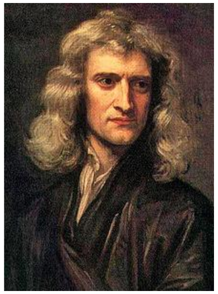
Newton, the apple and the moon

By looking at the sizes of shadows, the Ancient Greeks had reasoned the Moon must be about sixty Earth radii distant. This is impressively close to the modern measurements, which take into account that the orbital distance varies from *perigee* to *apogee* over time due to a slight *eccentricity* (i.e. not perfect circularity) of the orbit. Note also that the Moon is currently *receding* from the Earth at a rate of about 2.91 cm/year due to the tidal gravitational interaction between the moving water on Earth and the Moon.



Isaac Newton
1642-1726

Earth-Moon separation $363{,}104\text{km} \le R_M \le 406{,}696\text{km}$

$$\boxed{57.0R_\oplus \le R_M \le 63.8R_\oplus}$$

i.e. gravitational force or **weight** is mass x gravitational acceleration*

Galileo had shown experimentally that all falling bodies on Earth should *accelerate* a constant rate. In modern terms, we would say that if air resistance, lift, upthrust etc can be ignored, all objects fall at about $g = 9.81 \text{ m/s}^2$

From Galilean *kinematics*, one can predict how far an apple would fall in 1 second.

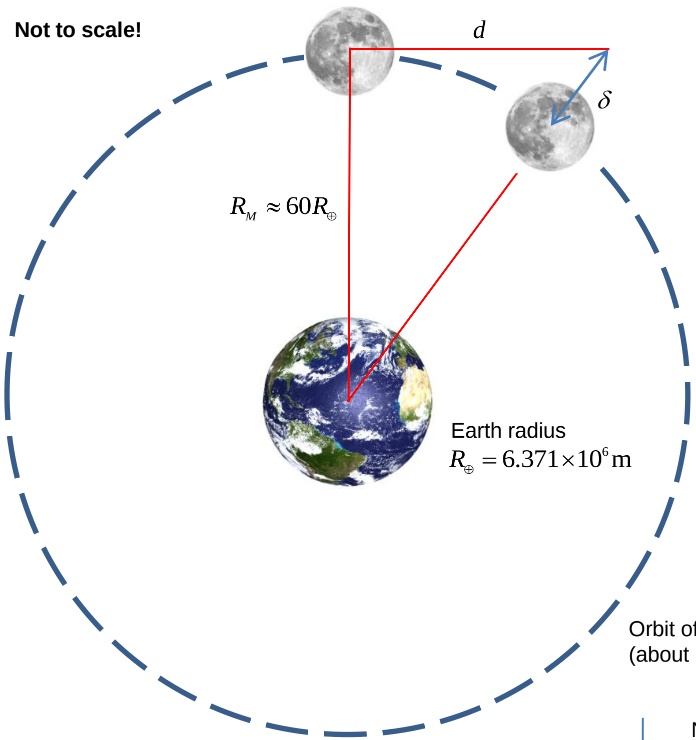
$$x = \tfrac{1}{2}gt^2 \qquad g = 9.81\text{ms}^{-2} \qquad t = 1\text{s}$$

$$\therefore \boxed{x = 4.9\text{m}}$$

i.e. constant acceleration motion means a *quadratic* dependence of displacement upon time

An apocryphal apple

Not to scale!



Newton then asked the question: **"How far does the *Moon* fall in 1 second?"**

If one assumes a circular orbit, the diagram on the left (scale highly exaggerated) represents *one second of movement*. From *Pythagoras' Theorem*, we can calculate the fall distance in terms of the Earth's Radii and the distance travelled by the Moon in one second. We shall assume over such short timescales the Moon *travels in a straight line.*

$$(60R_\oplus + \delta)^2 = d^2 + (60R_\oplus)^2$$

$$(60R_\oplus)^2 + 120R_\oplus\delta + \delta^2 = d^2 + (60R_\oplus)^2$$

$$120R_\oplus\delta + \delta^2 = d^2$$

Expect the square of the fall distance to be negligible compared to the other terms

$$120R_\oplus\delta \approx d^2$$

$$\boxed{\delta \approx \frac{d^2}{120R_\oplus}}$$

Galileo Galilei
1564-1642

In one second, the Moon will travel

$$d = \frac{2\pi\times60R_\oplus}{28\times24\times3600}$$

$$\boxed{d \approx 993\text{m}}$$

Hence in one second the Moon will fall

$$\delta \approx \frac{d^2}{120R_\oplus}$$

$$\delta \approx \frac{992.81^2}{120\times6.371\times10^6}$$

$$\boxed{\delta \approx 1.3\text{mm}}$$

Newton reasoned that to keep the Moon in orbit, a *central force* must act. He postulated that an **inverse-square law** would be appropriate, perhaps based upon how light rays might diverge from a circular source.

From a modern perspective, we might refer to how the radioactive power per unit area Φ received from the Sun follows an inverse square law, since the power per unit area times the area of a sphere surrounding the Sun must equal the total power output (or Luminosity L)

$$L = \Phi\times4\pi R^2 \quad \therefore \boxed{\Phi = \frac{L}{4\pi R^2}}$$

Note for the Sun

$$L \approx 3.846\times10^{26}\text{W}$$

$$\Phi_\oplus \approx 1{,}368\text{Wm}^{-2}$$

Based on the **inverse square law**

$$\frac{g_M}{g} = \frac{1}{60^2}$$

Hence:

$$\delta = \tfrac{1}{2}g_M t^2$$

$$g = 9.81\text{ms}^{-2} \qquad t = 1\text{s}$$

$$\therefore \delta = \tfrac{1}{2}\times\frac{9.81}{60^2} \approx \boxed{1.3\text{mm}}$$

which matches the expected distance the Moon falls to maintain its orbit.

Newton therefore proposed a **Universal Law of Gravitation** for the force acting between two masses separated by distance r

$$\boxed{F = \frac{GMm}{r^2}}$$

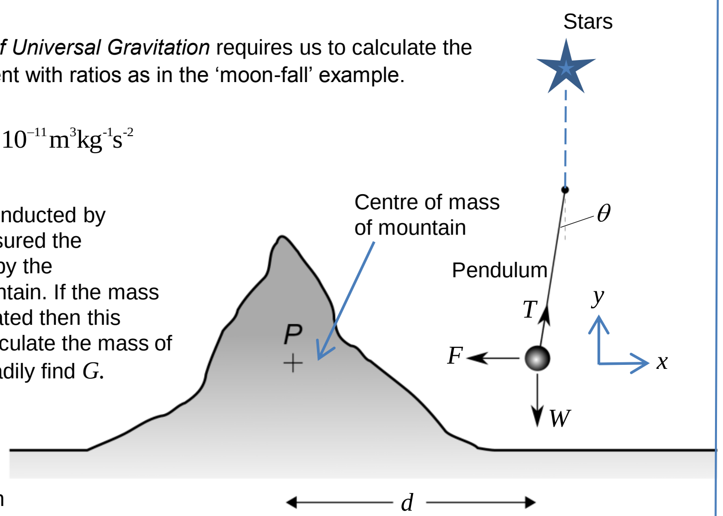
*It is assumed here that 'gravitational mass' in mg is the same as *inertia* in force = mass x acceleration. This deep connection shall be discussed later!

The Schiehallion experiment

To make use of *Newton's Law of Universal Gravitation* requires us to calculate the constant G, unless we are content with ratios as in the 'moon-fall' example.

$$\boxed{F = \frac{GMm}{r^2}} \qquad G = 6.67384 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

The Schiehallion experiment, conducted by the Royal Society in 1774, measured the defelection of a long pendulum by the gravitational attraction of a mountain. If the mass of the mountain could be calculated then this experiment could be used to calculate the mass of the Earth. From this one can readily find G.



The deflection θ was measured by comparing the line of the pendulum to the positions of several stars. These observation were performed by Nevil Maskelyne.

