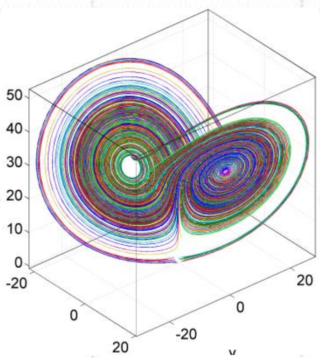
A computational cookbook of

CHAOS



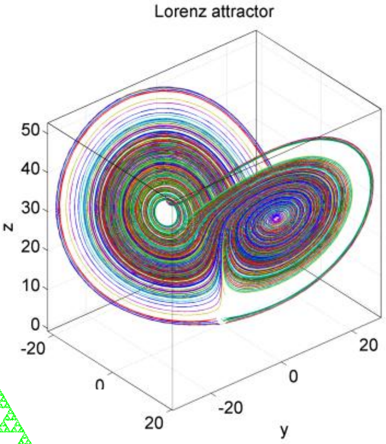
Dr Andrew French

Winchester College Studium

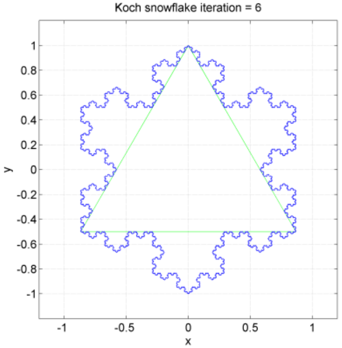
October 2017

1. **What is chaos?**
2. A short but chaotic history

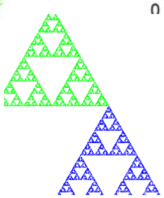
3. The **logistic map** and population modelling
4. Pendulums and **phase space**
5. Lorenz and Rössler **strange attractors**



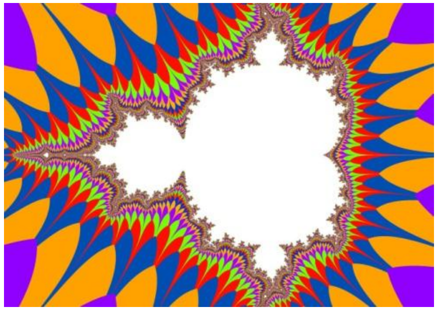
6. Shaw’s dripping faucet



7. **Fractals**
   - Koch snowflake
   - **Fractal dimension**
   - Barnsley fern and Sierpinski triangle

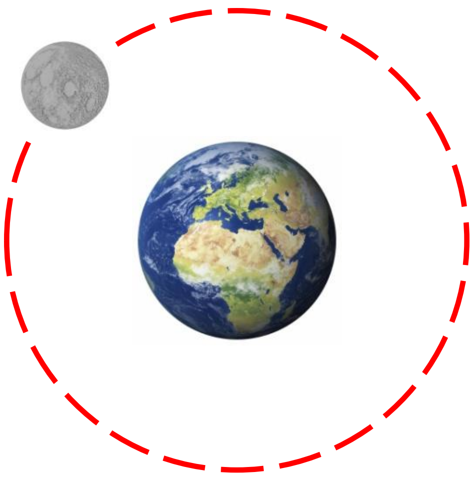


8. **Mandlebrot**, complex numbers and **iteration**
9. Chaos in planetary systems



10. Chaos in fluid flow
11. **Phase locking** & order from chaos
12. Further reading

What is Chaos?



Dynamics, the *physics of motion*, provides us with *equations* which can be used to **predict the future position of objects** if we know (i) their present **position** and **velocity** and (ii) the **forces** which act on each object.

This works *very well* for planetary motion, tides etc. *Not so well* for weather or indeed the position of pool balls....

This is because ***most* systems cannot be solved exactly**. An *approximate numerical method* is required to work out what happens next.
Many systems, even really simple ones, are ***highly sensitive to initial conditions.***

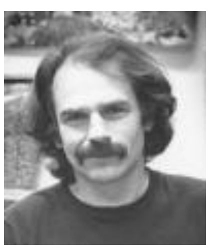
This means future behaviour becomes increasingly difficult to predict

Nonlinearity is often the problem!

What is Chaos?

i.e. where we *know the laws of motion*

"Simple *deterministic* systems with only a few elements can generate **random behaviour**. The randomness is fundamental; gathering more information does not make it go away. Randomness generated in this way has come to be called chaos."

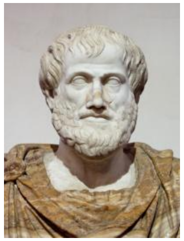


Robert Shaw of the "Santa Cruz Chaos Cabal" 1970s-1980s

Key references for this lecture are: Shaw *et al*; "Chaos", Scientific American 54:12 (1986) 46-57 and Gleick. J., *Chaos*. Vintage 1998.

Edward Norton
Lorenz
1917-2008

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?



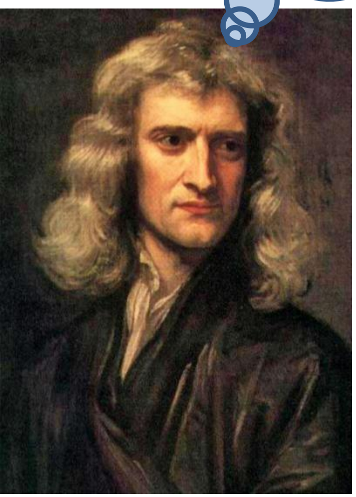
Hmmm. What does this "***Butterfly Effect"*** mean for Causality?

"... it is found that non-periodic solutions are ordinarily **unstable with respect to small modifications**, so that slightly differing initial states can evolve into considerably different states"

i.e. if I change pressure by even a tiny amount in a weather model, the effect may be profound after a relatively short time

A short but chaotic history

I give you:
Laws of motion
Calculus
Gravitation ...

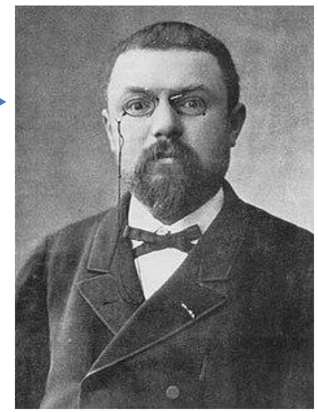


Isaac Newton
1642-1727

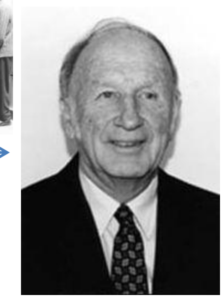
Pierre Simon Laplace
1749-1827

If we know the position and momentum of all particles in the Universe we could know the **past** *and* the **future!**

"Sensitive dependence on initial conditions"



Henri Poincaré
1854-1912



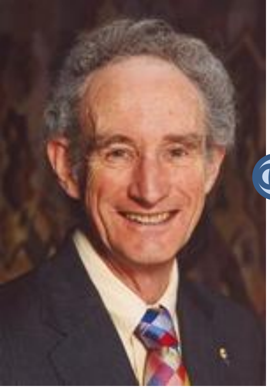
Edward Norton
Lorenz
1917-2008

Michel Hénon
1931-2013

Planetary dynamics can often be ***chaotic***

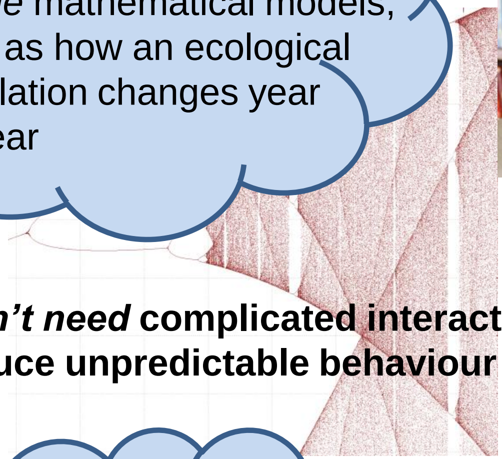
The *Uncertainty Principle* indeed sets a limit on what we can know for certain

But we can only know the initial situation *approximately*. And small errors can often **amplify** with interactions between many particles



Robert May
1936-

Chaos can be seen in *very simple* mathematical models, such as how an ecological population changes year on year



Mitchell Feigenbaum
1944-

I discovered *universal* mathematical truths about these systems

4.669201609...

You *don't need* complicated interactions to produce unpredictable behaviour

Gaston Julia
1893-1978

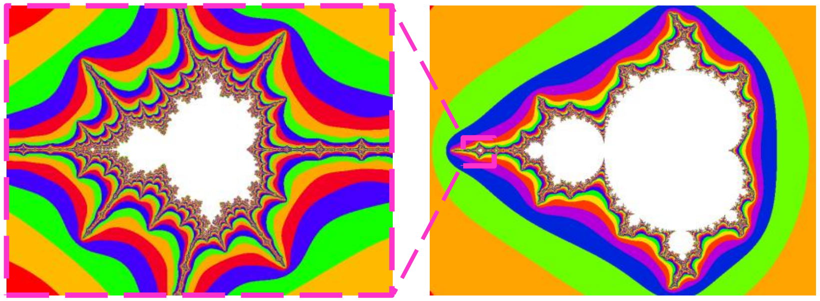
Very intricate geometry is hidden within the simplest of **quadratic equations** (if we use **complex numbers** and **iteration**)

$$z_{n+1} = z_n^2 + z_0$$

Doc Brown = Mitchell Feigenbaum?

Benoit Mandelbrot
1924-2010

Much of geometry in the natural world is **self similar on all scales**. We can use **fractional dimensions** to describe these ***fractal*** objects.

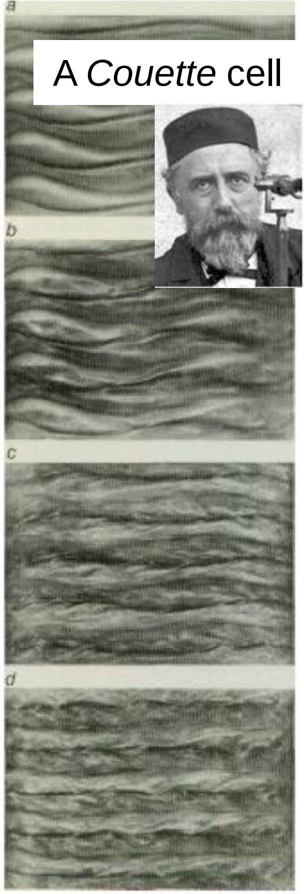


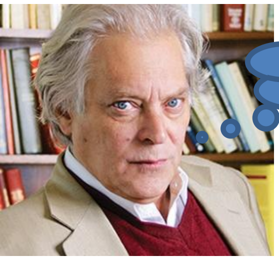
https://en.wikipedia.org/wiki/Chaos_theory

Jerry Gollub (1944-) & Harry Swinney (1939-)

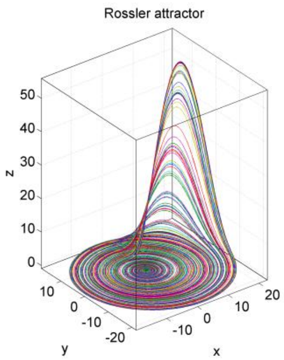
We measured a transition to chaos (turbulence) in fluid trapped between two rotating cylinders

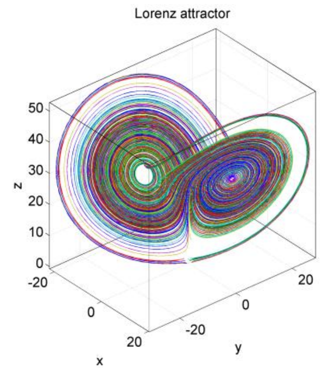
A *Couette* cell



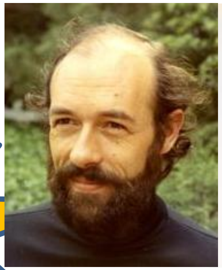


Universality!





David Ruelle (1935-)
& Floris Takens
(1940-2010)



How to investigate **nonlinear dynamics**?

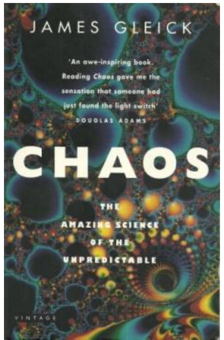
We propose a **strange attractor**

Otto Rössler
1940-

Edward Lorenz
1917-2008

James Gleick
1954-

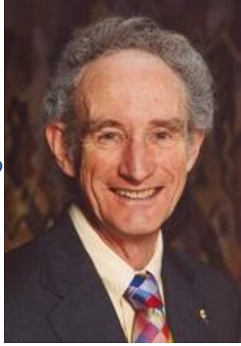
Robert Shaw James Crutchfield J. Doyne Farmer, Norman Packard
“Santa Cruz Chaos Cabal”
1970s-1980s



I wrote this all up in a book

The logistic map and population modelling

I published this model in 1976



Robert May
1936-

Assume an ecosystem can support a maximum number of rabbits. Let x be the fraction of this maximum at year n.

To account for **reproduction**, next year's population is proportional to the previous.

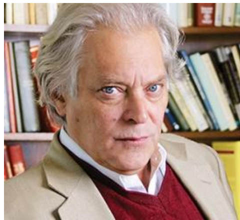
To account for **starvation**, next year's population is *also proportional* to the fraction of the maximum population as yet unfilled.

$$x_{n+1} = r x_n (1 - x_n)$$

Growth parameter

The population next year is predicted using this **iterative equation** called a **logistic map**

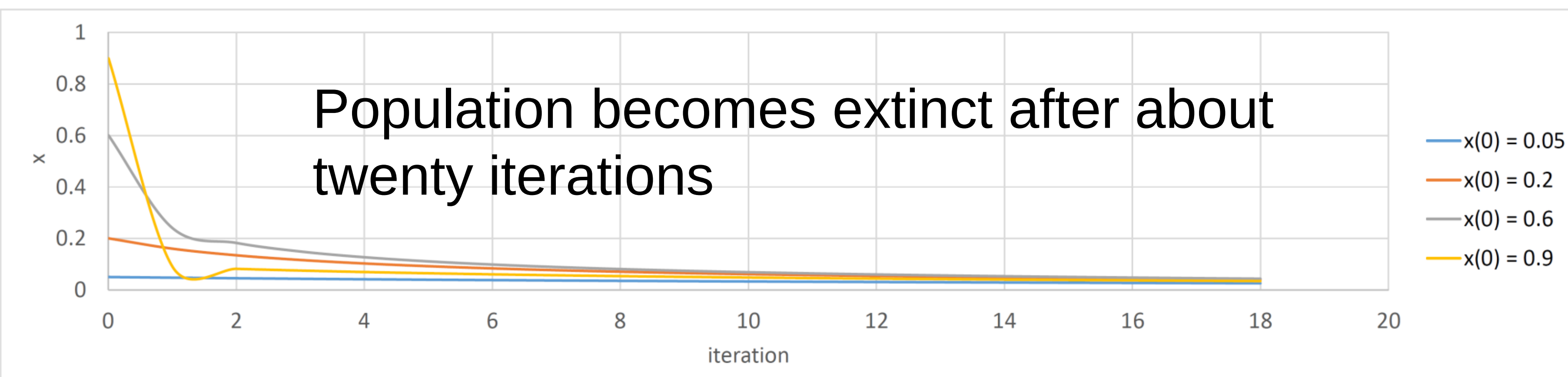
The pattern of x values with n is not always simple .....



$$r = 1 \qquad x_{n+1} = r x_n (1 - x_n)$$

iteration number n

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x(n)	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0.05	0.0475	0.045244	0.043197	0.041331	0.039623	0.038053	0.036605	0.035265	0.034021	0.032864	0.031784	0.030773	0.029826	0.028937	0.028099	0.02731	0.026564	0.025858
	0.1	0.09	0.0819	0.075192	0.069538	0.064703	0.060516	0.056854	0.053622	0.050746	0.048171	0.045851	0.043749	0.041835	0.040084	0.038478	0.036997	0.035628	0.034359
	0.15	0.1275	0.111244	0.098869	0.089094	0.081156	0.07457	0.069009	0.064247	0.060119	0.056505	0.053312	0.05047	0.047923	0.045626	0.043544	0.041648	0.039914	0.038321
	0.2	0.16	0.1344	0.116337	0.102802	0.092234	0.083727	0.076717	0.070831	0.065814	0.061483	0.057703	0.054373	0.051417	0.048773	0.046394	0.044242	0.042284	0.040496
	0.25	0.1875	0.152344	0.129135	0.112459	0.099812	0.08985	0.081777	0.075089	0.069451	0.064627	0.060451	0.056796	0.053571	0.050701	0.04813	0.045814	0.043715	0.041804
	0.3	0.21	0.1659	0.138377	0.119229	0.105013	0.093986	0.085152	0.077901	0.071833	0.066673	0.062228	0.058355	0.05495	0.05193	0.049234	0.04681	0.044619	0.042628
	0.35	0.2275	0.175744	0.144858	0.123874	0.108529	0.096751	0.08739	0.079753	0.073392	0.068006	0.063381	0.059364	0.05584	0.052722	0.049942	0.047448	0.045197	0.043154
	0.4	0.24	0.1824	0.14913	0.12689	0.110789	0.098515	0.08881	0.080923	0.074374	0.068843	0.064103	0.059994	0.056395	0.053214	0.050383	0.047844	0.045555	0.04348
	0.45	0.2475	0.186244	0.151557	0.128587	0.112053	0.099497	0.089597	0.08157	0.074916	0.069304	0.064501	0.06034	0.056699	0.053485	0.050624	0.048061	0.045751	0.043658
	0.5	0.25	0.1875	0.152344	0.129135	0.112459	0.099812	0.08985	0.081777	0.075089	0.069451	0.064627	0.060451	0.056796	0.053571	0.050701	0.04813	0.045814	0.043715
	0.55	0.2475	0.186244	0.151557	0.128587	0.112053	0.099497	0.089597	0.08157	0.074916	0.069304	0.064501	0.06034	0.056699	0.053485	0.050624	0.048061	0.045751	0.043658
	0.6	0.24	0.1824	0.14913	0.12689	0.110789	0.098515	0.08881	0.080923	0.074374	0.068843	0.064103	0.059994	0.056395	0.053214	0.050383	0.047844	0.045555	0.04348
	0.65	0.2275	0.175744	0.144858	0.123874	0.108529	0.096751	0.08739	0.079753	0.073392	0.068006	0.063381	0.059364	0.05584	0.052722	0.049942	0.047448	0.045197	0.043154
	0.7	0.21	0.1659	0.138377	0.119229	0.105013	0.093986	0.085152	0.077901	0.071833	0.066673	0.062228	0.058355	0.05495	0.05193	0.049234	0.04681	0.044619	0.042628
	0.75	0.1875	0.152344	0.129135	0.112459	0.099812	0.08985	0.081777	0.075089	0.069451	0.064627	0.060451	0.056796	0.053571	0.050701	0.04813	0.045814	0.043715	0.041804
	0.8	0.16	0.1344	0.116337	0.102802	0.092234	0.083727	0.076717	0.070831	0.065814	0.061483	0.057703	0.054373	0.051417	0.048773	0.046394	0.044242	0.042284	0.040496
	0.85	0.1275	0.111244	0.098869	0.089094	0.081156	0.07457	0.069009	0.064247	0.060119	0.056505	0.053312	0.05047	0.047923	0.045626	0.043544	0.041648	0.039914	0.038321
	0.9	0.09	0.0819	0.075192	0.069538	0.064703	0.060516	0.056854	0.053622	0.050746	0.048171	0.045851	0.043749	0.041835	0.040084	0.038478	0.036997	0.035628	0.034359
	0.95	0.0475	0.045244	0.043197	0.041331	0.039623	0.038053	0.036605	0.035265	0.034021	0.032864	0.031784	0.030773	0.029826	0.028937	0.028099	0.02731	0.026564	0.025858
	1	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16	-2.2E-16

