% systematic offset *and* a similar random error

So final result is: $$7.0\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} < G < 7.4\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$