Applying Newton's Law of Gravitation

$$W = \frac{GmM_\oplus}{R_\oplus^2} \qquad F = \frac{GmM_M}{d^2}$$

Earth mass and radius

Mountain mass

By Newton II, assuming equilibrium of pendulum bob:

$$// x: \quad 0 = T\sin\theta - \frac{GmM_M}{d^2}$$

$$// y: \quad 0 = T\cos\theta - \frac{GmM_\oplus}{R_\oplus^2}$$

$$T\sin\theta = \frac{GmM_M}{d^2}$$

$$T\cos\theta = \frac{GmM_\oplus}{R_\oplus^2}$$

$$\therefore \tan\theta = \frac{M_M}{d^2}\frac{R_\oplus^2}{M_\oplus}$$

$$\therefore \boxed{M_\oplus = M_M \frac{R_\oplus^2}{d^2\tan\theta}}$$

Nevil Maskelyne 1732-1811

$$M_\oplus = 5.972 \times 10^{24}\,\mathrm{kg}$$

$$W = mg$$

$$\therefore g = \frac{GM_\oplus}{R_\oplus^2}$$

$$\therefore \boxed{G = \frac{gR_\oplus^2}{M_\oplus}}$$

Note g can readily be measured, and a reasonable estimate of the radius of the Earth has been known since the time of Eratosthenes (276-195BC)

In the published report it was the *density* of the earth that was reported

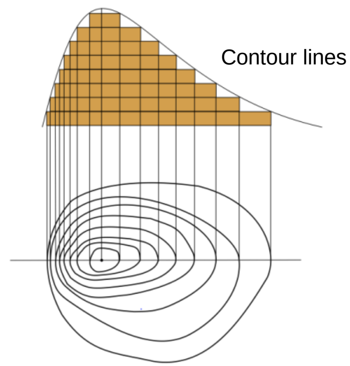
$$\rho_\oplus \approx \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^2}$$

Charles Hutton 1737-1823

Calculating the mass of Schiehallion required a detail survey of the mountain. Its density was taken as 2500 kgm^{-3}, so an accurate calculation of volume was required. Charles Hutton performed this arduous mapping task, and invented contour lines in the process!

The Schiehallion experiment reported that the Earth average density was about 4500kgm^{-3}. A modern (2007) repeat of the experiment using a digital elevation model yielded 5480kgm^{-3}. The actual value is 5515kgm^{-3}.

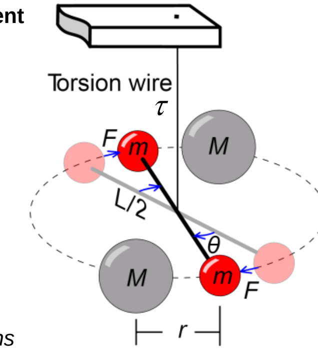
This means the Earth is not hollow, and must contain denser material at depth, possibly metallic.



Schiehallion, Perthshire, Scotland 1,083m

Measuring G via the Cavendish experiment

A more accurate value of G can be found by performing a sensitive experiment in the laboratory, using a *torsion pendulum*.



Henry Cavendish 1731-1810

Balance the torsion force (the twist) on the wire with the torque resulting from the gravitational attraction of masses M and m

$$\tau\theta = L \times \frac{GmM}{r^2}$$

$$\therefore G = \frac{\tau r^2\theta}{LmM}$$

Angle in *radians*

The torsion constant τ can be found by measuring the period P of small oscillations of the pendulum

$$P = 2\pi\sqrt{\frac{I}{\tau}}$$

$$P = 2\pi\sqrt{\frac{m}{k}}$$

For a mass on a spring of stiffness k

$$I = 2 \times m\left(\tfrac{1}{2}L\right)^2 = \tfrac{1}{2}mL^2$$

Moment of inertia I of the pendulum *about* the wire axis

$$\frac{P^2}{4\pi^2} = \frac{\frac{1}{2}mL^2}{\tau}$$

$$\therefore \boxed{\tau = \frac{2\pi^2 mL^2}{P^2}}$$

Cavendish measured $G = 6.74 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$

We can now combine the expressions to find G

$$G = \frac{2\pi^2 mL^2}{P^2} \times \frac{r^2\theta}{LmM}$$

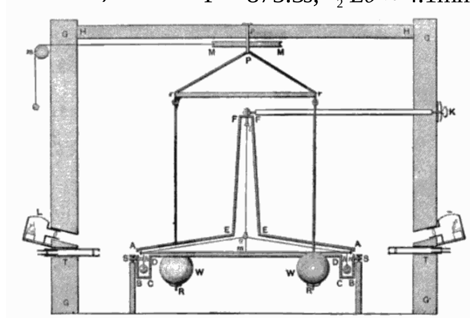
$$\boxed{G = \frac{2\pi^2 Lr^2\theta}{MP^2}}$$

In the original experiment (see diagram below)

$$m = 0.73\mathrm{kg}, \quad M = 158\mathrm{kg}$$

$$L = 1.8\mathrm{m}, \quad r = (230 - 4.1)\mathrm{mm}$$

$$P = 875.3\mathrm{s}, \quad \tfrac{1}{2}L\theta \approx 4.1\mathrm{mm}$$



https://en.wikipedia.org/wiki/Cavendish_experiment

Kepler's Three Laws of Orbital Motion

Inspired by the heliocentric model of Copernicus, and using the astronomical data obtained by Tycho Brahe, Kepler discovered three laws of planetary motion.

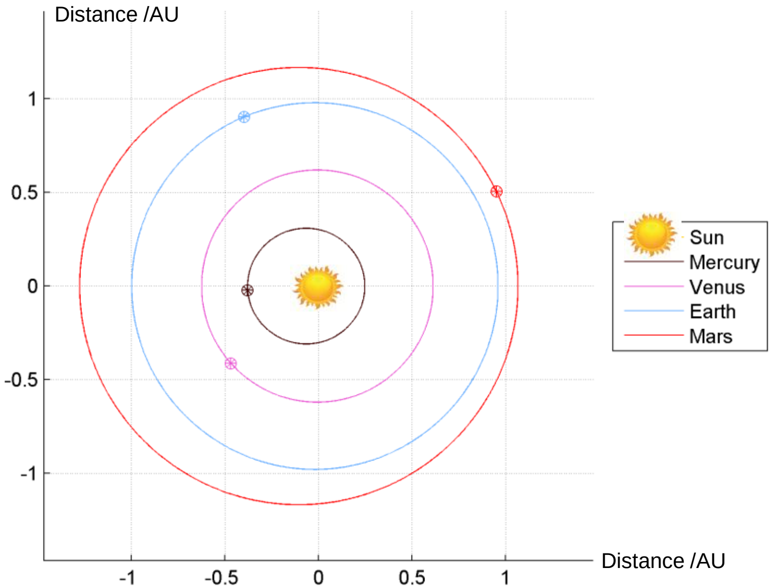
1. The orbit of every planet in the solar system is an **ellipse** with the Sun at one of the two foci.
2. A line joining a planet and the Sun sweeps out **equal areas during equal intervals of time**
3. The **square** of the orbital **period** of a planet is directly proportional to the **cube** of the **semi-major axis** of its orbit.

The wording of Kepler's Laws implies a specific application to the solar system. However, the laws are more generally applicable to *any* system of two masses whose mutual attraction is an inverse-square law.

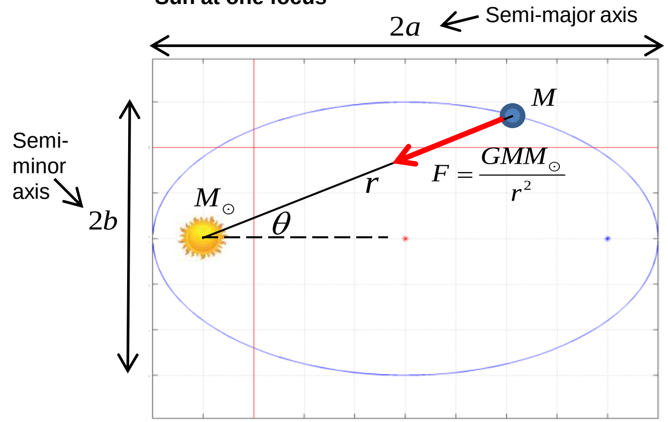

Johannes Kepler
1571-1630


Tycho Brahe
1546-1601

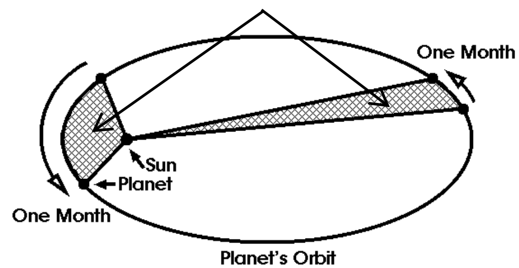

Nicolaus
Copernicus
1473-1543



Kepler I: Orbits of the planets are ellipses with the Sun at one focus



Kepler II: Equal areas swept out in equal times



$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

i.e. a **circle has zero eccentricity**. As eccentricity tends to unity, the ellipse becomes more elongated.

$$P^2 = \frac{4\pi^2}{G(M+M_\odot)}a^3$$

Kepler III

Assume $a \geq b$ without loss of generality – since we can rotate the ellipse!