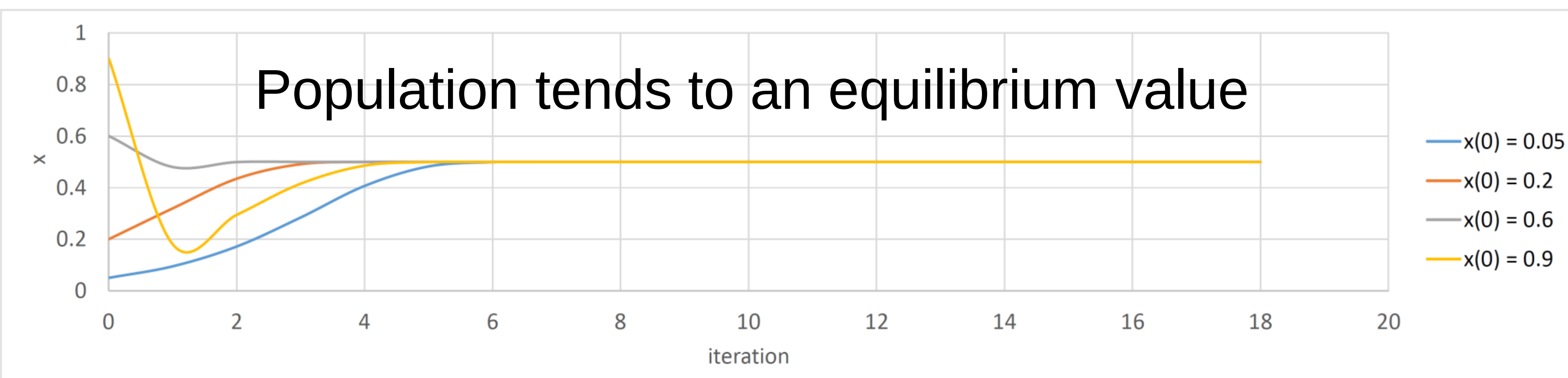


$$r = 2 \qquad x_{n+1} = r x_n \left(1 - x_n\right)$$

iteration number n

x(n)

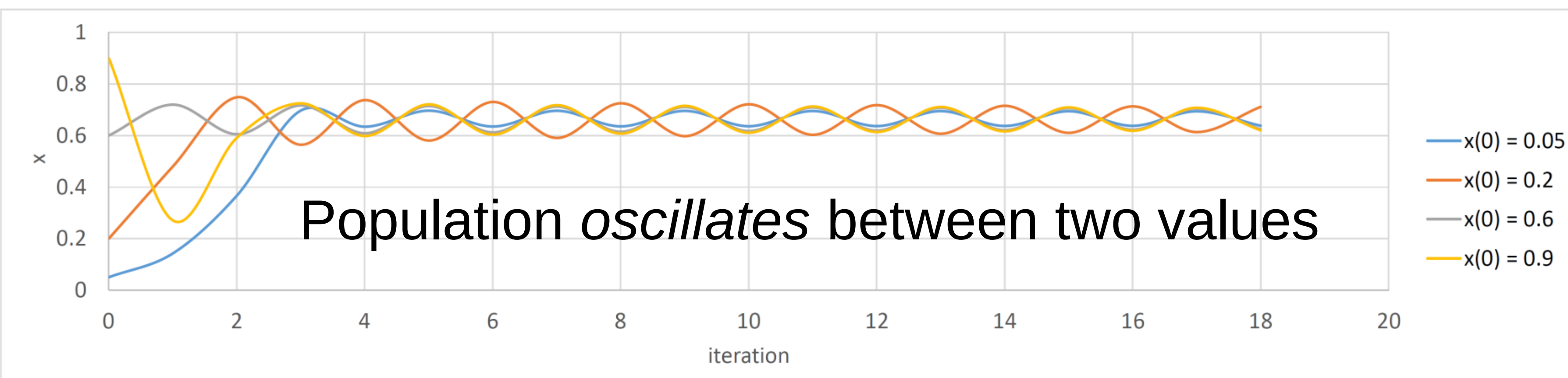
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.05	0.095	0.17195	0.284766	0.407349	0.482832	0.49941	0.499999	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.1	0.18	0.2952	0.416114	0.485926	0.499604	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.15	0.255	0.37995	0.471176	0.498338	0.499994	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.2	0.32	0.4352	0.491602	0.499859	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.25	0.375	0.46875	0.498047	0.499992	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.3	0.42	0.4872	0.499672	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.35	0.455	0.49595	0.499967	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.4	0.48	0.4992	0.499999	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.45	0.495	0.49995	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.55	0.495	0.49995	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.48	0.4992	0.499999	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.65	0.455	0.49595	0.499967	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.7	0.42	0.4872	0.499672	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.75	0.375	0.46875	0.498047	0.499992	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.8	0.32	0.4352	0.491602	0.499859	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.85	0.255	0.37995	0.471176	0.498338	0.499994	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.9	0.18	0.2952	0.416114	0.485926	0.499604	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.95	0.095	0.17195	0.284766	0.407349	0.482832	0.49941	0.499999	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
1	-4.4E-16	-8.9E-16	-1.8E-15	-3.6E-15	-7.1E-15	-1.4E-14	-2.8E-14	-5.7E-14	-1.1E-13	-2.3E-13	-4.5E-13	-9.1E-13	-1.8E-12	-3.6E-12	-7.3E-12	-1.5E-11	-2.9E-11	-5.8E-11



$$r = 3 \qquad x_{n+1} = r x_n \left(1 - x_n\right)$$

iteration number n

x(n)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0.05	0.1425	0.366581	0.696598	0.634047	0.696094	0.634641	0.695615	0.635204	0.695159	0.635738	0.694725	0.636246	0.694311	0.63673	0.693915	0.637191	0.693536	0.637632
	0.1	0.27	0.5913	0.724993	0.598135	0.721109	0.603333	0.717967	0.607471	0.71535	0.610873	0.713121	0.613738	0.711191	0.616195	0.709496	0.618334	0.707991	0.620219
	0.15	0.3825	0.708581	0.619482	0.707172	0.621239	0.705904	0.622811	0.704752	0.62423	0.703701	0.625518	0.702736	0.626694	0.701846	0.627775	0.701021	0.628772	0.700253
	0.2	0.48	0.7488	0.564296	0.737598	0.580641	0.730491	0.590622	0.725363	0.597634	0.721403	0.602943	0.718208	0.607155	0.715553	0.61061	0.713296	0.613514	0.711343
	0.25	0.5625	0.738281	0.579666	0.73096	0.589973	0.725715	0.597158	0.721681	0.602573	0.718436	0.606857	0.715745	0.610362	0.71346	0.613304	0.711487	0.61582	0.709757
	0.3	0.63	0.6993	0.630839	0.698644	0.631622	0.698027	0.632356	0.697446	0.633046	0.696897	0.633695	0.696377	0.634308	0.695884	0.634889	0.695415	0.635439	0.694969
	0.35	0.6825	0.650081	0.682427	0.650161	0.682355	0.65024	0.682284	0.650318	0.682213	0.650395	0.682144	0.65047	0.682076	0.650545	0.682009	0.650619	0.681942	0.650691
	0.4	0.72	0.6048	0.717051	0.608667	0.714575	0.611873	0.712453	0.614591	0.710607	0.616934	0.708979	0.618983	0.707529	0.620795	0.706226	0.622413	0.705045	0.62387
	0.45	0.7425	0.573581	0.733757	0.586072	0.727775	0.594356	0.723291	0.600424	0.719745	0.605136	0.716839	0.608942	0.714395	0.612105	0.712298	0.614789	0.71047	0.617107
	0.5	0.75	0.5625	0.738281	0.579666	0.73096	0.589973	0.725715	0.597158	0.721681	0.602573	0.718436	0.606857	0.715745	0.610362	0.71346	0.613304	0.711487	0.61582
	0.55	0.7425	0.573581	0.733757	0.586072	0.727775	0.594356	0.723291	0.600424	0.719745	0.605136	0.716839	0.608942	0.714395	0.612105	0.712298	0.614789	0.71047	0.617107
	0.6	0.72	0.6048	0.717051	0.608667	0.714575	0.611873	0.712453	0.614591	0.710607	0.616934	0.708979	0.618983	0.707529	0.620795	0.706226	0.622413	0.705045	0.62387
	0.65	0.6825	0.650081	0.682427	0.650161	0.682355	0.65024	0.682284	0.650318	0.682213	0.650395	0.682144	0.65047	0.682076	0.650545	0.682009	0.650619	0.681942	0.650691
	0.7	0.63	0.6993	0.630839	0.698644	0.631622	0.698027	0.632356	0.697446	0.633046	0.696897	0.633695	0.696377	0.634308	0.695884	0.634889	0.695415	0.635439	0.694969
	0.75	0.5625	0.738281	0.579666	0.73096	0.589973	0.725715	0.597158	0.721681	0.602573	0.718436	0.606857	0.715745	0.610362	0.71346	0.613304	0.711487	0.61582	0.709757
	0.8	0.48	0.7488	0.564296	0.737598	0.580641	0.730491	0.590622	0.725363	0.597634	0.721403	0.602943	0.718208	0.607155	0.715553	0.61061	0.713296	0.613514	0.711343
	0.85	0.3825	0.708581	0.619482	0.707172	0.621239	0.705904	0.622811	0.704752	0.62423	0.703701	0.625518	0.702736	0.626694	0.701846	0.627775	0.701021	0.628772	0.700253
	0.9	0.27	0.5913	0.724993	0.598135	0.721109	0.603333	0.717967	0.607471	0.71535	0.610873	0.713121	0.613738	0.711191	0.616195	0.709496	0.618334	0.707991	0.620219
	0.95	0.1425	0.366581	0.696598	0.634047	0.696094	0.634641	0.695615	0.635204	0.695159	0.635738	0.694725	0.636246	0.694311	0.63673	0.693915	0.637191	0.693536	0.637632
	1	-6.7E-16	-2E-15	-6E-15	-1.8E-14	-5.4E-14	-1.6E-13	-4.9E-13	-1.5E-12	-4.4E-12	-1.3E-11	-3.9E-11	-1.2E-10	-3.5E-10	-1.1E-09	-3.2E-09	-9.6E-09	-2.9E-08	-8.6E-08

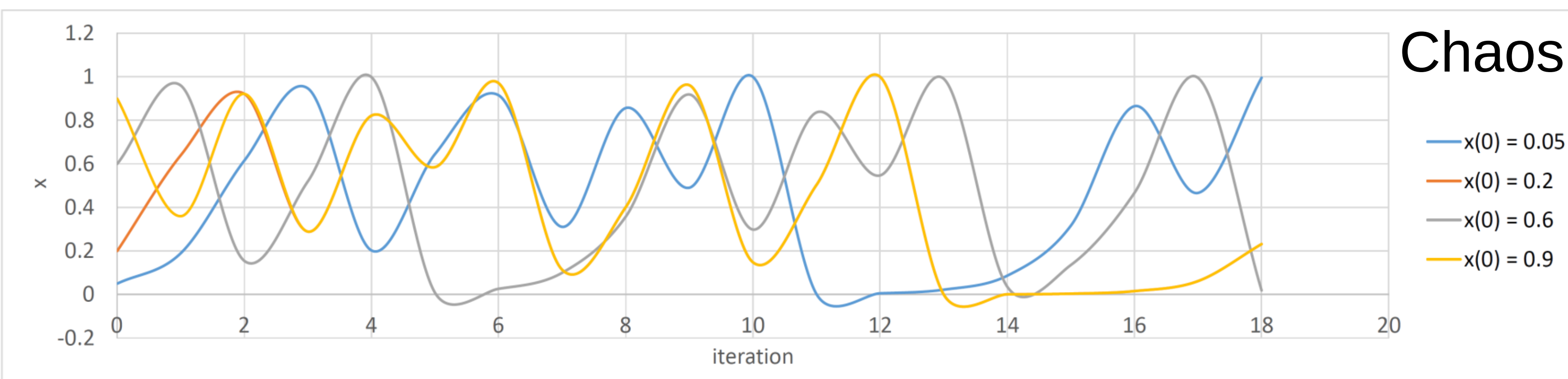


$$r = 4 \qquad x_{n+1} = r x_n (1 - x_n)$$

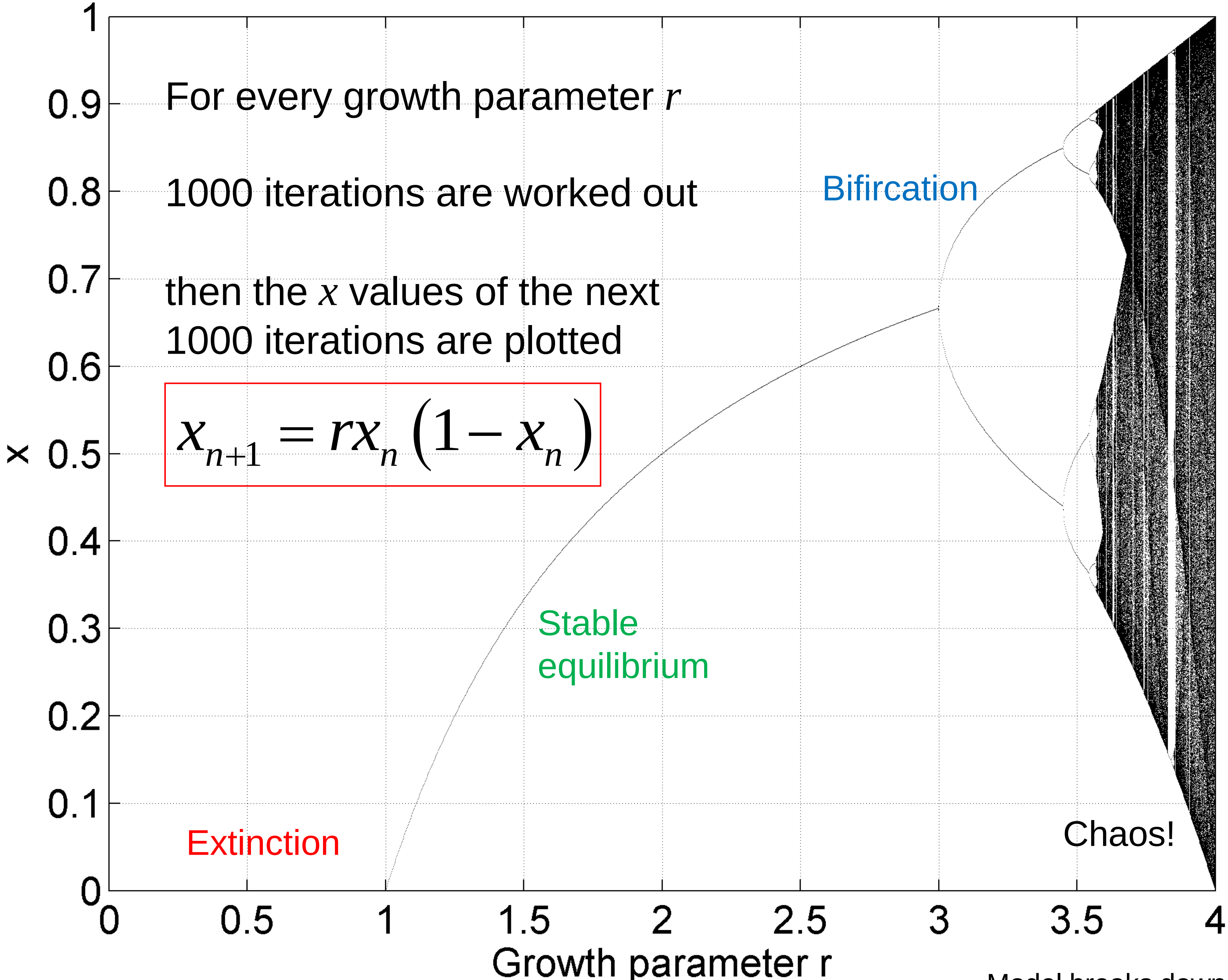
iteration number n

x(n)