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(M+M_\odot)\left(1-\varepsilon^2\right)a}$$

Applying Newton's Law of Gravitation we can show the rate of change of area A swept is a constant.

Using Newton's Law of Universal Gravitation to characterize circular orbits

If a planet orbits a massive object such as a star, to a good approximation the orbit is a **perfect circle** centred on the centre of the star. (In general in a two-mass closed system where relativistic effects can be ignored, both objects will orbit in an *elliptical* fashion about their common centre of mass or *barycenter**).

The only force binding the planet to the star is **gravity**, which is a *central force* i.e. acts entirely radially. If we ignore any mass asymmetries for the planet and the star, we can conclude that there will no tangential forces which might speed up the orbital rotation rate.

The *orbital velocity* is therefore a *constant*. If the period is P and the orbital radius r, the orbital velocity is

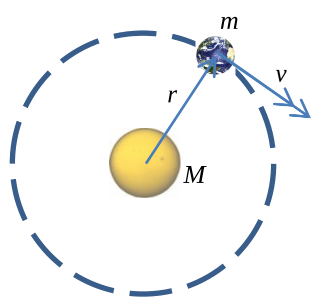
$$\boxed{v = \frac{2\pi r}{P}}$$

If the planet is executing circular motion its acceleration is radially towards the center of the star and has magnitude

$$\boxed{a = \frac{v^2}{r}}$$

$$\therefore a = \frac{4\pi^2 r^2}{P^2}\frac{1}{r}$$

$$\therefore a = \frac{4\pi^2 r}{P^2}$$



Applying **Newton's Second Law**, and using the **Universal Law of Gravitation**

$$ma = \frac{GMm}{r^2}$$

$$m\frac{4\pi^2 r}{P^2} = \frac{GMm}{r^2}$$

$$\boxed{\frac{4\pi^2}{GM}r^3 = P^2}$$

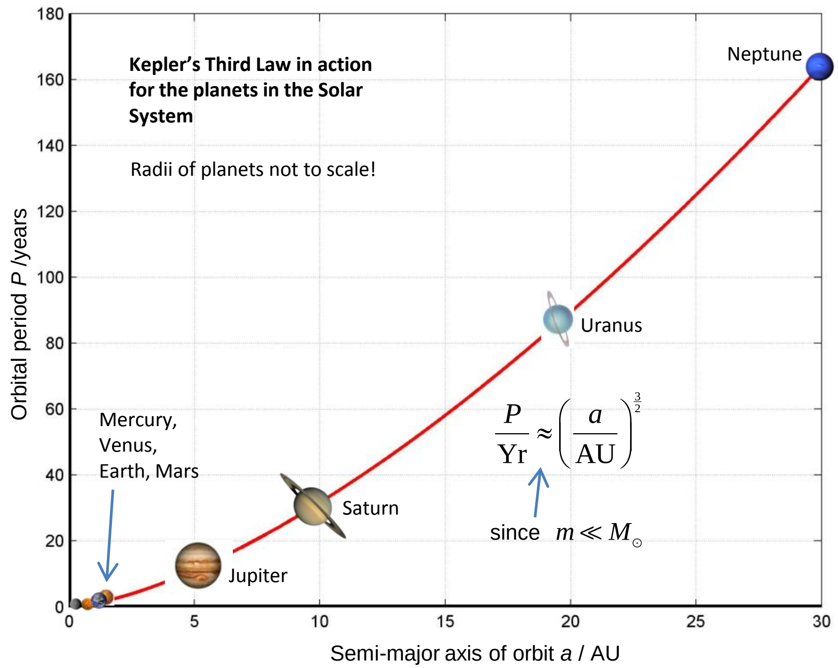
Which is **Kepler's Third Law**. Indeed the form is quite general. For a two body system the M is the total mass and the r is the maximum separation (the 'semi-major axis').

Kepler III as a *ratio* between different planets

$$\frac{4\pi^2}{GM}r^3 = P^2$$

$$\therefore \frac{4\pi^2}{GM}R^3 = P_R^{\ 2}$$

$$\therefore \boxed{\left(\frac{r}{R}\right)^3 = \left(\frac{P}{P_R}\right)^2}$$



Kepler's Third Law

$$\boxed{P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3}$$

$M_\odot = 1.99\times10^{30}\,\text{kg}$ Sun mass

$G = 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$

$\text{AU} = 1.49597871\times10^{11}\,\text{m}$

$24\times3600\text{s} = 1\text{day}$

$M_\odot = 332{,}837 m_\oplus$

$m_\oplus = 5.972\times10^{24}\,\text{kg}$ Earth mass

An **Astronomical Unit** (AU) is the average Earth-Sun separation.

Planet	*T* / years	*r* / AU	*m* / Earth masses	Rotation period /days	Orbital eccentricity
Mercury	0.241	0.387	0.055	58.646	0.21
Venus	0.615	0.723	0.815	243.018	0.01
Earth	1.000	1.000	1.000	1.000	0.02
Mars	1.881	1.523	0.107	1.026	0.09
Jupiter	11.861	5.202	317.85	0.413	0.05
Saturn	29.628	9.576	95.159	0.444	0.06
Uranus	84.747	19.293	14.5	0.718	0.05
Neptune	166.344	30.246	17.204	0.671	0.01
Pluto	248.348	39.509	0.003	6.387	0.25

[Note Pluto orbits in a *different plane* to the other planets, and is officially a 'dwarf planet', not a planet]

These orbit in the *"plane of the ecliptic"*

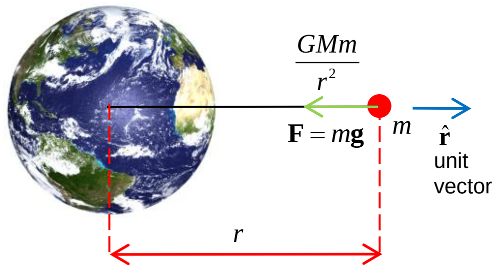
*Note the barycenter of the Pluto-Charon system is actually *outside* Pluto

Gravitational field strength and gravitational potential

Newton's Law of Universal Gravitation tells us the force $\mathbf{F}$ on a mass m at distance r from the centre of a mass M acts radially inwards along the line joining the centres of the masses.

The *gravitational field strength* $\mathbf{g}$ is defined to be

$$\boxed{\begin{aligned}&\mathbf{F} = m\mathbf{g}\\ &\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}\end{aligned}}$$



The Newtonian model of gravity is that of a force which permeates all space, and whose magnitude and direction is computable from the spatial distribution of mass.

Gravity is therefore a **field of vectors** – at any point in space we can draw an arrow pointing the direction of gravitational force, and with a length proportional to the strength.

For many calculations it is useful t p compute the **Gravitational Potential Energy**, that is a measure of the work done against gravity to move a particle to a particular point. This is useful when combined with the Law of Conservation of Energy, as we can work out the speed of a gravitationally bound object based upon **scalar** parameters, rather than needing to worry about directions of the vector quantities involved like force, velocity and displacement.

The work done in moving an object of mass m from distance a to b against gravity is

$$W = \int_a^b \frac{GMm}{r^2}\,dr$$

$$W = \left[-\frac{GMm}{r}\right]_a^b$$

$$W = GMm\left(\frac{1}{a} - \frac{1}{b}\right)$$

The maximum work done is when b is infinite:

$$W_{\max} = \frac{GMm}{a}$$

If mass m is launched radially from distance a with kinetic energy E, we would expect gravity to slow it down. If at an infinite distance away the mass has zero speed, then by conservation of energy:

$$E = W_{\max} = \frac{GMm}{a}$$

But the total energy 'at infinity' must be zero since the mass has no speed and will not be affected by gravity.

Therefore in order to conserve energy everywhere, the total energy at any radius *must be zero everywhere.*

We can therefore define a *gravitational potential energy* (GPE) $m\phi$ such that

$$E + m\phi = 0$$

$$E = \frac{GMm}{a}$$

$$\therefore \phi = -\frac{GM}{a}$$

i.e. it makes sense for GPE to be *negative*

In general the mass may not have enough energy to escape to infinity, or indeed have more than enough. Let the total energy be U

$$\boxed{U = \tfrac{1}{2}mv^2 - \frac{GMm}{r}}$$

The definition of GPE we have adopted allows us to make a very general connection between **field strength** and **potential**

$$\mathbf{g} = -\frac{GM}{r^2}, \quad \phi = -\frac{GM}{r}$$

$$\therefore \boxed{\mathbf{g} = -\frac{d\phi}{dr}\hat{\mathbf{r}}}$$

This provides up with a powerful tool if we wish to generalize the problem to many masses. We can *sum the gravitational potentials* and then **take the negative gradient to find the field strength**.

In 2D or 3D we need the *vector operator "grad"* as potential ϕ might vary with all x,y,z coordinates

$$\mathbf{g} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right)$$

$$\boxed{\mathbf{g} = -\nabla\phi}$$

Escape velocity

In order to escape, the total energy of the system must be *positive* at an infinite distance from the body. In other words, it will have some kinetic energy and will never be gravitationally attracted back towards the body.

For a mass m blasting off with velocity v, it will escape the gravitational influence of M if:

$$\tfrac{1}{2}mv^2 - \frac{GMm}{R} > 0 \qquad \therefore \boxed{v > \sqrt{\frac{2GM}{R}}}$$

For Earth, the escape velocity is:

$$v_{escape} = \sqrt{\frac{2GM}{R}}$$

$$v_{escape} = \sqrt{\frac{2\times 6.67\times 10^{-11}\times 5.97\times 10^{24}}{6.38\times 10^{6}}}$$

$$\approx \boxed{11.2\,\mathrm{kms}^{-1}}$$

It is interesting to work out the radius of a star of mass *M* such that the escape velocity exceeds that of the speed of light. Since this is not possible, the star becomes a *Black Hole.* This inequality defines the maximum radius of a Black Hole, which is called the *Schwarzschild radius.* Alternatively, this is the *event horizon*, or 'point of no return' from the centre of a Black Hole.

$$\sqrt{\frac{2GM}{R}} > c$$

$$\frac{2GM}{R} > c^2$$

$$\boxed{R < \frac{2GM}{c^2}}$$

For the Sun to become a Black Hole (M = 2 x 10^{30} kg, R = 6.96 x 10^{8} m) its radius would have to shrink to less than 2.97 km. This is a mind-blowing density of 1.8 x 10^{19} kgm^{-3}

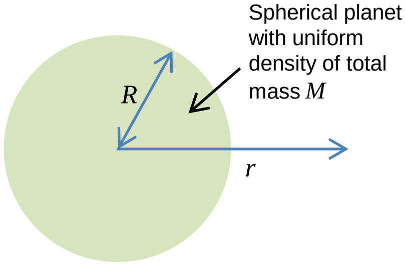
$$\rho_{Black\ hole} > \frac{M}{\frac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3}$$

$$\boxed{\rho_{Black\ hole} > \frac{3c^6}{32\pi G^3 M^2}}$$

As enormous as this sounds, it is not *entirely* outrageous given the density of the nucleus of a typical atom is approximately

$$\rho \approx \frac{2\times 10^{-27}}{\frac{4}{3}\pi\times\left(10^{-15}\right)^3} \approx \boxed{5\times 10^{17}\,\mathrm{kgm}^{-3}}$$

Gravitational field strength inside and outside a uniform sphere



We can generalize our definition of gravitational potential to be **Gauss' Law of Gravity**

$$\boxed{\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G m}$$

where m is the mass enclosed within closed surface S, whose surface normal area vector is $d\mathbf{S}$

Now $m(r)$ is the mass enclosed within radius r, hence

$0 < r \le R$

$$m = \tfrac{4}{3}\pi r^3 \times \frac{M}{\tfrac{4}{3}\pi R^3}$$

$$m = \frac{Mr^3}{R^3}$$

$$\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G m$$

$$\mathbf{g} = -g\hat{\mathbf{r}}$$

$$d\mathbf{S} = dS\hat{\mathbf{r}}$$

$$\int_S \mathbf{g}\cdot d\mathbf{S} = \int_S -g\hat{\mathbf{r}}\cdot dS\hat{\mathbf{r}} = -g\int_S dS$$

$$\therefore -g \times 4\pi r^2 = -4\pi G\frac{Mr^3}{R^3}$$

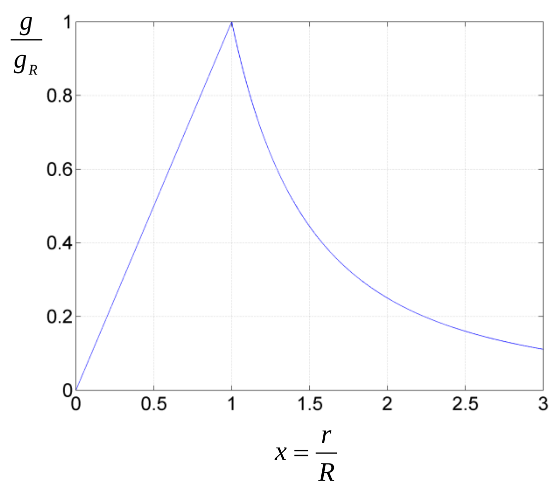
$$\therefore \boxed{g = \frac{GMr}{R^3}}$$

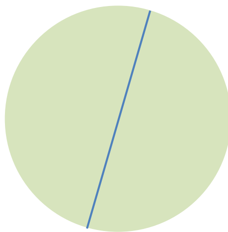
$r \ge R$

$$\boxed{g = \frac{GM}{r^2}}$$

To construct a generic plot define:

$$\boxed{\begin{aligned} g_R &= \frac{GM}{R^2} \\ x &= \frac{r}{R} \\ \therefore g &= \begin{cases} g_R x & 0 < x \le 1 \\ \dfrac{g_R}{x^2} & x \ge 1 \end{cases} \end{aligned}}$$





This means that if a tunnel could be drilled through a planet of uniform density, *Newton's Second Law* means for a mass dropped into the tunnel at rest at the surface

$$\ddot{r} = -\frac{GM}{R^3}r$$

This is the equation of **Simple Harmonic Motion (SHM)**

$$\ddot{r} = -\left(\frac{2\pi}{P}\right)^2 r$$

where P is the period of the resulting oscillatory motion

$$\left(\frac{2\pi}{P}\right)^2 = \frac{GM}{R^3}$$

$$\therefore \boxed{P = 2\pi\sqrt{\frac{R^3}{GM}}}$$

$$r = R\cos\left(\frac{2\pi t}{P}\right)$$

$$\boxed{r = R\cos\left(t\sqrt{\frac{GM}{R^3}}\right)}$$

For the Earth this would be a period of

$$P = 2\pi\sqrt{\frac{\left(6.371\times 10^6\right)^3}{6.67384\times 10^{-11}\times 5.972\times 10^{24}}}$$

$$P = 5061\text{s} \approx \boxed{84\text{mins}}$$

So a 42 minute trip to Australasia from Europe without any jet fuel required...

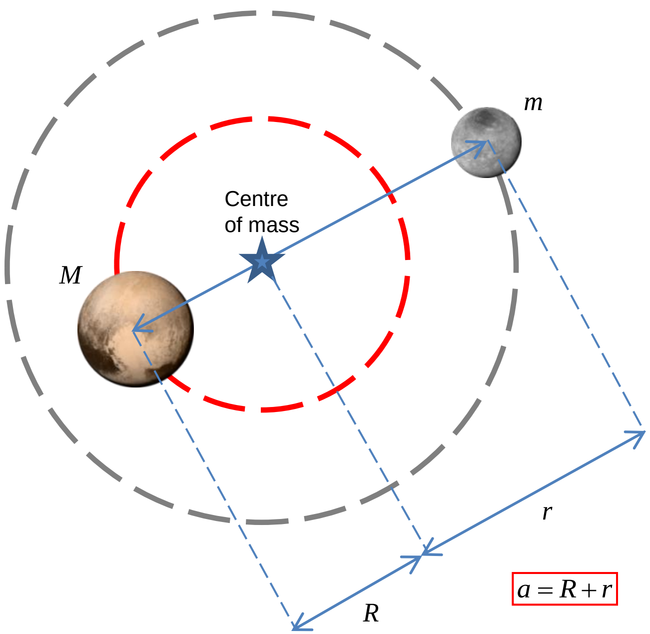
$$R_\oplus = 6.371\times 10^6\,\text{m}$$

$$G = 6.67384\times 10^{-11}\,\text{Nm}^2\text{kg}^{-2}$$

$$m_\oplus = 5.972\times 10^{24}\,\text{kg}$$

Two-body Kepler problem with circular orbits

The essential features of the more general Kepler problem can be obtained by considering circular orbits of gravitationally bound masses about a common centre of mass.