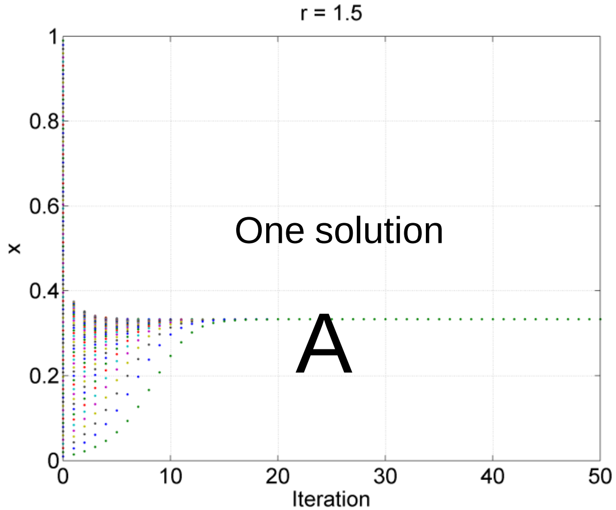
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.05	0.19	0.6156	0.946547	0.202385	0.6457	0.915085	0.310816	0.856838	0.490667	0.999652	0.001393	0.005565	0.022137	0.086589	0.316366	0.865114	0.466766	0.995582
0.1	0.36	0.9216	0.289014	0.821939	0.585421	0.970813	0.113339	0.401974	0.961563	0.147837	0.503924	0.999938	0.000246	0.000985	0.003936	0.015682	0.061745	0.23173
0.15	0.51	0.9996	0.001599	0.006387	0.025386	0.098965	0.356683	0.917841	0.301635	0.842605	0.530488	0.996282	0.014817	0.058389	0.219918	0.686217	0.861293	0.47787
0.2	0.64	0.9216	0.289014	0.821939	0.585421	0.970813	0.113339	0.401974	0.961563	0.147837	0.503924	0.999938	0.000246	0.000985	0.003936	0.015682	0.061745	0.23173
0.25	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
0.3	0.84	0.5376	0.994345	0.022492	0.087945	0.320844	0.871612	0.447617	0.989024	0.043422	0.166146	0.554165	0.988265	0.046391	0.176954	0.582565	0.972732	0.106097
0.35	0.91	0.3276	0.881113	0.419012	0.973764	0.102192	0.366996	0.92924	0.263011	0.775345	0.69674	0.845174	0.523421	0.997806	0.008757	0.034722	0.134065	0.464367
0.4	0.96	0.1536	0.520028	0.998395	0.006408	0.025467	0.099273	0.35767	0.918969	0.29786	0.836557	0.546917	0.991195	0.034909	0.134761	0.466403	0.995485	0.017978
0.45	0.99	0.0396	0.152127	0.515939	0.998984	0.00406	0.016176	0.063657	0.238418	0.7263	0.795154	0.651537	0.908147	0.333665	0.889331	0.393686	0.954789	0.172666
0.5	1	4.44E-16	1.78E-15	7.11E-15	2.84E-14	1.14E-13	4.55E-13	1.82E-12	7.28E-12	2.91E-11	1.16E-10	4.66E-10	1.86E-09	7.45E-09	2.98E-08	1.19E-07	4.77E-07	1.91E-06
0.55	0.99	0.0396	0.152127	0.515939	0.998984	0.00406	0.016176	0.063657	0.238418	0.7263	0.795154	0.651537	0.908147	0.333665	0.889331	0.393686	0.954789	0.172666
0.6	0.96	0.1536	0.520028	0.998395	0.006408	0.025467	0.099273	0.35767	0.918969	0.29786	0.836557	0.546917	0.991195	0.034909	0.134761	0.466403	0.995485	0.017978
0.65	0.91	0.3276	0.881113	0.419012	0.973764	0.102192	0.366996	0.92924	0.263011	0.775345	0.69674	0.845174	0.523421	0.997806	0.008757	0.034722	0.134065	0.464367
0.7	0.84	0.5376	0.994345	0.022492	0.087945	0.320844	0.871612	0.447617	0.989024	0.043422	0.166146	0.554165	0.988265	0.046391	0.176954	0.582565	0.972732	0.106097
0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
0.8	0.64	0.9216	0.289014	0.821939	0.585421	0.970813	0.113339	0.401974	0.961563	0.147837	0.503924	0.999938	0.000246	0.000985	0.003936	0.015682	0.061745	0.23173
0.85	0.51	0.9996	0.001599	0.006387	0.025386	0.098965	0.356683	0.917841	0.301635	0.842605	0.530488	0.996282	0.014817	0.058389	0.219918	0.686217	0.861293	0.47787
0.9	0.36	0.9216	0.289014	0.821939	0.585421	0.970813	0.113339	0.401974	0.961563	0.147837	0.503924	0.999938	0.000246	0.000985	0.003936	0.015682	0.061745	0.23173
0.95	0.19	0.6156	0.946547	0.202385	0.6457	0.915085	0.310816	0.856838	0.490667	0.999652	0.001393	0.005565	0.022137	0.086589	0.316366	0.865114	0.466766	0.995582
1	-8.9E-16	-3.6E-15	-1.4E-14	-5.7E-14	-2.3E-13	-9.1E-13	-3.6E-12	-1.5E-11	-5.8E-11	-2.3E-10	-9.3E-10	-3.7E-09	-1.5E-08	-6E-08	-2.4E-07	-9.5E-07	-3.8E-06	-1.5E-05

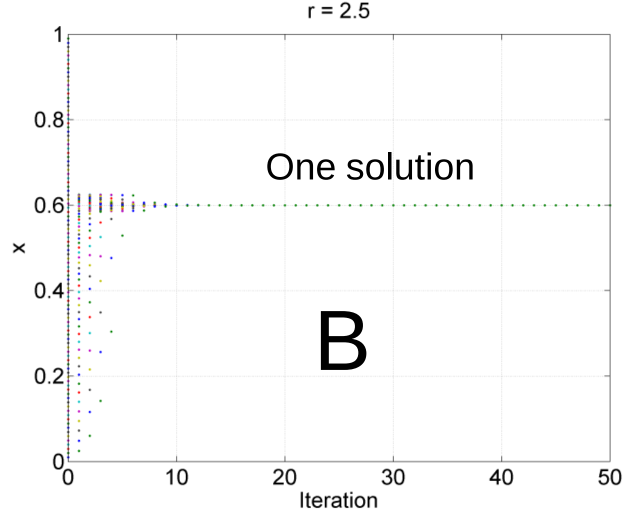


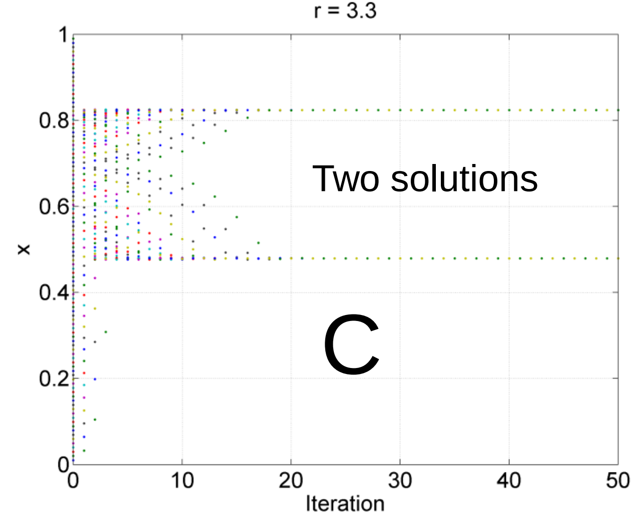
May Bifurcations Logistic map

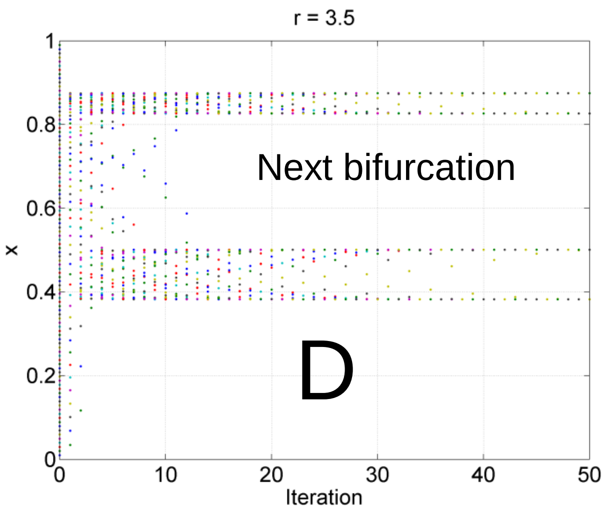


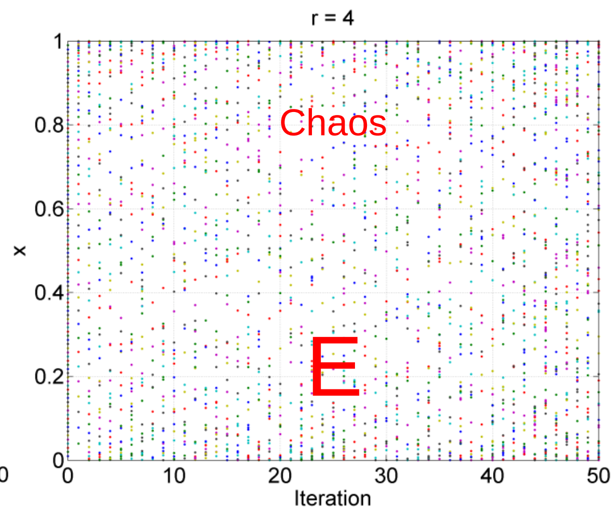
Model breaks down for $r > 4$



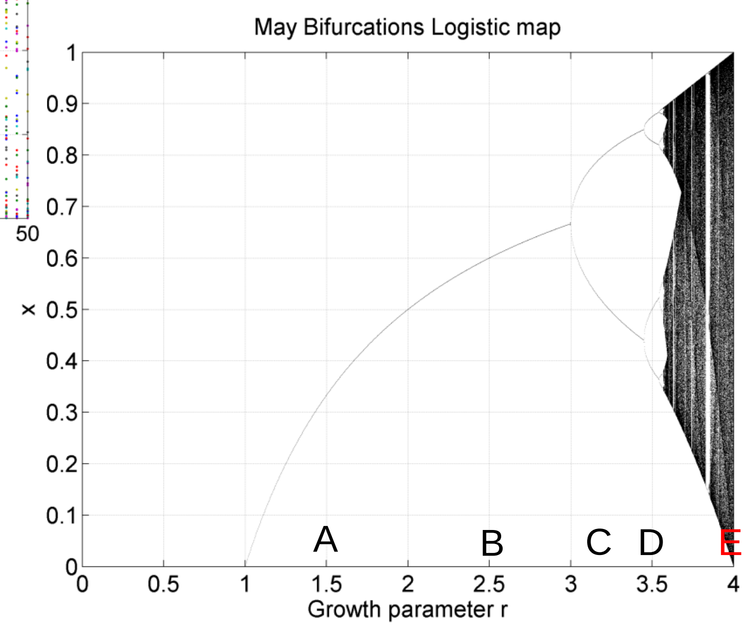






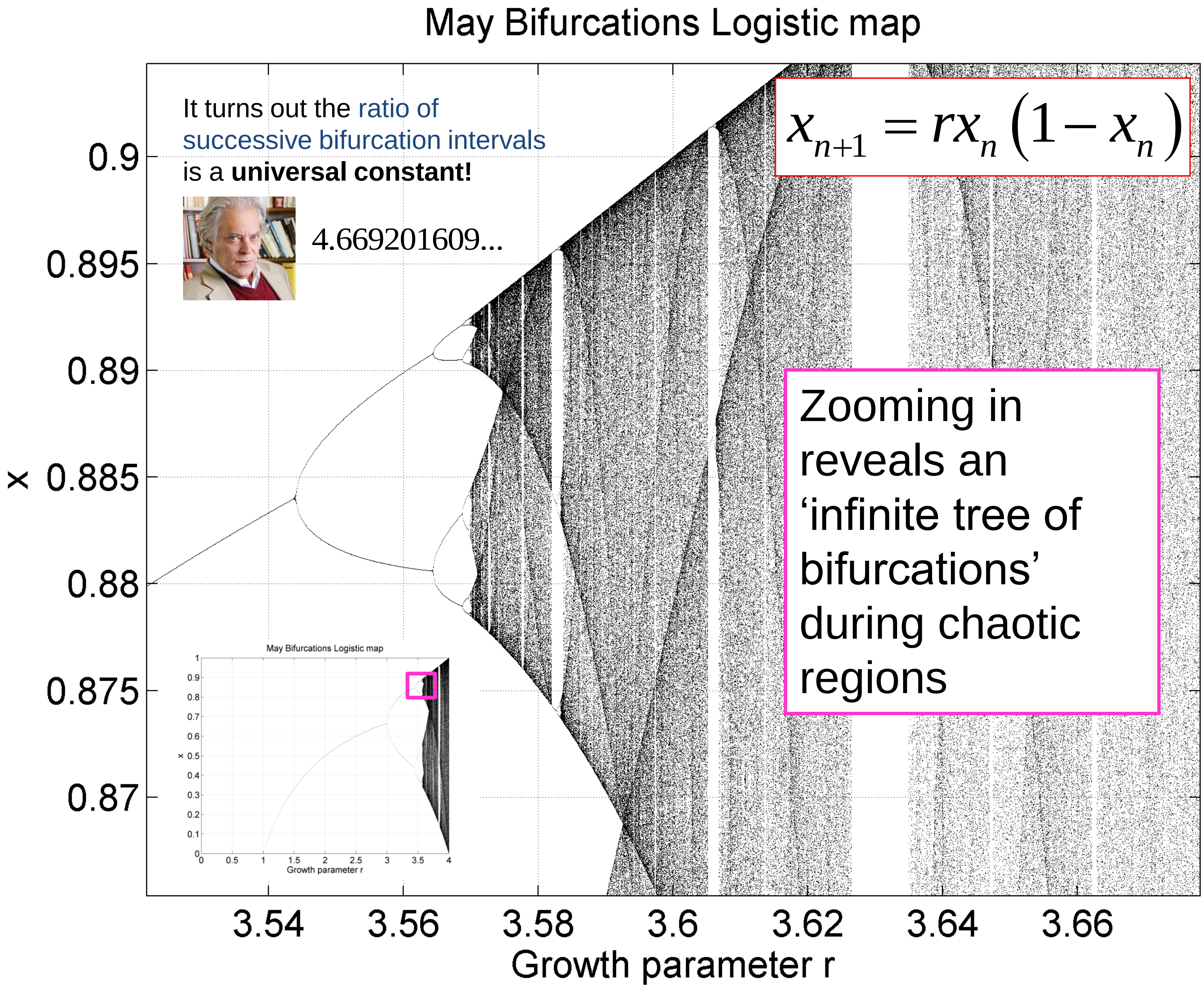


$$x_{n+1} = rx_n(1 - x_n)$$

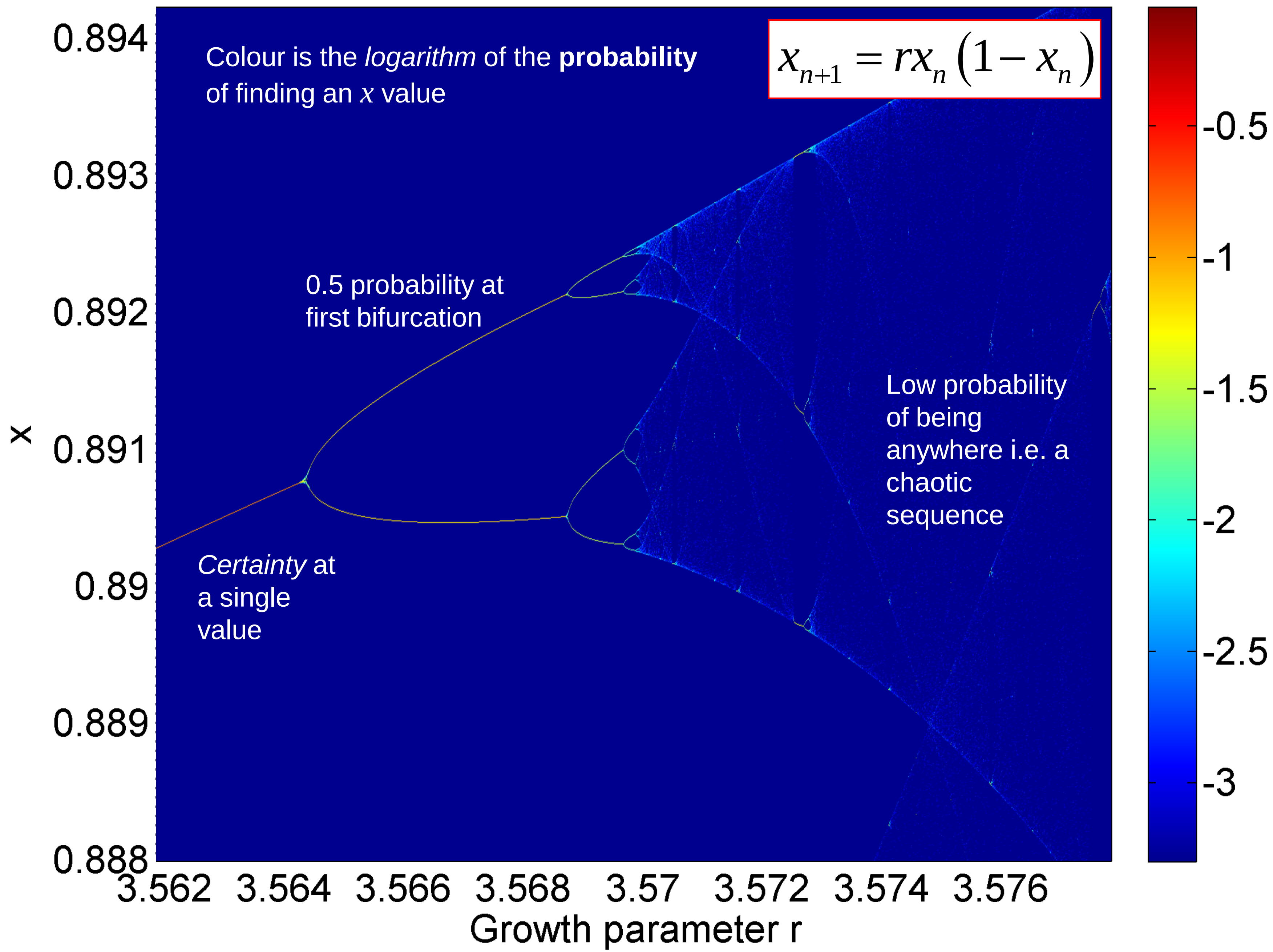


Tracking the bifurcations maps the 'road to chaos'. The ratio of successive bifurcation intervals
is a **universal constant!**

4.669201609...