Since mass M gravitationally attracts mass m with a force along the lines connecting the centres (and vice versa), there is no net torque on the system. Hence the angular acceleration is zero and therefore the angular speed ω of (both masses) is a constant. Let P be the orbital period.

$$\boxed{\omega = \frac{2\pi}{P}}$$

By Newton's second law $\quad MR\omega^2 = \dfrac{GMm}{a^2}, \quad mr\omega^2 = \dfrac{GMm}{a^2}$

$$\therefore R\omega^2 = \frac{Gm}{a^2}, \quad r\omega^2 = \frac{GM}{a^2}$$

$$\therefore R\omega^2 + r\omega^2 = \frac{G(m+M)}{a^2}$$

$$(R+r)\omega^2 = \frac{G(m+M)}{a^2}$$

$$\therefore \omega^2 = \frac{G(m+M)}{a^3}$$

$$\frac{4\pi^2}{P^2} = \frac{G(m+M)}{a^3}$$

$$\therefore \boxed{P^2 = \frac{4\pi^2}{G(m+M)}a^3}$$

Kepler's Third Law

We can now work out the energy of the combined system:

$$E = \tfrac{1}{2}M(R\omega)^2 + \tfrac{1}{2}m(r\omega)^2 - \frac{GMm}{a}$$

$$E = \tfrac{1}{2}\omega^2\left(MR^2 + mr^2\right) - \frac{GMm}{a}$$

By Newton's second law $\quad MR\omega^2 = \dfrac{GMm}{a^2}, \quad mr\omega^2 = \dfrac{GMm}{a^2}$

Include expression for radial acceleration for circular motion

$$\therefore R = \frac{Gm}{a^2\omega^2}, \quad r = \frac{GM}{a^2\omega^2}$$

$$\therefore R^2 = \frac{G^2m^2}{a^4\omega^4}, \quad r^2 = \frac{G^2M^2}{a^4\omega^4}$$

Hence

$$E = \tfrac{1}{2}\omega^2\left(MR^2 + mr^2\right) - \frac{GMm}{a}$$

$$E = \tfrac{1}{2}\omega^2\left(M\frac{G^2m^2}{a^4\omega^4} + m\frac{G^2M^2}{a^4\omega^4}\right) - \frac{GMm}{a}$$

$$E = \tfrac{1}{2}\frac{G^2Mm}{a^4\omega^2}(m+M) - \frac{GMm}{a}$$

Now from above

$$\omega^2 = \frac{G(m+M)}{a^3}$$

Therefore:

$$E = \tfrac{1}{2}\frac{G^2Mm}{a^4}\frac{a^3}{G(m+M)}(m+M) - \frac{GMm}{a}$$

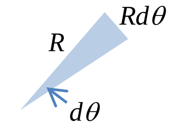
$$E = \tfrac{1}{2}\frac{GMm}{a} - \frac{GMm}{a}$$

$$\boxed{E = -\frac{GMm}{2a}}$$

Note the total negative energy is indicative of a *bound* orbit. *Parabolic* or *hyperbolic* trajectories will have a positive total energy

The *rate of area swept* by the masses is

$$\boxed{\begin{aligned}\frac{dA_M}{dt} &= \tfrac{1}{2}R^2\omega \\ \frac{dA_m}{dt} &= \tfrac{1}{2}r^2\omega\end{aligned}}$$



$$\omega dt = Rd\theta$$

e.g. $\quad dA_M = \tfrac{1}{2}R^2 d\theta$

$$\frac{dA_M}{dt} = \tfrac{1}{2}R^2\frac{d\theta}{dt} = \tfrac{1}{2}R^2\omega$$

From above:

$$R^2 = \frac{G^2m^2}{a^4\omega^4}, \quad r^2 = \frac{G^2M^2}{a^4\omega^4}$$

Hence:

$$\frac{dA_M}{dt} = \tfrac{1}{2}R^2\omega = \tfrac{1}{2}\frac{G^2m^2}{a^4\omega^4}\omega = \tfrac{1}{2}\frac{G^2m^2}{a^4\omega^3}$$

$$\frac{dA_m}{dt} = \tfrac{1}{2}r^2\omega = \tfrac{1}{2}\frac{G^2M^2}{a^4\omega^4}\omega = \tfrac{1}{2}\frac{G^2M^2}{a^4\omega^3}$$

The rate of area swept is therefore a *constant* for each mass, i.e.

Kepler's Second Law

Note: $\quad \dfrac{dA_m}{dt} = \tfrac{1}{2}\dfrac{G^2M^2}{a^4\omega^3}$

$$\frac{dA_m}{dt} = \tfrac{1}{2}\frac{G^2M^2}{a^4}\left(\frac{a^3}{G(m+M)}\right)^{\frac{3}{2}}$$

$$m \ll M$$

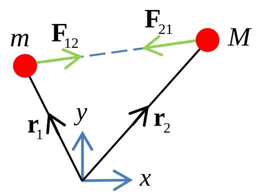
$$\omega^2 = \frac{G(m+M)}{a^3} \longrightarrow \therefore \frac{dA_m}{dt} \approx \tfrac{1}{2}\frac{G^2M^2}{a^4}\left(\frac{a^3}{GM}\right)^{\frac{3}{2}}$$

$$\frac{dA_m}{dt} \approx \tfrac{1}{2}G^{\frac{1}{2}}M^{\frac{1}{2}}a^{\frac{1}{2}} = \boxed{\tfrac{1}{2}\sqrt{GMa}}$$

This is consistent with the more general result for elliptical orbits

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

Two body Kepler problem



$$\mathbf{F}_{12} = \frac{GMm}{|\mathbf{r}_2-\mathbf{r}_1|^2}\times\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|}$$

$$\mathbf{F}_{12} = \frac{GMm}{|\mathbf{r}_2-\mathbf{r}_1|^3}(\mathbf{r}_2-\mathbf{r}_1)$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Newton II

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_{12}$$

$$m\ddot{\mathbf{r}}_1 = \frac{GMm}{|\mathbf{r}_2-\mathbf{r}_1|^3}(\mathbf{r}_2-\mathbf{r}_1)$$

$$M\ddot{\mathbf{r}}_2 = \mathbf{F}_{21}$$

$$M\ddot{\mathbf{r}}_2 = -\frac{GMm}{|\mathbf{r}_2-\mathbf{r}_1|^3}(\mathbf{r}_2-\mathbf{r}_1)$$

Define centre of mass vector

$$\boxed{\mathbf{R} = \frac{m\mathbf{r}_1+M\mathbf{r}_2}{m+M}}$$

and separation vector

$$\boxed{\mathbf{r} = \mathbf{r}_2-\mathbf{r}_1}$$

Hence:

$$\boxed{\begin{aligned}\mathbf{r}_1 &= \mathbf{R}-\frac{M}{m+M}\mathbf{r}\\ \mathbf{r}_2 &= \mathbf{R}+\frac{m}{m+M}\mathbf{r}\end{aligned}}$$

$$\ddot{\mathbf{R}} = \frac{m\ddot{\mathbf{r}}_1+M\ddot{\mathbf{r}}_2}{m+M}$$

$$\therefore (m+M)\ddot{\mathbf{R}} = m\ddot{\mathbf{r}}_1+M\ddot{\mathbf{r}}_2$$

From above

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_{12}$$

$$M\ddot{\mathbf{r}}_2 = \mathbf{F}_{21} = -\mathbf{F}_{12}$$

$$\therefore m\ddot{\mathbf{r}}_1+M\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Therefore $\boxed{\dot{\mathbf{R}} = \text{constant}}$

which means the centre of mass of the system moves at a constant velocity. Without loss of generality we can define a reference frame co-moving with the centre of mass. So from now on we will set $\boxed{\mathbf{R} = \mathbf{0}}$

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2-\ddot{\mathbf{r}}_1$$

$$\therefore \ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r} = -\frac{G(M+m)}{r^2}\hat{\mathbf{r}}$$

Using the previous Newton II expressions.

This means the two body problem is basically a *one body problem*, with the separation vector $\mathbf{r}$ being the displacement from a total mass $m + M$

Angular Momentum Note $\dot{\mathbf{R}} = \mathbf{0}$

$$\mathbf{J} = m\mathbf{r}_1\times\dot{\mathbf{r}}_1+M\mathbf{r}_2\times\dot{\mathbf{r}}_2$$

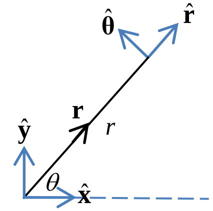
$$\mathbf{J} = m\left(\mathbf{R}-\frac{M}{m+M}\mathbf{r}\right)\left(-\frac{M}{m+M}\dot{\mathbf{r}}\right)+M\left(\mathbf{R}+\frac{m}{m+M}\mathbf{r}\right)\times\frac{m}{m+M}\dot{\mathbf{r}}$$

$$\therefore \boxed{\mathbf{J} = \frac{mM}{m+M}\mathbf{r}\times\dot{\mathbf{r}}} \quad \Rightarrow \dot{\mathbf{J}} = \frac{mM}{m+M}(\dot{\mathbf{r}}\times\dot{\mathbf{r}}+\mathbf{r}\times\ddot{\mathbf{r}})$$

$$\therefore \dot{\mathbf{J}} = \frac{mM}{m+M}\left(\dot{\mathbf{r}}\times\dot{\mathbf{r}}-\mathbf{r}\times\mathbf{r}\frac{G(M+m)}{r^3}\right) = \mathbf{0}$$

both zero

$\therefore J = |\mathbf{J}|$ is a constant for the system



$$\mathbf{J} = \frac{mM}{m+M}\mathbf{r}\times\dot{\mathbf{r}}$$

$$\mathbf{J} = \frac{mM}{m+M}r\hat{\mathbf{r}}\times\left(\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{J} = \frac{mM}{m+M}r^2\dot{\theta}\left(\mathbf{r}\times\hat{\boldsymbol{\theta}}\right)$$

$$\therefore J^2 = |\mathbf{J}|^2 = \frac{m^2M^2}{(m+M)^2}r^4\dot{\theta}^2$$

Since angular momentum is a constant

$$\dot{\theta}^2 = \frac{(m+M)^2J^2}{m^2M^2r^4}$$

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^2}\hat{\mathbf{r}}$$

$$\ddot{\mathbf{r}} = \left(\ddot{r}-r\dot{\theta}^2\right)\hat{\mathbf{r}}+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

$$\therefore \frac{\ddot{\theta}}{\dot{\theta}} = -\frac{2\dot{r}}{r}$$

$$\therefore \ddot{r}-r\dot{\theta}^2 = -\frac{G(M+m)}{r^2}$$

$$\therefore \ddot{r} = r\frac{(m+M)^2J^2}{m^2M^2r^4}-\frac{G(M+m)}{r^2}$$

We will now solve the Kepler problem using **plane polar coordinates**

$$\boxed{\begin{aligned}\mathbf{r} &= r\hat{\mathbf{r}}\\ \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}\\ \ddot{\mathbf{r}} &= \left(\ddot{r}-r\dot{\theta}^2\right)\hat{\mathbf{r}}+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{\mathbf{r}} &= \hat{\mathbf{x}}\cos\theta+\hat{\mathbf{y}}\sin\theta\\ \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{x}}\sin\theta+\hat{\mathbf{y}}\cos\theta\end{aligned}}$$

To simplify let us define $\boxed{u = \frac{1}{r}}$ $\quad \therefore \dot{u} = -\frac{1}{r^2}\dot{r}, \quad \ddot{u} = -\frac{\ddot{r}}{r^2}+\frac{2\dot{r}^2}{r^3}$

$$\dot{r} = -r^2\dot{u}, \quad \ddot{r} = \frac{2\dot{r}^2}{r}-\ddot{u}r^2 = 2r^3\dot{u}^2-\ddot{u}r^2 = \frac{2\dot{u}^2}{u^3}-\frac{\ddot{u}}{u^2}$$

$$\therefore \frac{2\dot{u}^2}{u^3}-\frac{\ddot{u}}{u^2} = u^3\frac{(m+M)^2J^2}{m^2M^2}-u^2G(M+m)$$

$$\therefore \frac{2\dot{u}^2}{u}-\ddot{u} = u^5\frac{(m+M)^2J^2}{m^2M^2}-u^4G(M+m)$$

$$\frac{du}{d\theta} = \frac{du}{dt}\times\frac{dt}{d\theta} = \frac{\dot{u}}{\dot{\theta}}$$ by the Chain Rule

$$\frac{d^2u}{d\theta^2} = \frac{\dot{\theta}\frac{d\dot{u}}{d\theta}-\dot{u}\frac{d\dot{\theta}}{d\theta}}{\dot{\theta}^2} = \frac{\dot{\theta}\frac{d\dot{u}}{dt}\frac{dt}{d\theta}-\dot{u}\frac{d\dot{\theta}}{dt}\frac{dt}{d\theta}}{\dot{\theta}^2}$$

$$\frac{d^2u}{d\theta^2} = \frac{\dot{\theta}\ddot{u}\frac{1}{\dot{\theta}}-\dot{u}\ddot{\theta}\frac{1}{\dot{\theta}}}{\dot{\theta}^2} = \frac{\ddot{u}-\dot{u}\frac{\ddot{\theta}}{\dot{\theta}}}{\dot{\theta}^2}$$

From above: $\frac{\ddot{\theta}}{\dot{\theta}} = -\frac{2\dot{r}}{r}, \quad r\dot{u} = -\frac{\dot{r}}{r} \quad \therefore \frac{\ddot{\theta}}{\dot{\theta}} = 2r\dot{u} = \frac{2\dot{u}}{u}$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{\ddot{u}-\dot{u}\frac{\ddot{\theta}}{\dot{\theta}}}{\dot{\theta}^2} = \frac{\ddot{u}-\frac{2\dot{u}^2}{u}}{\dot{\theta}^2} \qquad \dot{\theta}^2 = \frac{(m+M)^2J^2}{m^2M^2}u^4$$

$$\therefore \frac{2\dot{u}^2}{u}-\ddot{u} = -\frac{(m+M)^2J^2}{m^2M^2}u^4\frac{d^2u}{d\theta^2}$$

Hence: $$-\frac{(m+M)^2J^2}{m^2M^2}u^4\frac{d^2u}{d\theta^2} = u^5\frac{(m+M)^2J^2}{m^2M^2}-u^4G(M+m)$$

$$\therefore \boxed{\frac{d^2u}{d\theta^2}+u = \frac{Gm^2M^2}{(M+m)J^2}}$$

$$\boxed{\frac{d^2u}{d\theta^2}+u=\frac{Gm^2M^2}{(M+m)J^2}}$$

If the orbits are ellipses, the equation of an ellipse in polar coordinates is

$$\boxed{r=\frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}}$$

(assume use left focus)

$$r(0)=\frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon}=\frac{a(1-\varepsilon)(1+\varepsilon)}{1-\varepsilon}=a(1+\varepsilon)$$

$$u=\frac{1-\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}$$

$$\frac{du}{d\theta}=\frac{\varepsilon\sin\theta}{a\left(1-\varepsilon^2\right)}$$

$$\frac{d^2u}{d\theta^2}=\frac{\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}$$

$$\frac{d^2u}{d\theta^2}+u=\frac{Gm^2M^2}{(M+m)J^2}$$

$$\Rightarrow\frac{\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}+\frac{1-\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}=\frac{Gm^2M^2}{(M+m)J^2}$$

$$\frac{1}{a\left(1-\varepsilon^2\right)}=\frac{Gm^2M^2}{(M+m)J^2}$$

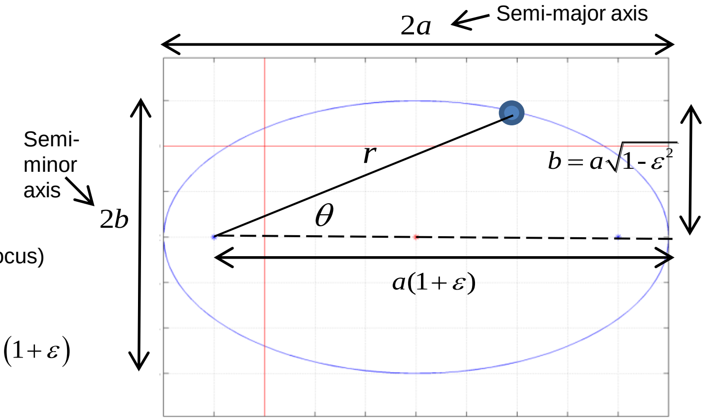
$$\boxed{J^2=\frac{Gm^2M^2\left(1-\varepsilon^2\right)a}{(M+m)}}$$