May Bifurcations Logistic map
It turns out the ratio of successive bifurcation intervals is a **universal constant!**
4.669201609...
$x_{n+1} = rx_n(1 - x_n)$
Zooming in reveals an ‘infinite tree of bifurcations’ during chaotic regions
0.9
0.895
0.89
0.885
0.88
0.875
0.87
x
3.54
3.56
3.58
3.6
3.62
3.64
3.66
Growth parameter r
May Bifurcations Logistic map
Growth parameter r

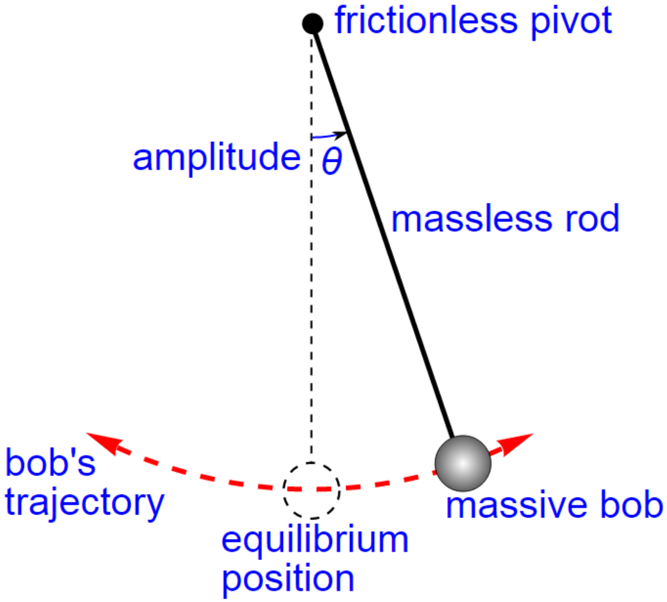
May Bifurcations Logistic map probability

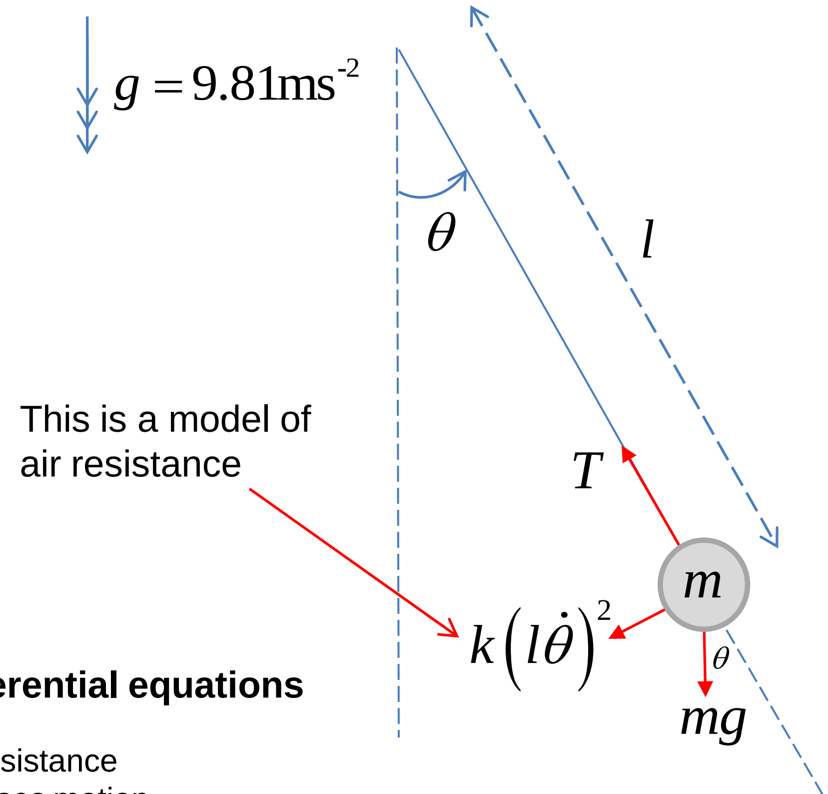


Pendulums and phase space

$$\dot{\theta} = \frac{d\theta}{dt}$$

Although we can't fully 'solve' a chaotic system, we can create a **diagram which describes the motion.** In **phase space**, patterns often emerge, which are hidden in the randomness of a **time series**.





We can use **Newton's Second Law** to write down **differential equations** for the motion of the pendulum bob

So that air resistance always *opposes* motion

$$ml\frac{d\dot{\theta}}{dt} = -mg\sin\theta - kl^2\dot{\theta}\left|\dot{\theta}\right|$$

$$\therefore \frac{d\dot{\theta}}{dt} = -\frac{g}{l}\sin\theta - \frac{kl}{m}\dot{\theta}\left|\dot{\theta}\right|$$

If angles are small and we ignore air resistance:

$$\frac{d\dot{\theta}}{dt} \approx -\frac{g}{l}\theta$$

We *can* solve this!

$$\theta = \theta_0\cos\left(2\pi\frac{t}{P}\right) \qquad P = 2\pi\sqrt{\frac{l}{g}}$$

To keep things simple (!) let's use the **period** P of a frictionless, small angle ideal pendulum to **define a time scale.** We can then make our pendulum equation in terms of **dimensionless numbers.**

$$t \to P\tau \qquad \dot{\theta} \to \frac{\dot{\theta}}{P} \qquad \text{i.e. } \dot{\theta} = \frac{d\theta}{d\tau}$$

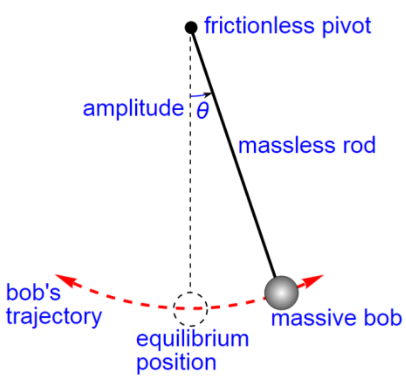
using this *dimensionless time scale* $P = 2\pi\sqrt{\frac{l}{g}}$

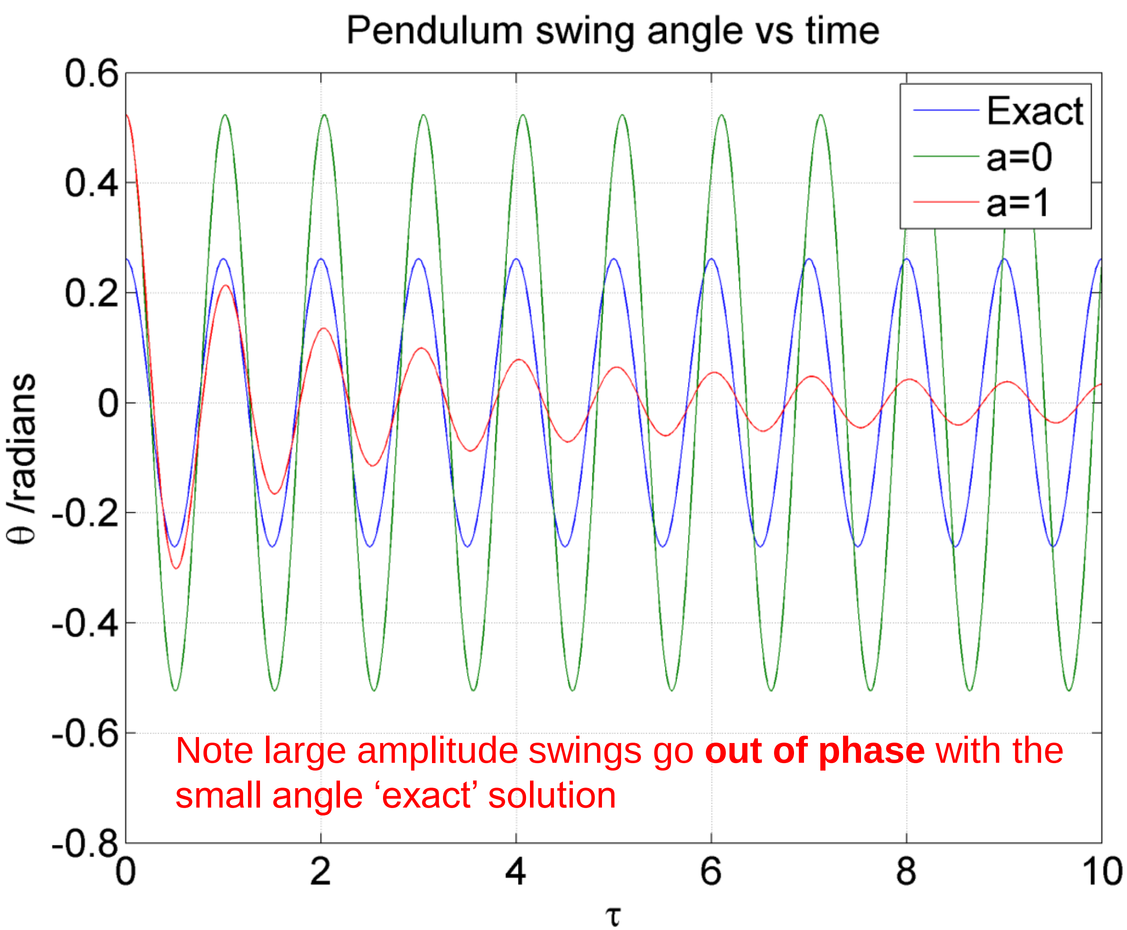
$$\frac{1}{P^2}\frac{d\dot{\theta}}{d\tau} = -\frac{g}{l}\sin\theta - \frac{1}{P^2}\frac{kl}{m}\dot{\theta}\left|\dot{\theta}\right|$$

$$\therefore \boxed{\frac{d\dot{\theta}}{d\tau} = 4\pi^2\sin\theta - a\dot{\theta}\left|\dot{\theta}\right|}$$

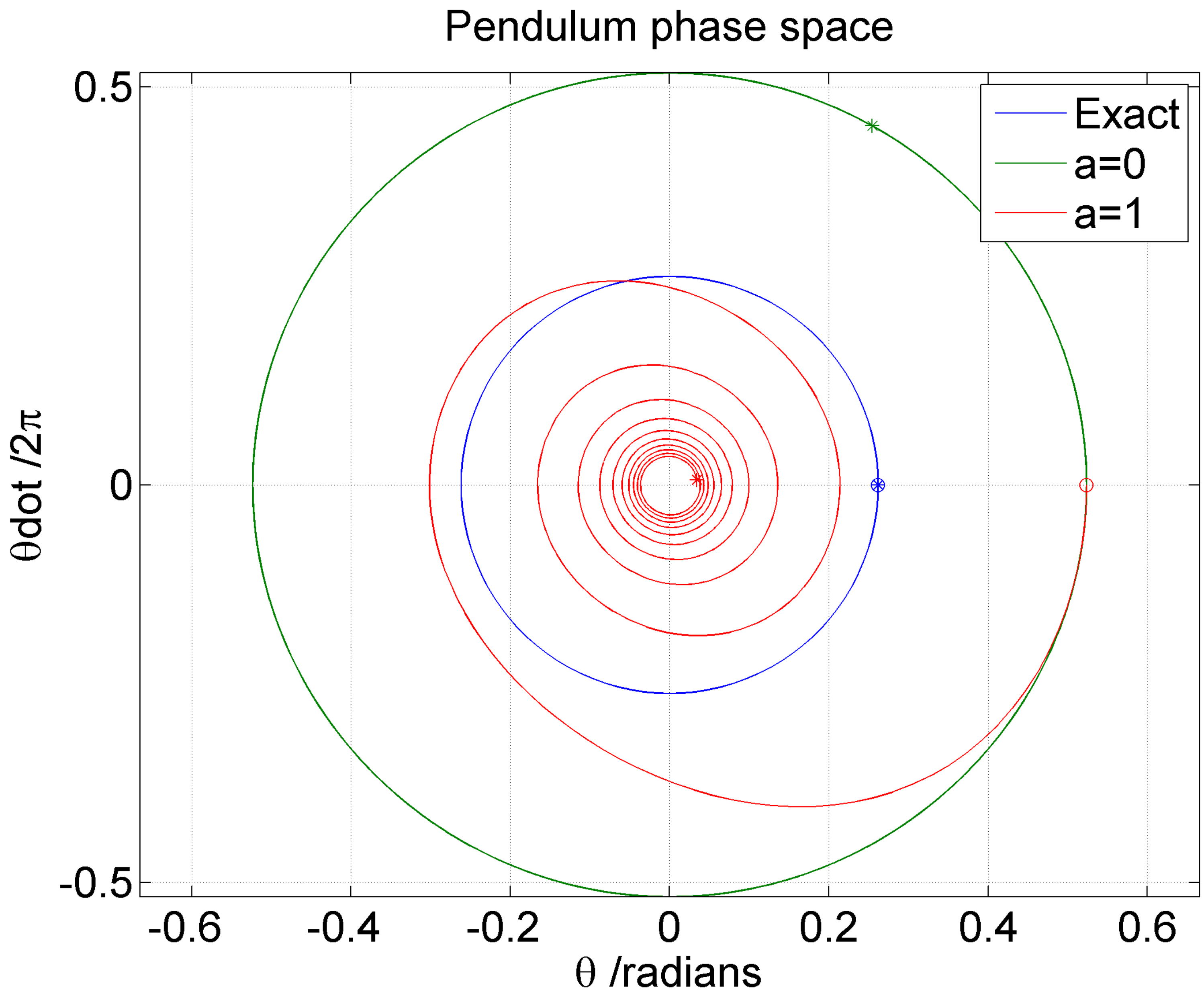
$$a = \frac{gk}{4\pi^2 m}$$

a is now simply a *number* which sets the effect of air resistance





Perhaps a more informative picture of the motion is the **phase portrait**, or **Poincaré diagram**



Henri Poincaré
1854-1912

Recall 'Exact' means small angles, and no air resistance

The frictionless oscillations are circles whereas air resistance causes an inspiralling to zero angle and zero angular speed

HARDCORE MATHS ALERT!!

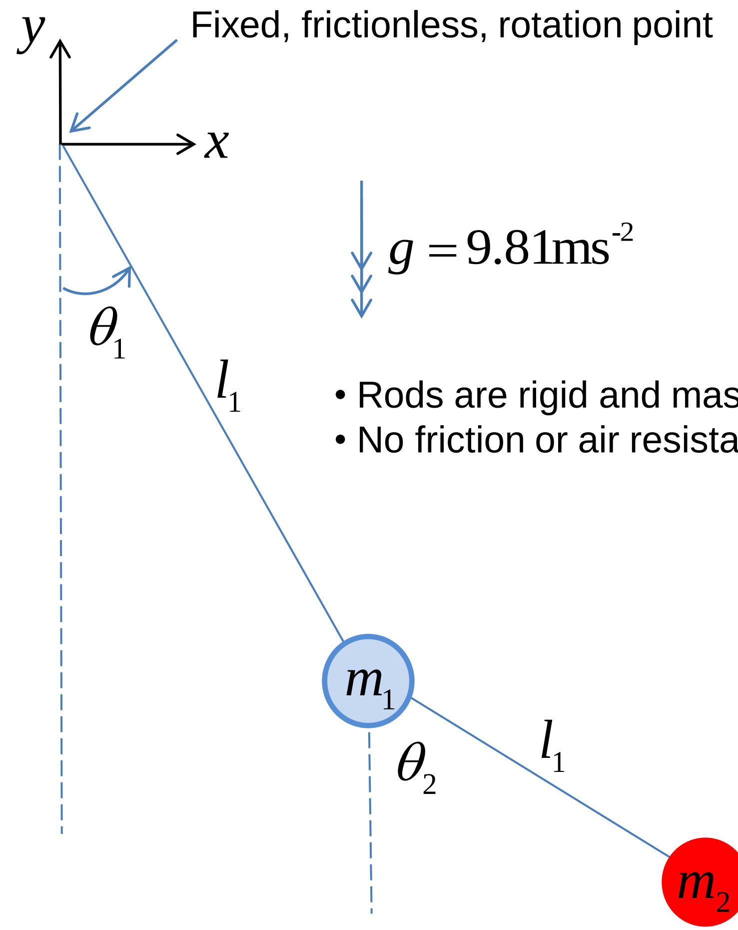
The double pendulum

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

x,y coordinates

$$v_{x1} = l_1 \cos\theta_1 \, \dot{\theta}_1$$
$$v_{y1} = l_1 \sin\theta_1 \dot{\theta}_1$$
$$v_{x2} = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$$
$$v_{y2} = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$$

Velocities



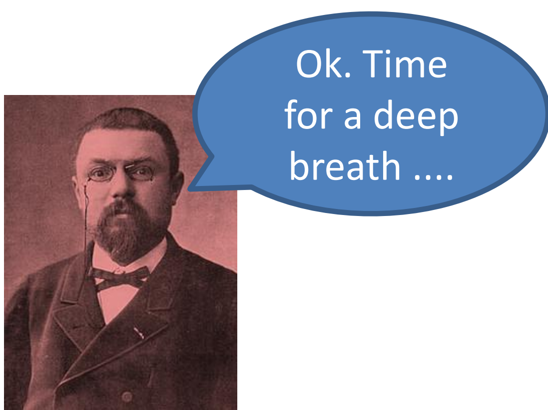
- Rods are rigid and massless
- No friction or air resistance

Potential energy

$$V = m_1 g y_1 + m_2 g y_2$$
$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

Kinetic energy

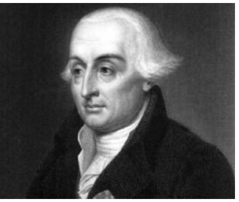
$$T = \tfrac{1}{2} m_1 \left(v_{x1}^2 + v_{y1}^2\right) + \tfrac{1}{2} m_2 \left(v_{x2}^2 + v_{y2}^2\right)$$
$$T = \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) \right]$$



We need to compute the **Lagrangian** L and then solve the **Euler-Lagrange equations**!

$$L = T - V$$

$$L = \tfrac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$



Joseph Louis Lagrange 1736-1813

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1}$$

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 = 0 \quad [1]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2}$$

$$m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 l_2 \ddot{\theta}_2 - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \quad [2]$$

$$\frac{d\theta_1}{dt} = \omega_1$$

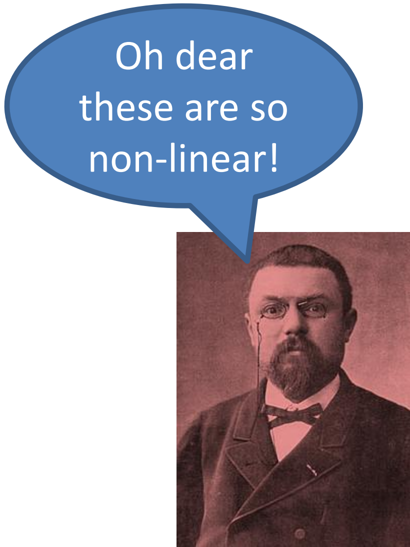
Four coupled non-linear differential equations. A mere bagatelle!

$$\frac{d\omega_1}{dt} = \frac{m_2 l_1 \omega_1^2 \sin\Delta \cos\Delta + m_2 g \sin\theta_2 \cos\Delta + m_2 l_2 \omega_2^2 \sin\Delta - (m_1 + m_2) g \sin\theta_1}{(m_1 + m_2) l_1 - m_2 l_1 \cos^2\Delta}$$

$$\frac{d\theta_2}{dt} = \omega_2$$

$$\frac{d\omega_2}{dt} = \frac{-m_2 l_2 \omega_2^2 \sin\Delta \cos\Delta + (m_1 + m_2)(g \sin\theta_1 \cos\Delta - l_1 \omega_1^2 \sin\Delta - g \sin\theta_2)}{(m_1 + m_2) l_2 - m_2 l_2 \cos^2\Delta}$$

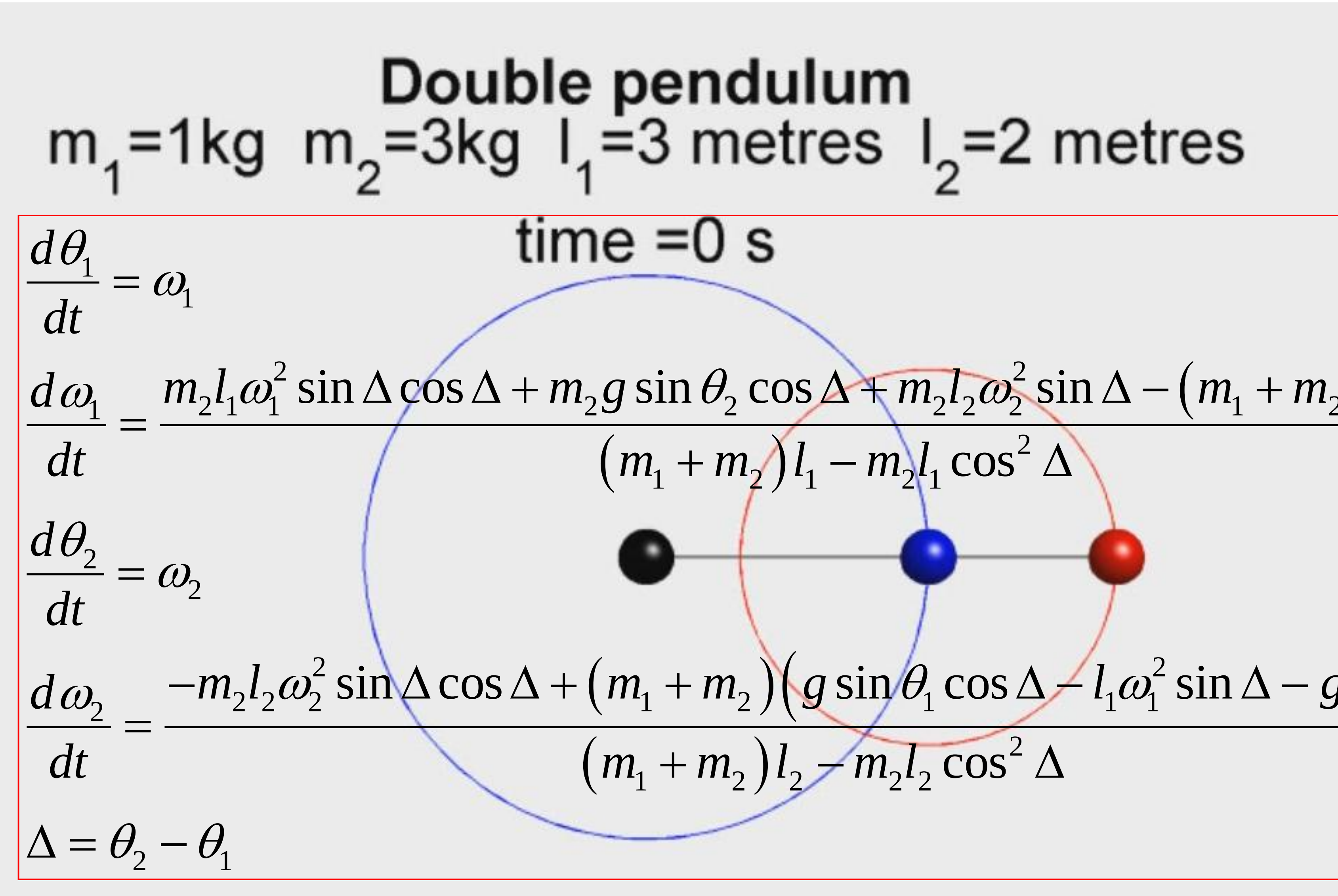
$$\Delta = \theta_2 - \theta_1$$



http://www.physics.usyd.edu.au/~wheat/dpend_html/ http://scienceworld.wolfram.com/physics/DoublePendulum.html

We can (approximately) solve the equations for the angles and angular velocities of the double pendulum using a *numeric method*. **Runge-Kutta** is a popular scheme. This has been implemented in MATLAB in order to generate the following plots.

But first a rather boring pendulum scenario to check my simulation makes sense....



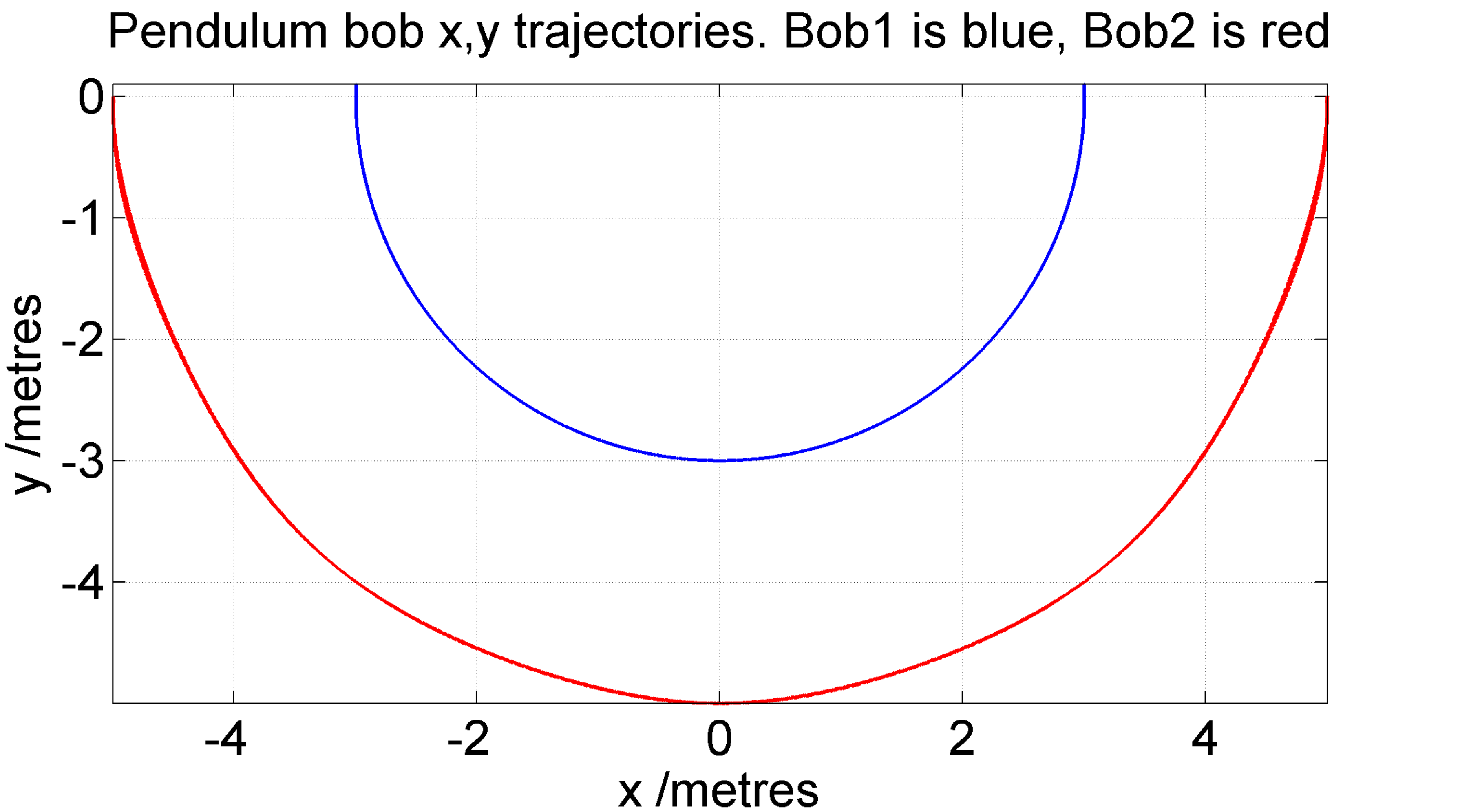
$$\frac{d\theta_1}{dt} = \omega_1$$

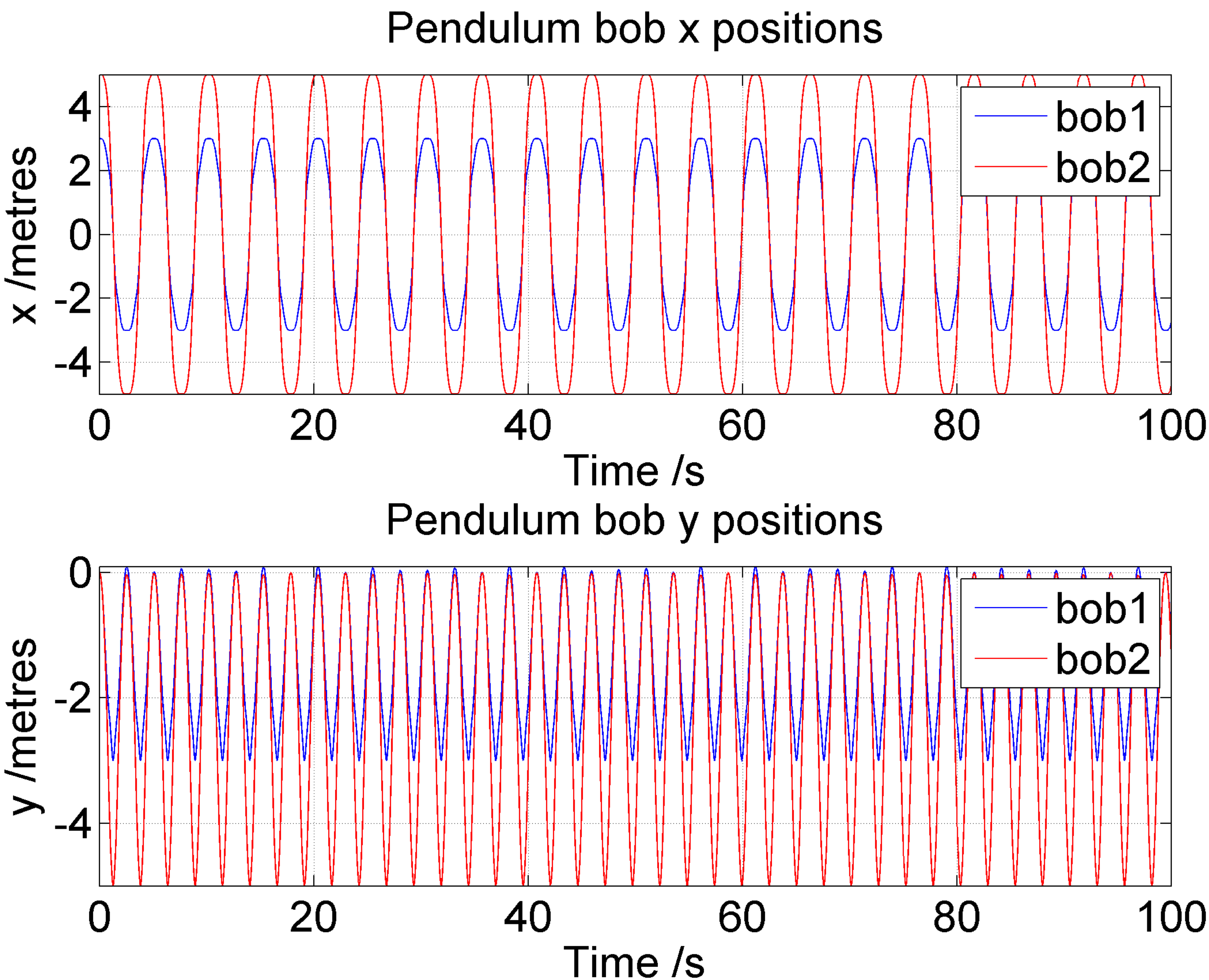
$$\frac{d\omega_1}{dt} = \frac{m_2 l_1 \omega_1^2 \sin\Delta\cos\Delta + m_2 g \sin\theta_2 \cos\Delta + m_2 l_2 \omega_2^2 \sin\Delta - (m_1 + m_2) g \sin\theta_1}{(m_1 + m_2) l_1 - m_2 l_1 \cos^2\Delta}$$

$$\frac{d\theta_2}{dt} = \omega_2$$

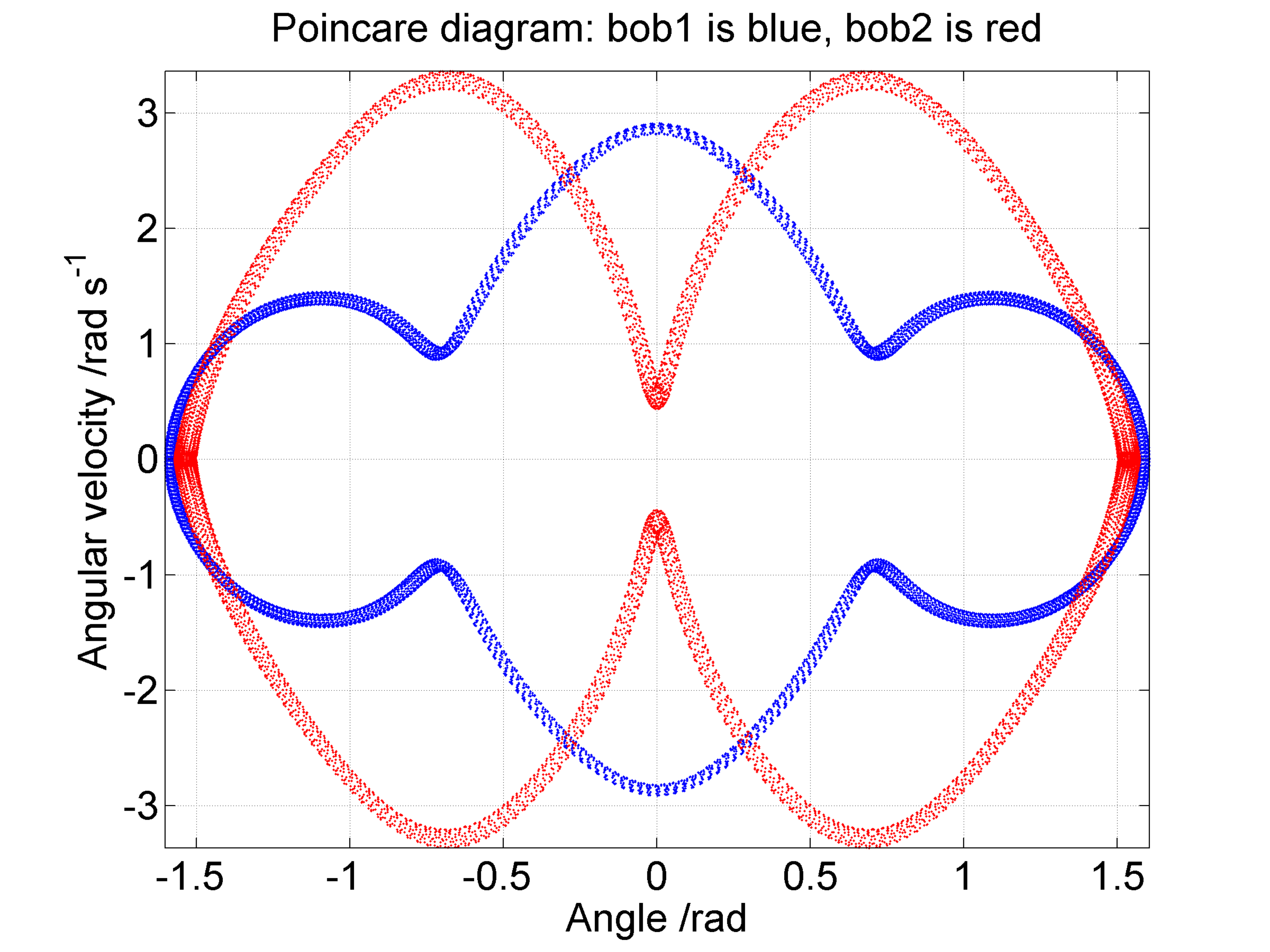
$$\frac{d\omega_2}{dt} = \frac{-m_2 l_2 \omega_2^2 \sin\Delta\cos\Delta + (m_1 + m_2)\left(g \sin\theta_1 \cos\Delta - l_1 \omega_1^2 \sin\Delta - g \sin\theta_2\right)}{(m_1 + m_2) l_2 - m_2 l_2 \cos^2\Delta}$$