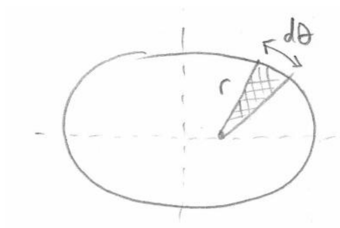
Which is certainly a constant i.e. independent of polar angle.

Since **ellipses** are solutions of

$$\frac{d^2u}{d\theta^2}+u=\frac{Gm^2M^2}{(M+m)J^2}$$

we have therefore proved **Kepler's First Law**





Area dA swept out by an orbit radial in time dt is

$$dA=\tfrac{1}{2}r^2d\theta$$

$$\therefore\boxed{\frac{dA}{dt}=\tfrac{1}{2}r^2\dot{\theta}}$$

$$\dot{\theta}^2=\frac{(m+M)^2J^2}{m^2M^2r^4}$$

$$J^2=\frac{Gm^2M^2\left(1-\varepsilon^2\right)a}{(M+m)}$$

$$\therefore\dot{\theta}^2=\frac{(m+M)^2}{m^2M^2r^4}\frac{Gm^2M^2\left(1-\varepsilon^2\right)a}{(M+m)}$$

$$\therefore\dot{\theta}^2=\frac{G(m+M)\left(1-\varepsilon^2\right)a}{r^4}$$

$$\therefore r^2\dot{\theta}=\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

$$\therefore\boxed{\frac{dA}{dt}=\tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}$$

So equal areas are swept out in equal times
Kepler's Second Law

Since equal areas are swept out in equal times, the orbital period is the area of the ellipse divided by the rate of area sweep

$$P=\frac{\pi ab}{\frac{dA}{dt}}\Rightarrow P=\frac{\pi a^2\sqrt{1-\varepsilon^2}}{\tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}$$

$$\boxed{P^2=\frac{4\pi^2}{G(m+M)}a^3}$$

Kepler's Third Law: The **square** of the orbital **period** of a planet is directly proportional to the **cube** of the **semi-major axis** of its orbit.

$$r^2\frac{d\theta}{dt}=\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

$$\therefore\int_{\theta_0}^{\theta}r^2d\theta=t\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

$$\therefore t=\frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{(1-\varepsilon\cos\theta)^2}$$

$$\therefore t=\frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{(1-\varepsilon\cos\theta)^2}$$

$$\therefore\boxed{t=\sqrt{\frac{a^3\left(1-\varepsilon^2\right)^3}{G(m+M)}}\int_{\theta_0}^{\theta}\frac{d\theta}{(1-\varepsilon\cos\theta)^2}}$$

Evaluate this numerically

Summary of orbital dynamics

Displacement

$$\mathbf{r}_1=\mathbf{R}-\frac{M}{m+M}\mathbf{r}$$

$$\mathbf{r}_2=\mathbf{R}+\frac{m}{m+M}\mathbf{r}$$

$$r=\frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\mathbf{r}=r\hat{\mathbf{r}}$$

$$\dot{\mathbf{r}}=\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{r}}=-\frac{a\left(1-\varepsilon^2\right)}{(1-\varepsilon\cos\theta)^2}(\varepsilon\sin\theta)\dot{\theta}\hat{\mathbf{r}}+\frac{1}{r}r^2\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{r}}=-\frac{1}{r}\frac{\varepsilon\sin\theta}{(1-\varepsilon\cos\theta)}r^2\dot{\theta}\hat{\mathbf{r}}+\frac{1}{r}r^2\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{r}}=\frac{r^2\dot{\theta}}{r}\left(\hat{\boldsymbol{\theta}}-\frac{\varepsilon\sin\theta}{1-\varepsilon\cos\theta}\hat{\mathbf{r}}\right)$$

$$r^2\dot{\theta}=\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

$$\dot{\mathbf{r}}=\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}\,\frac{1-\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}\left(\hat{\boldsymbol{\theta}}-\frac{\varepsilon\sin\theta}{1-\varepsilon\cos\theta}\hat{\mathbf{r}}\right)$$

$$\boxed{\dot{\mathbf{r}}=(1-\varepsilon\cos\theta)\sqrt{\frac{G(m+M)}{\left(1-\varepsilon^2\right)a}}\left(\hat{\boldsymbol{\theta}}-\frac{\varepsilon\sin\theta}{1-\varepsilon\cos\theta}\hat{\mathbf{r}}\right)}$$

Velocity

$$\therefore\dot{\mathbf{r}}(\theta=0)=(1-\varepsilon)\sqrt{\frac{G(m+M)}{\left(1-\varepsilon^2\right)a}}\hat{\boldsymbol{\theta}}$$

$$\ddot{\mathbf{r}}=-\frac{G(M+m)}{r^2}\hat{\mathbf{r}}$$

Acceleration

Example two-body simulation: Pluto and Charon http://nssdc.gsfc.nasa.gov/planetary/factsheet/plutofact.html

$m = 1.586 \times 10^{21}\,\text{kg}$ Charon

$R_C = 606\,\text{km}$

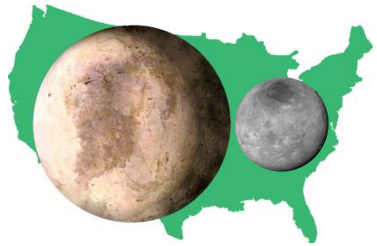
$a = 19{,}596\,\text{km}$

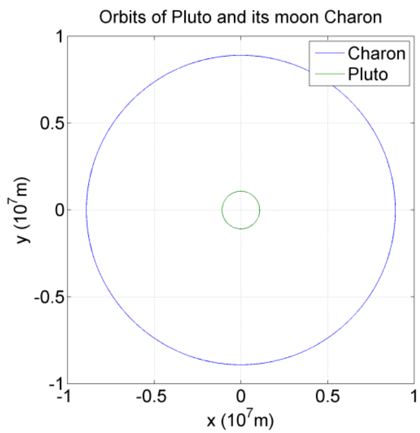
$M = 1.303 \times 10^{22}\,\text{kg}$ Pluto

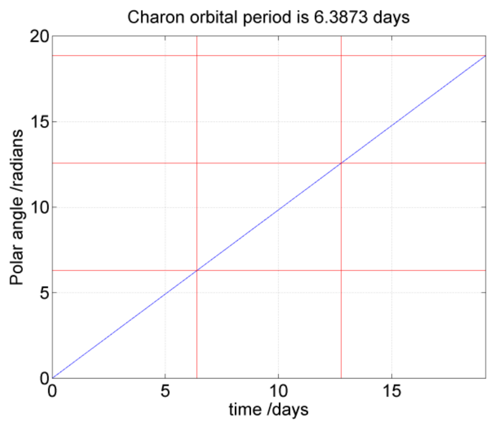
$R_P = 1187\,\text{km}$

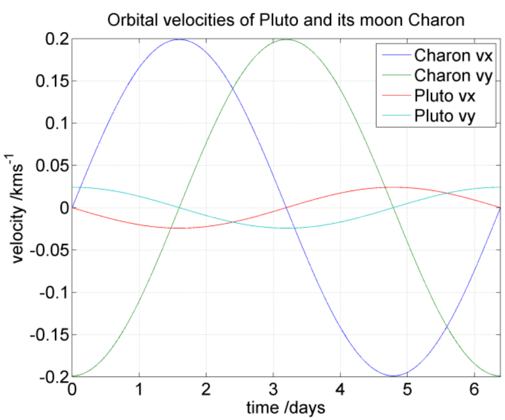
$P = 6.387\,\text{days}$

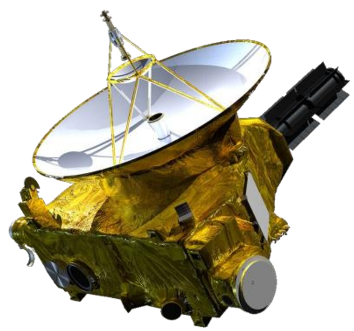
$\varepsilon = 0.00$











The Spacecraft *New Horizons* made a 12,500km approach of Pluto on July 14 2015.

Alternative simulation

$m = 1.0 \times 10^{22}\,\text{kg}$ Planet Y

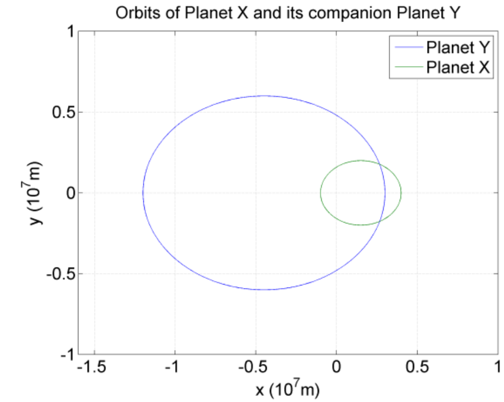
$a = 20{,}000\,\text{km}$

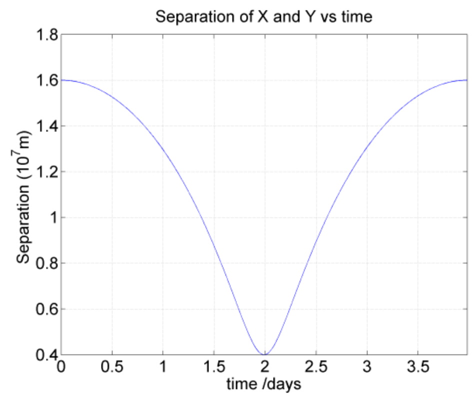
$M = 3.0 \times 10^{22}\,\text{kg}$ Planet X

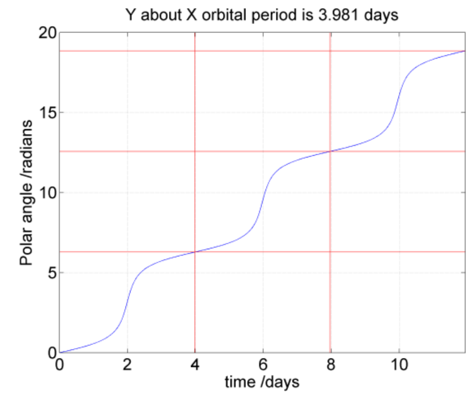
$P = 3.981$

$\varepsilon = 0.6$

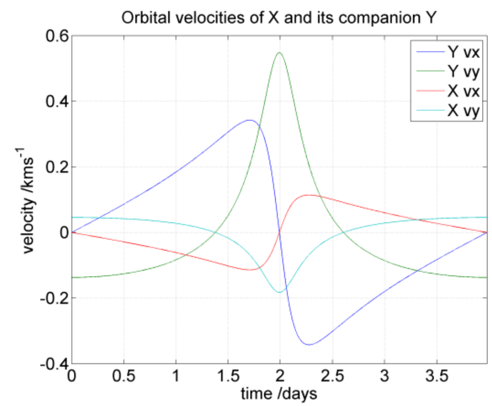
It looks like the orbits might collide... But the 'intersection points' will occur at different times for each planet. Plotting the separation magnitude r vs time shows how far apart the planets get over each orbit.



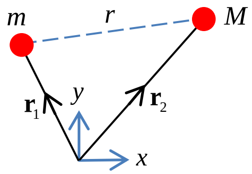




For orbits with zero eccentricity the separation will be constant i.e. the separation vector $\mathbf{r}$ races a circular orbit.



Energy in the Kepler problem



$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$r = |\mathbf{r}|$$

Recap of previous results

$$\mathbf{r}_1 = \mathbf{R} - \frac{M}{m+M}\mathbf{r}$$

$$\mathbf{r}_2 = \mathbf{R} + \frac{m}{m+M}\mathbf{r}$$

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\dot{\mathbf{r}} = \left(1-\varepsilon\cos\theta\right)\sqrt{\frac{G\left(m+M\right)}{\left(1-\varepsilon^2\right)a}}\left(\hat{\boldsymbol{\theta}} - \frac{\varepsilon\sin\theta}{1-\varepsilon\cos\theta}\hat{\mathbf{r}}\right)$$

$$|\dot{\mathbf{r}}|^2 = \left(1-\varepsilon\cos\theta\right)^2\frac{G\left(m+M\right)}{\left(1-\varepsilon^2\right)a}\left(1+\frac{\varepsilon^2\sin^2\theta}{\left(1-\varepsilon\cos\theta\right)^2}\right)$$

$$\boxed{E = \tfrac{1}{2}m|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}M|\dot{\mathbf{r}}_2|^2 - \frac{GMm}{r}}$$

Total energy is the sum of the kinetic energy of the masses and the gravitational potential energy

$$\dot{\mathbf{r}}_1 = -\frac{M}{m+M}\dot{\mathbf{r}}$$

$$\dot{\mathbf{r}}_2 = \frac{m}{m+M}\dot{\mathbf{r}}$$

$$\therefore E = \tfrac{1}{2}\frac{mM}{m+M}|\dot{\mathbf{r}}|^2 - \frac{GMm}{r}$$

$$E = \tfrac{1}{2}\frac{mM}{m+M}\left(1-\varepsilon\cos\theta\right)^2\frac{G\left(m+M\right)}{\left(1-\varepsilon^2\right)a}\left(1+\frac{\varepsilon^2\sin^2\theta}{\left(1-\varepsilon\cos\theta\right)^2}\right) - GMm\frac{1-\varepsilon\cos\theta}{a\left(1-\varepsilon^2\right)}$$

$$E = -\frac{GmM}{2a}\left\{-\frac{\left(1-\varepsilon\cos\theta\right)^2}{\left(1-\varepsilon^2\right)} - \frac{\left(1-\varepsilon\cos\theta\right)^2}{\left(1-\varepsilon^2\right)}\frac{\varepsilon^2\sin^2\theta}{\left(1-\varepsilon\cos\theta\right)^2} + \frac{2\left(1-\varepsilon\cos\theta\right)}{\left(1-\varepsilon^2\right)}\right\}$$

$$E = -\frac{GmM}{2a}\left\{\frac{-\left(1-\varepsilon\cos\theta\right)^2 - \varepsilon^2\sin^2\theta + 2\left(1-\varepsilon\cos\theta\right)}{1-\varepsilon^2}\right\}$$

$$E = -\frac{GmM}{2a}\left\{\frac{-\left(1-2\varepsilon\cos\theta+\varepsilon^2\cos^2\theta\right) - \varepsilon^2\sin^2\theta + 2 - 2\varepsilon\cos\theta}{1-\varepsilon^2}\right\}$$

$$E = -\frac{GmM}{2a}\left\{\frac{-1+2\varepsilon\cos\theta-\varepsilon^2\cos^2\theta-\varepsilon^2\sin^2\theta+2-2\varepsilon\cos\theta}{1-\varepsilon^2}\right\}$$

$$E = -\frac{GmM}{2a}\left\{\frac{1-\varepsilon^2\left(\cos^2\theta+\sin^2\theta\right)}{1-\varepsilon^2}\right\}$$

$$\boxed{E = -\frac{GmM}{2a}}$$

Hence

$$-\frac{GmM}{2a} = \tfrac{1}{2}\frac{mM}{m+M}|\dot{\mathbf{r}}|^2 - \frac{GMm}{r}$$

$$\tfrac{1}{2}\frac{mM}{m+M}|\dot{\mathbf{r}}|^2 = GMm\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$$|\dot{\mathbf{r}}|^2 = 2G\left(m+M\right)\left(\frac{2a-r}{2ar}\right)$$

$$\boxed{|\dot{\mathbf{r}}| = \sqrt{\frac{G\left(m+M\right)\left(2a-r\right)}{ar}}}$$