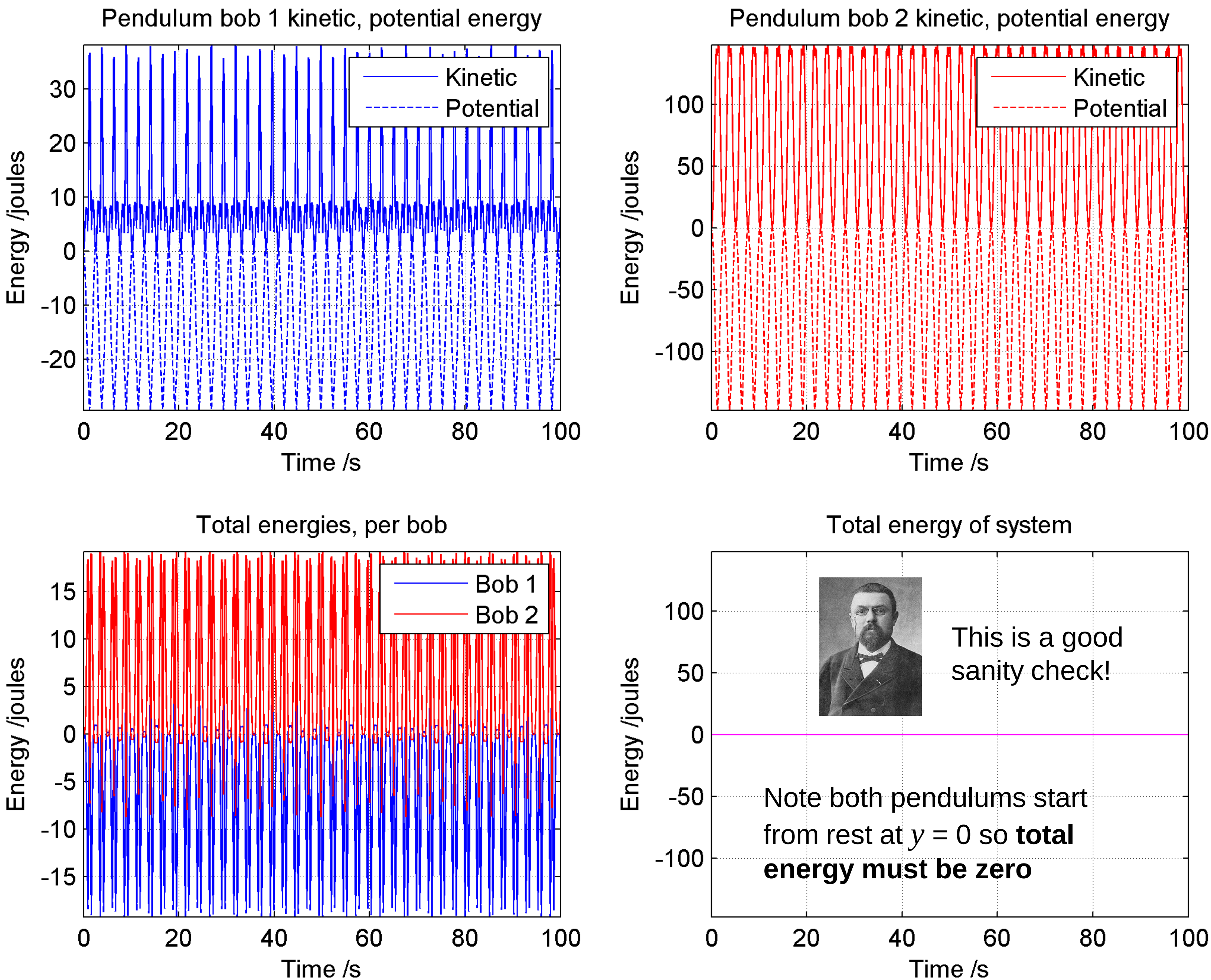
$$\Delta = \theta_2 - \theta_1$$


Pendulum bob x,y trajectories. Bob1 is blue, Bob2 is red
0
-1
-2
-3
-4
y /metres
-4
-2
0
2
4
x /metres



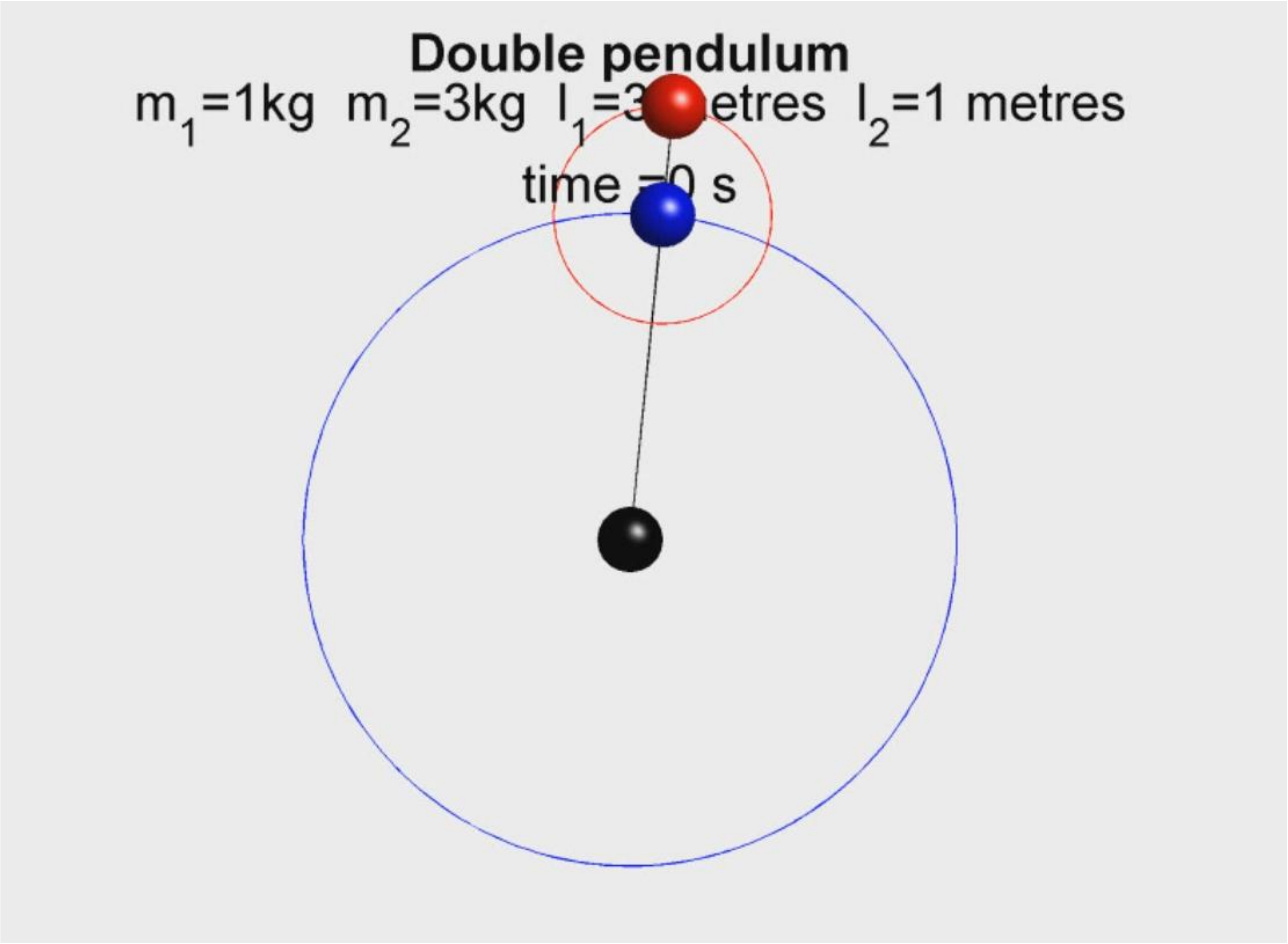
Pendulum bob x positions
x /metres
4
2
0
-2
-4
0
20
40
60
80
100
Time /s
bob1
bob2
Pendulum bob y positions
y /metres
0
-2
-4
0
20
40
60
80
100
Time /s
bob1
bob2

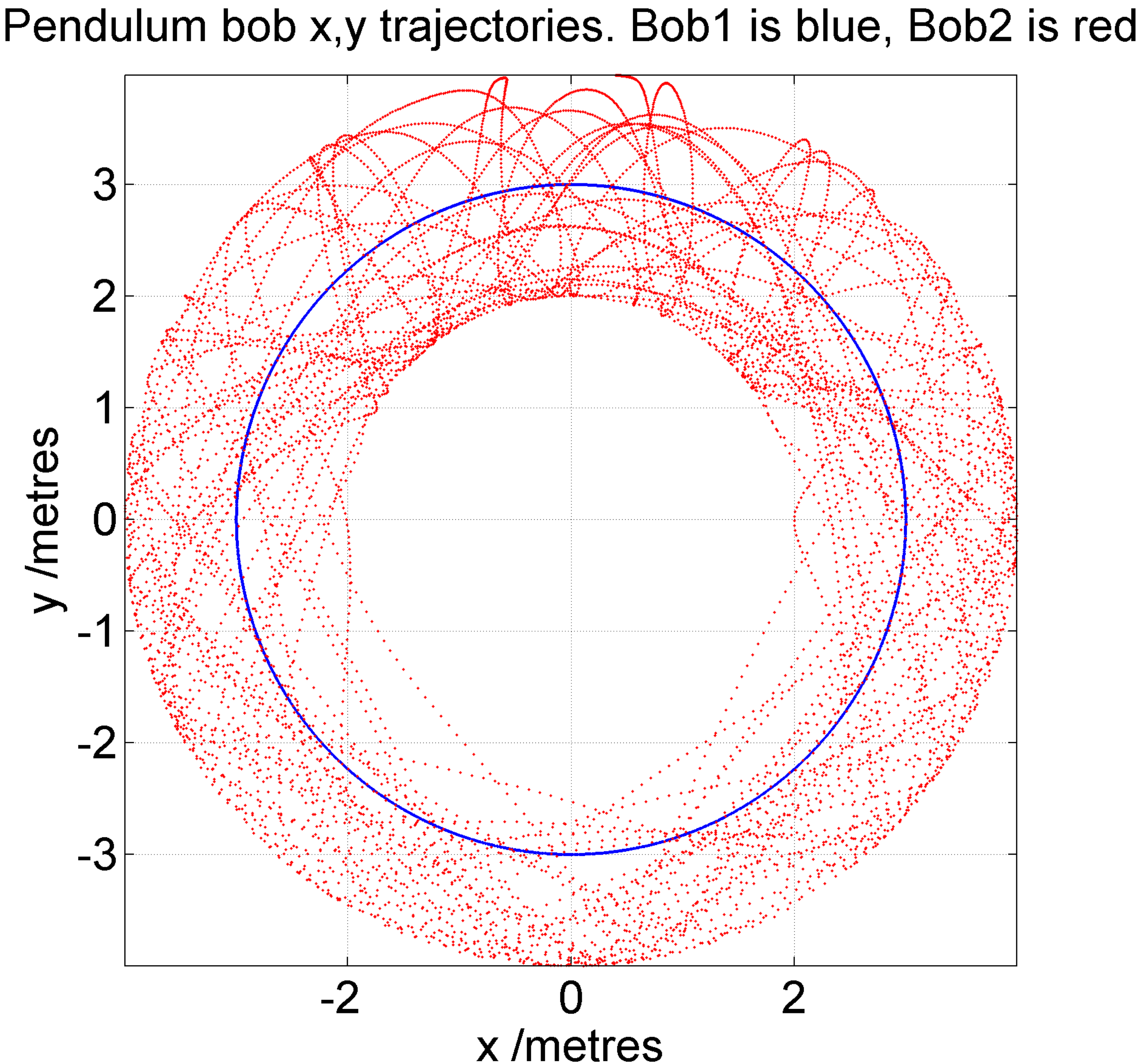

Poincare diagram: bob1 is blue, bob2 is red
Angular velocity /rad s^{-1}
3
2
1
0
-1
-2
-3
-1.5
-1
-0.5
0
0.5
1
1.5
Angle /rad

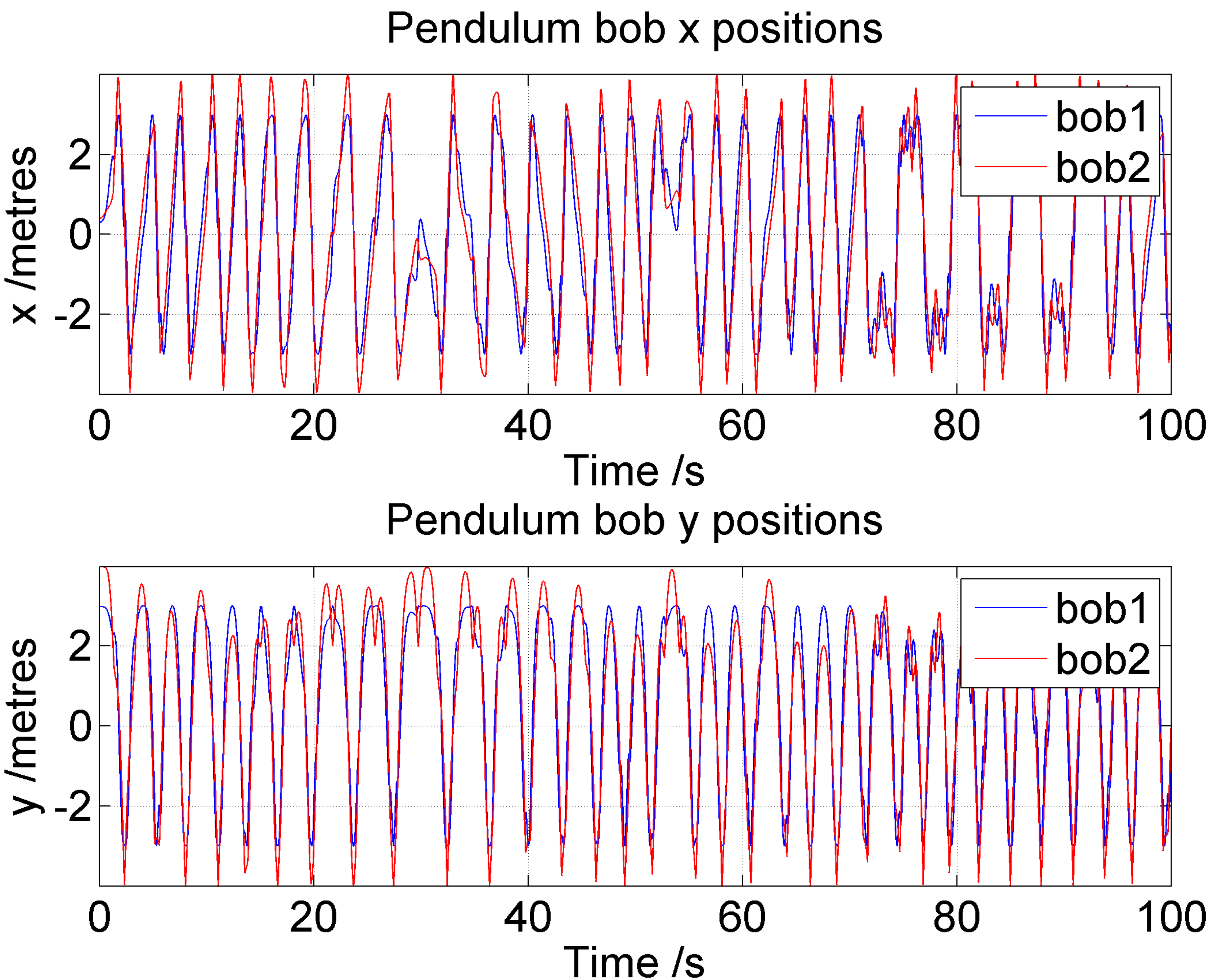


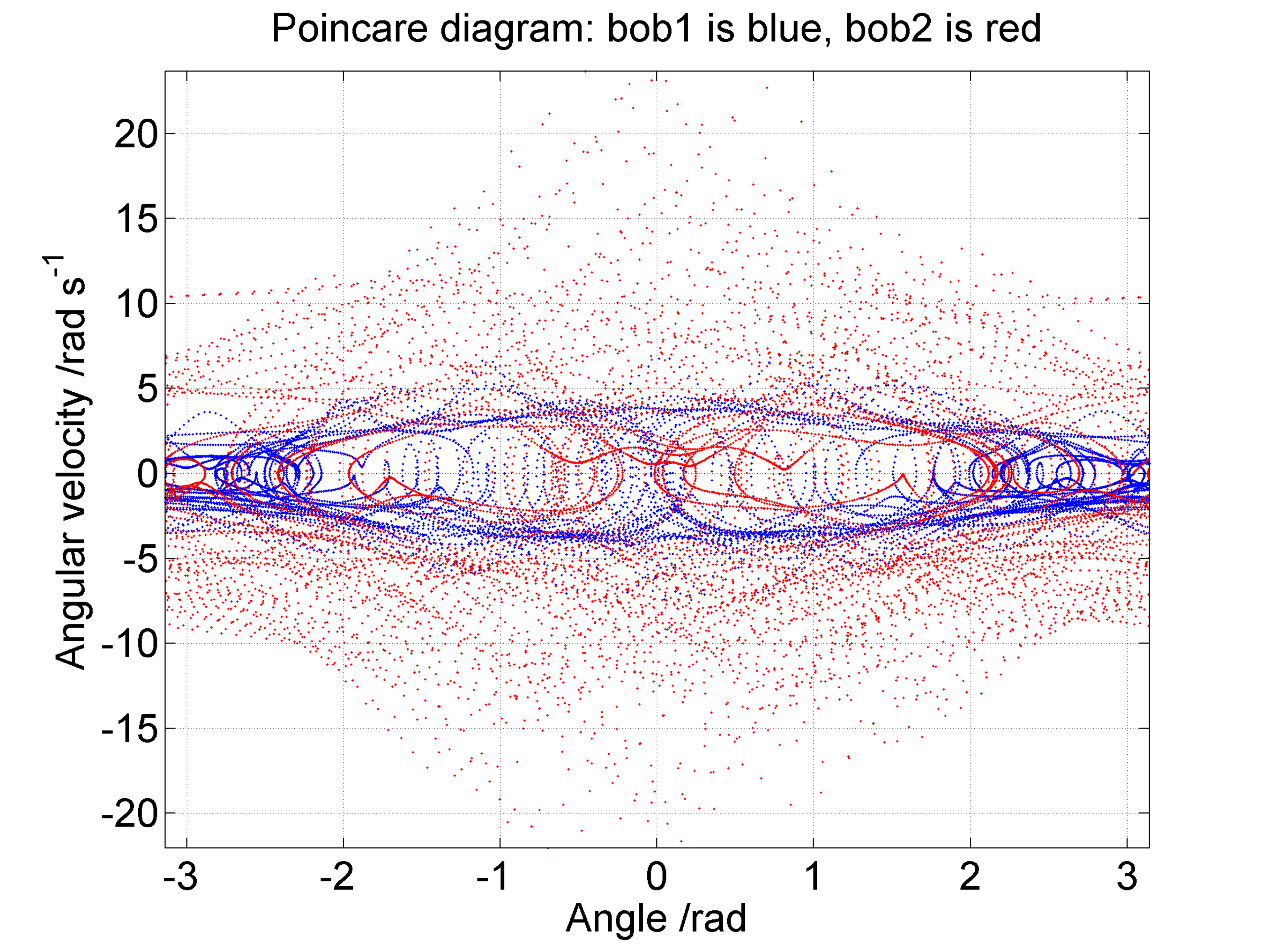
Pendulum bob 1 kinetic, potential energy
Kinetic
Potential
Energy /joules
Time /s
Pendulum bob 2 kinetic, potential energy
Kinetic
Potential
Energy /joules
Time /s
Total energies, per bob
Bob 1
Bob 2
Energy /joules
Time /s
Total energy of system
This is a good sanity check!
Note both pendulums start from rest at y = 0 so total energy must be zero
Energy /joules
Time /s

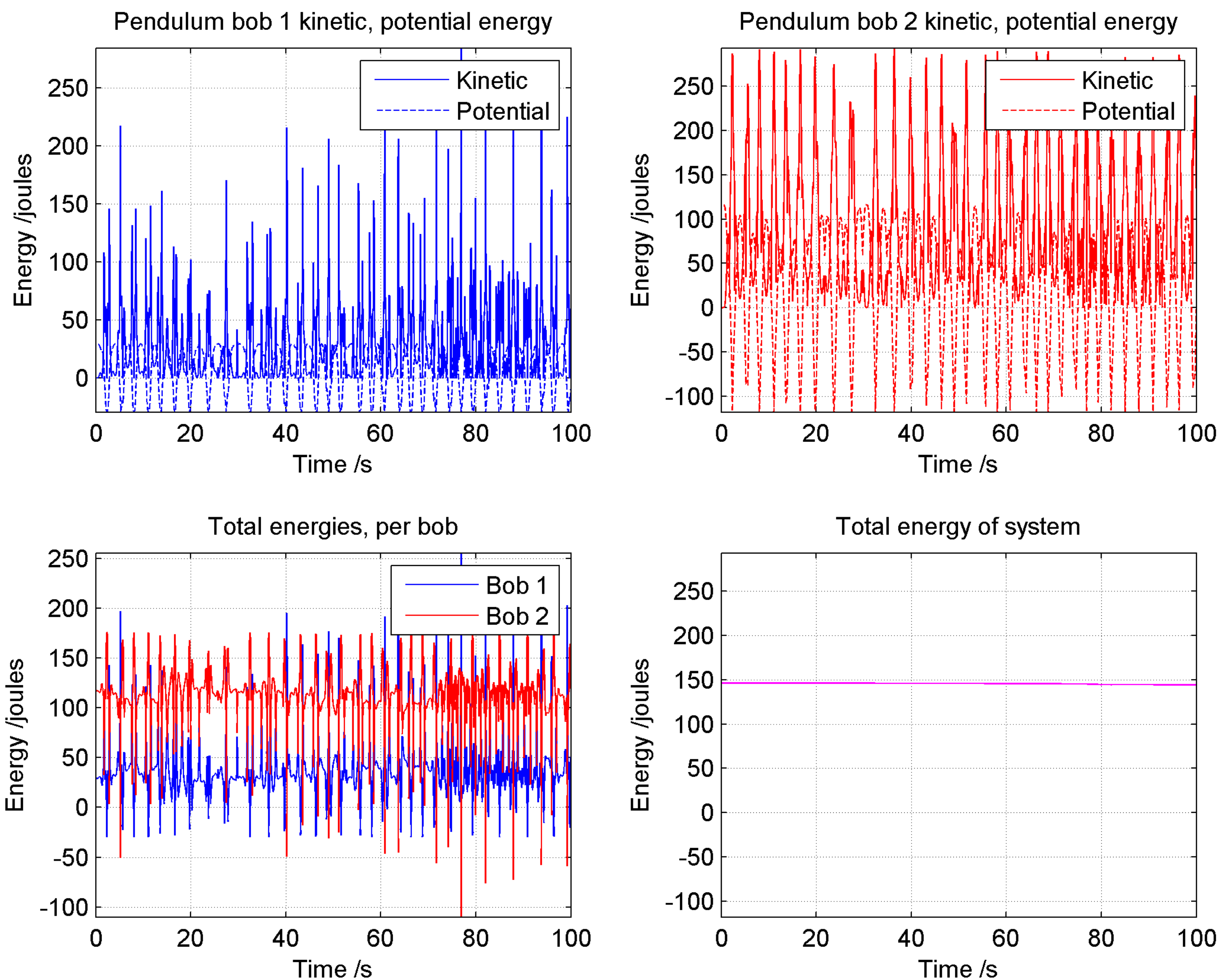
And now for **chaotic motion!**




Pendulum bob x,y trajectories. Bob1 is blue, Bob2 is red
y /metres
x /metres
3
2
1
0
-1
-2
-3
-2
0
2


Pendulum bob x positions
x /metres
Time /s
bob1
bob2
0
20
40
60
80
100
2
0
-2
Pendulum bob y positions
y /metres
Time /s
bob1
bob2
0
20
40
60
80
100
2
0
-2


Poincare diagram: bob1 is blue, bob2 is red
Angular velocity /rad s^-1
Angle /rad
20
15
10
5
0
-5
-10
-15
-20
-3
-2
-1
0
1
2
3


Pendulum bob 1 kinetic, potential energy
Kinetic
Potential
Energy /joules
Time /s
Pendulum bob 2 kinetic, potential energy
Kinetic
Potential
Energy /joules
Time /s
Total energies, per bob
Bob 1
Bob 2
Energy /joules
Time /s
Total energy of system
Energy /joules
Time /s

Lorenz and Rössler **strange attractors**

Edward Lorenz was using a Royal McBee LGP-30 computer in 1961 to model weather patterns. He accidentally fed in 3 digit precision numbers into the model from a printout rather than the 6 digits used by the computer. These tiny errors created a hugely different weather forecast....

Lorenz's weather model was *very* sensitive to initial conditions.

His equations looked a bit like these:

$$\frac{dx}{dt} = s(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz$$

$$s = 10$$

$$r = 28$$

$$b = \tfrac{8}{3}$$

Edward Lorenz
1917-2008

Although x, y, z trajectories are chaotic, they tend to *gravitate towards a particular region.*

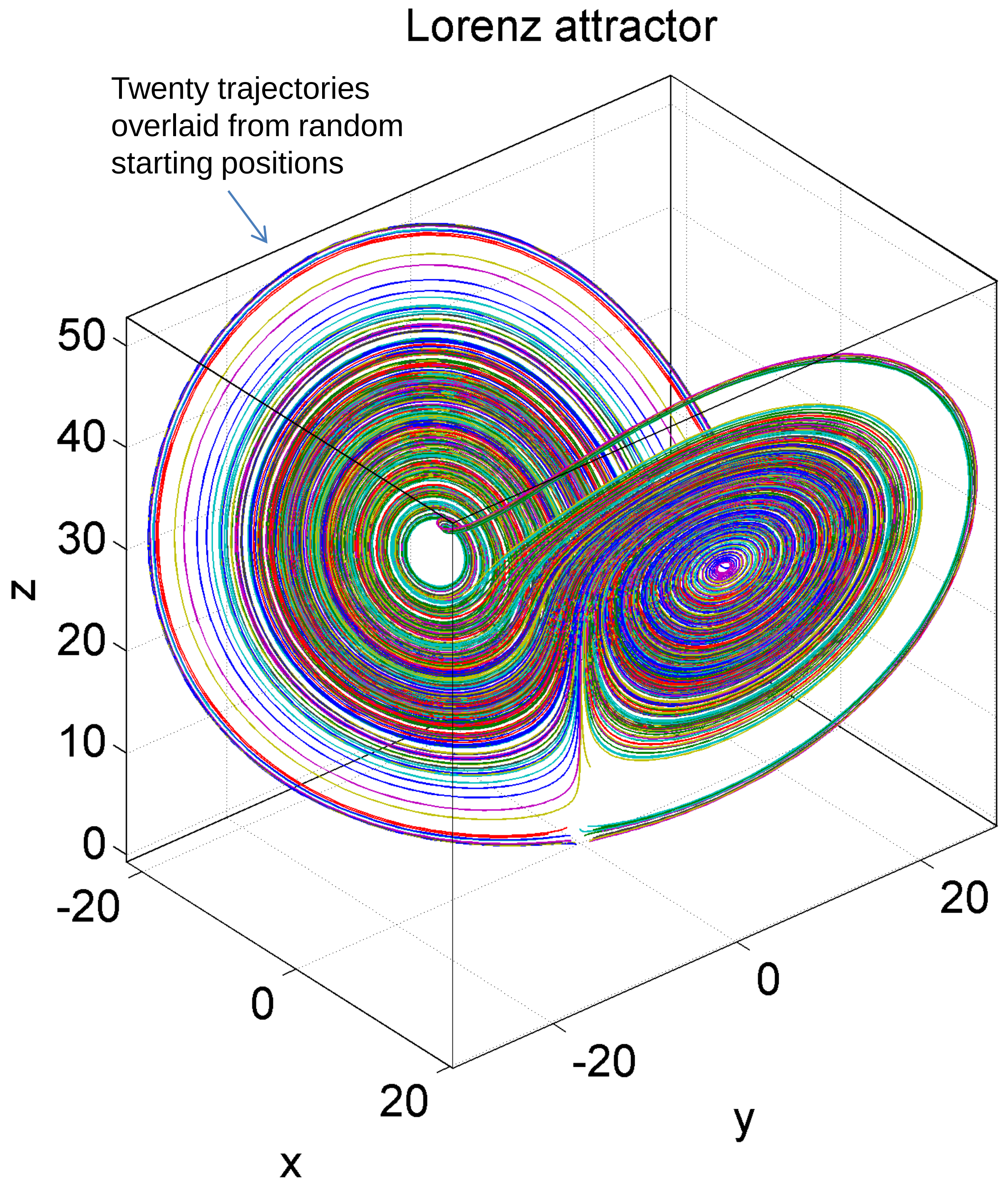
This region is called a **strange attractor.**

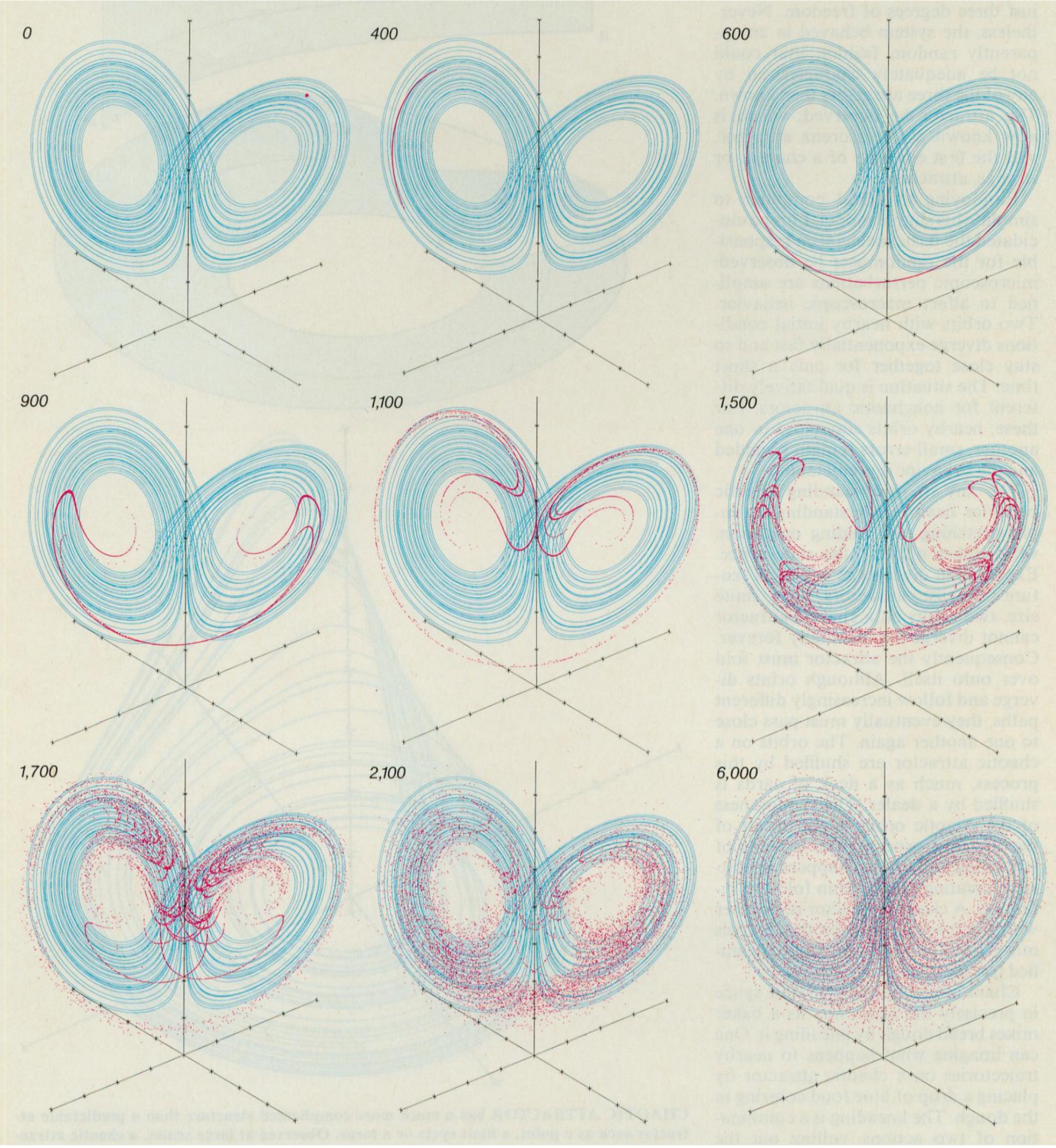
$$\frac{dx}{dt} = s(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz$$

$$s = 10 \quad r = 28 \quad b = \tfrac{8}{3}$$





Applying the Lorenz equations, a cluster of initial x,y,z values separated by a *tiny* random deviation will eventually **spread out evenly throughout the strange attractor.**

Shaw *et al*; “Chaos”, Scientific American 54:12 (1986) 46-57

Another chaotic system with a **strange attractor** is the solution set of the **Rössler equations**

Otto Rössler
1940-

$$a = \frac{1}{10}$$

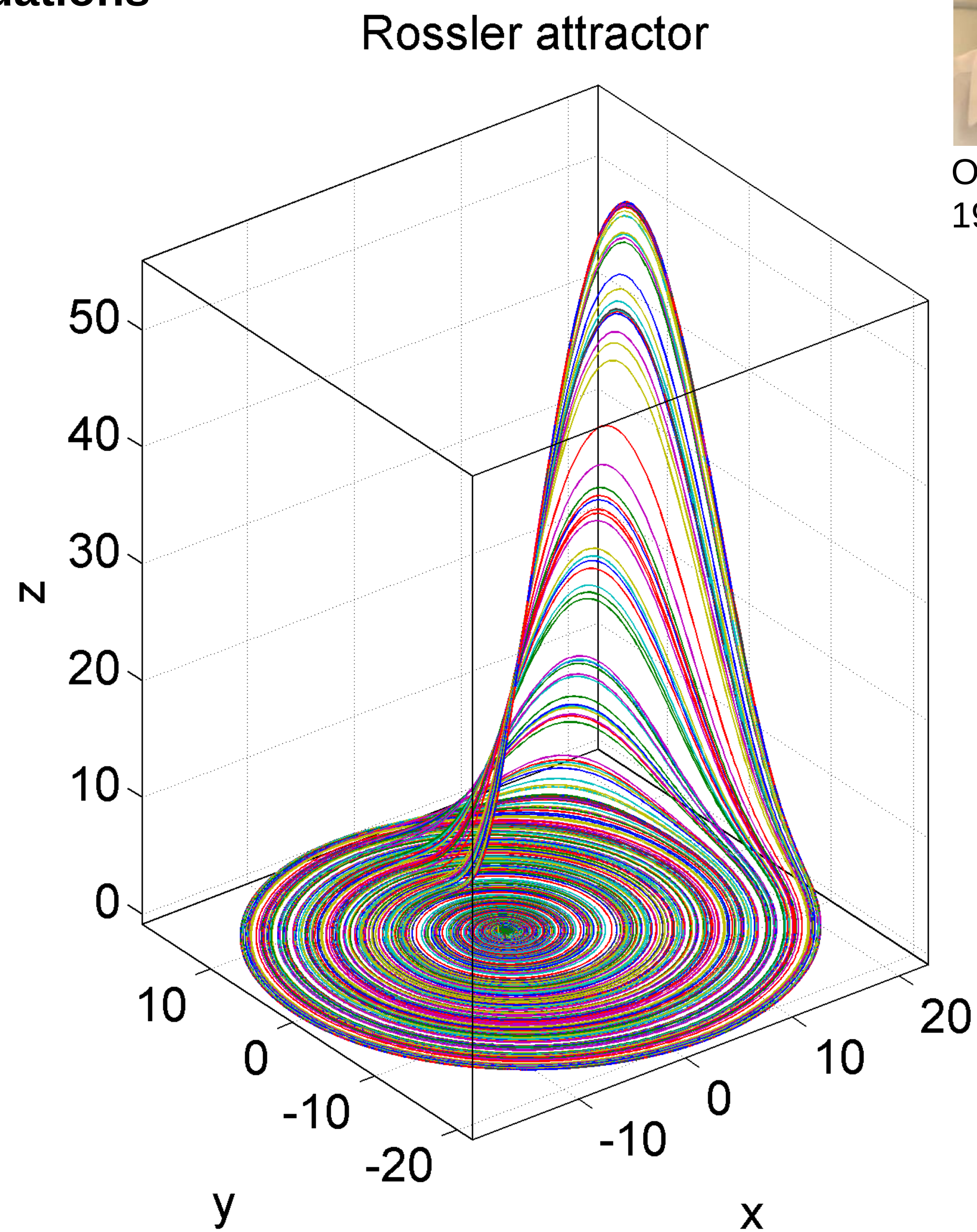
$$b = \frac{1}{10}$$

$$c = 14$$

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

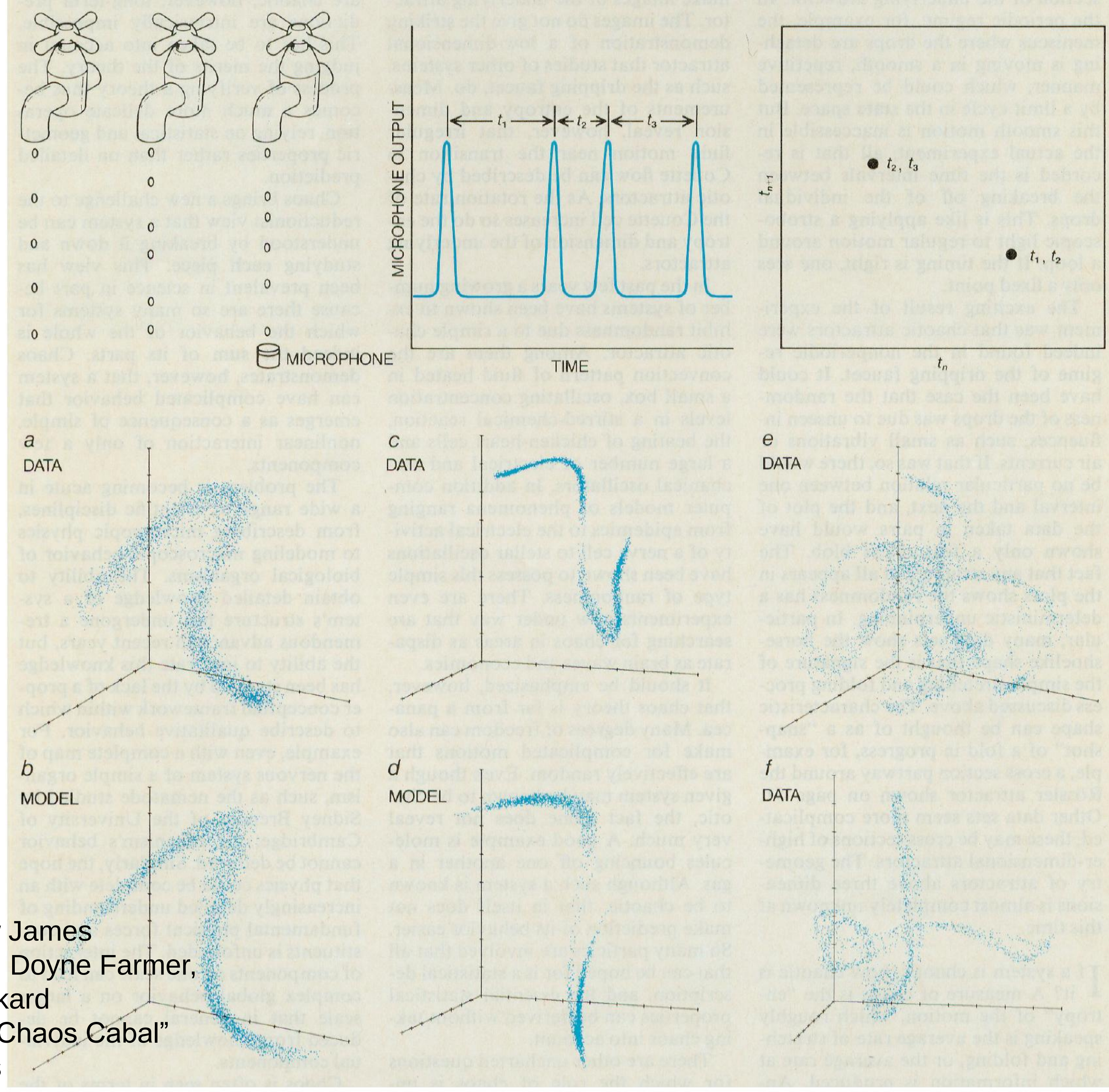
$$\frac{dz}{dt} = z(x - c) + b$$

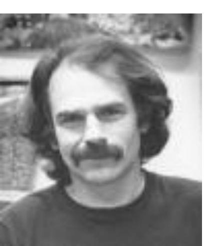


Shaw's dripping faucet

Construct *x*,*y*,*z* coordinates from time differences between drips

Seemingly random drips form a strange attractor, whose shape depends on the flow rate





Robert Shaw James Crutchfield J. Doyne Farmer, Norman Packard
"Santa Cruz Chaos Cabal"
1970s-1980s

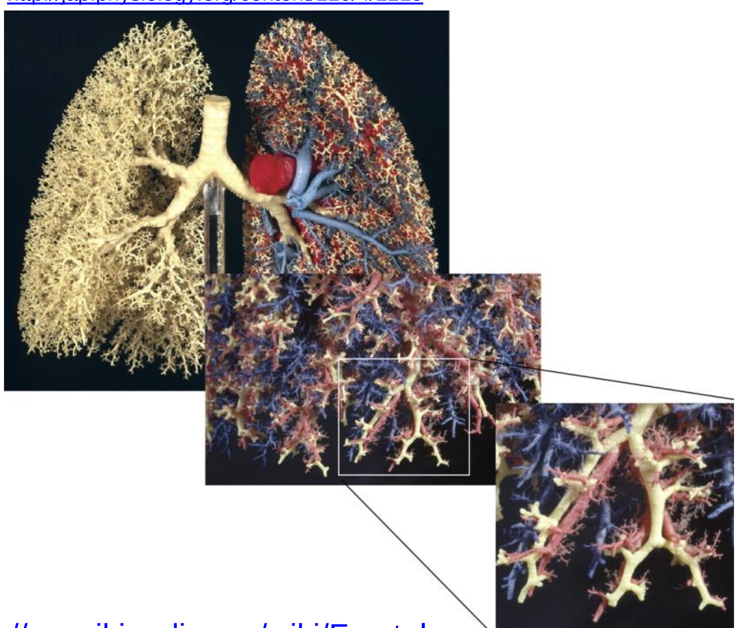
Fractals

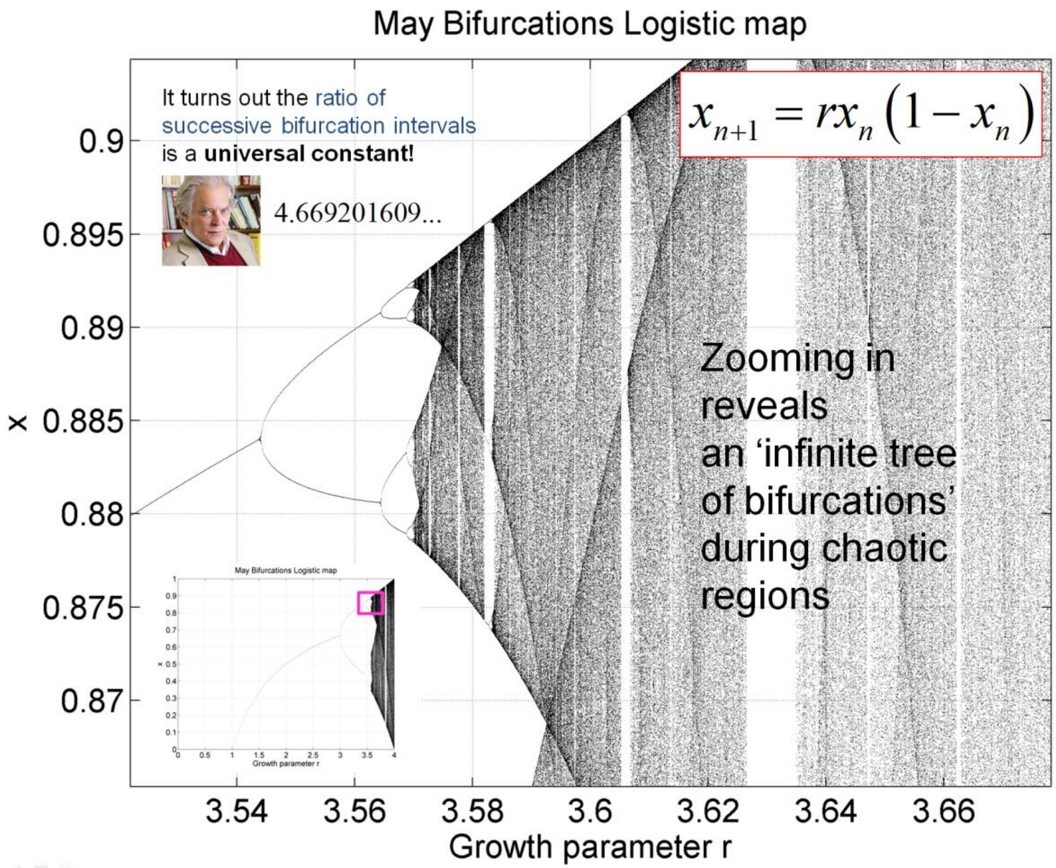
A *fractal* is a structure which is **geometrically similar over a wide range of scales.** In other words, zoom in and it looks the same.

Fractals are *everywhere* in **natural forms**, from the branching structure of our lungs and trees, to the shape of coastlines, to river networks, to eddies in turbulent fluids ....

And it is also a feature of the bifurcation diagrams we have already met ....

http://jap.physiology.org/content/110/4/1119



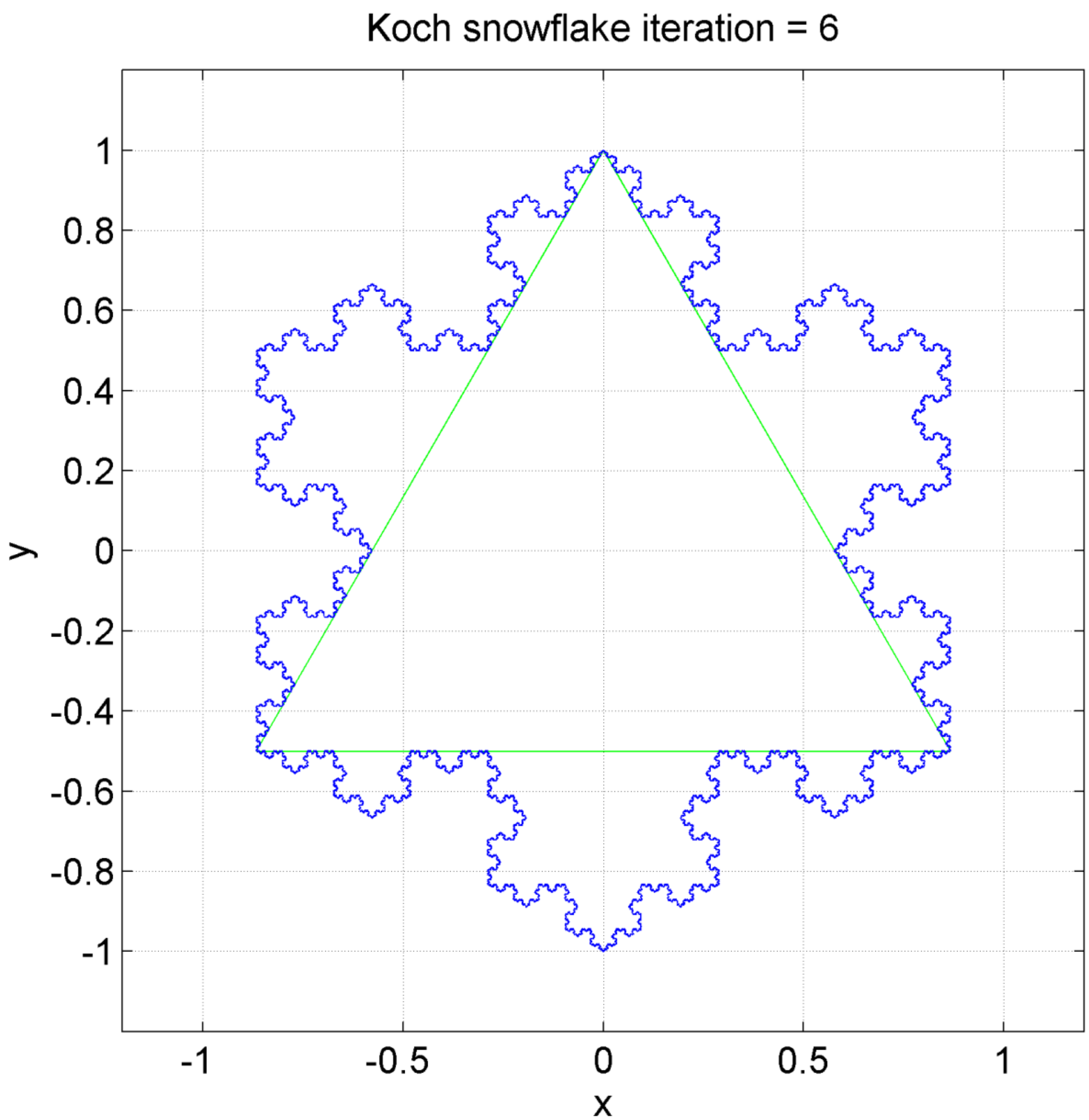


https://en.wikipedia.org/wiki/Fractal

http://fractalfoundation.org/resources/what-are-fractals/

The Koch Snowflake

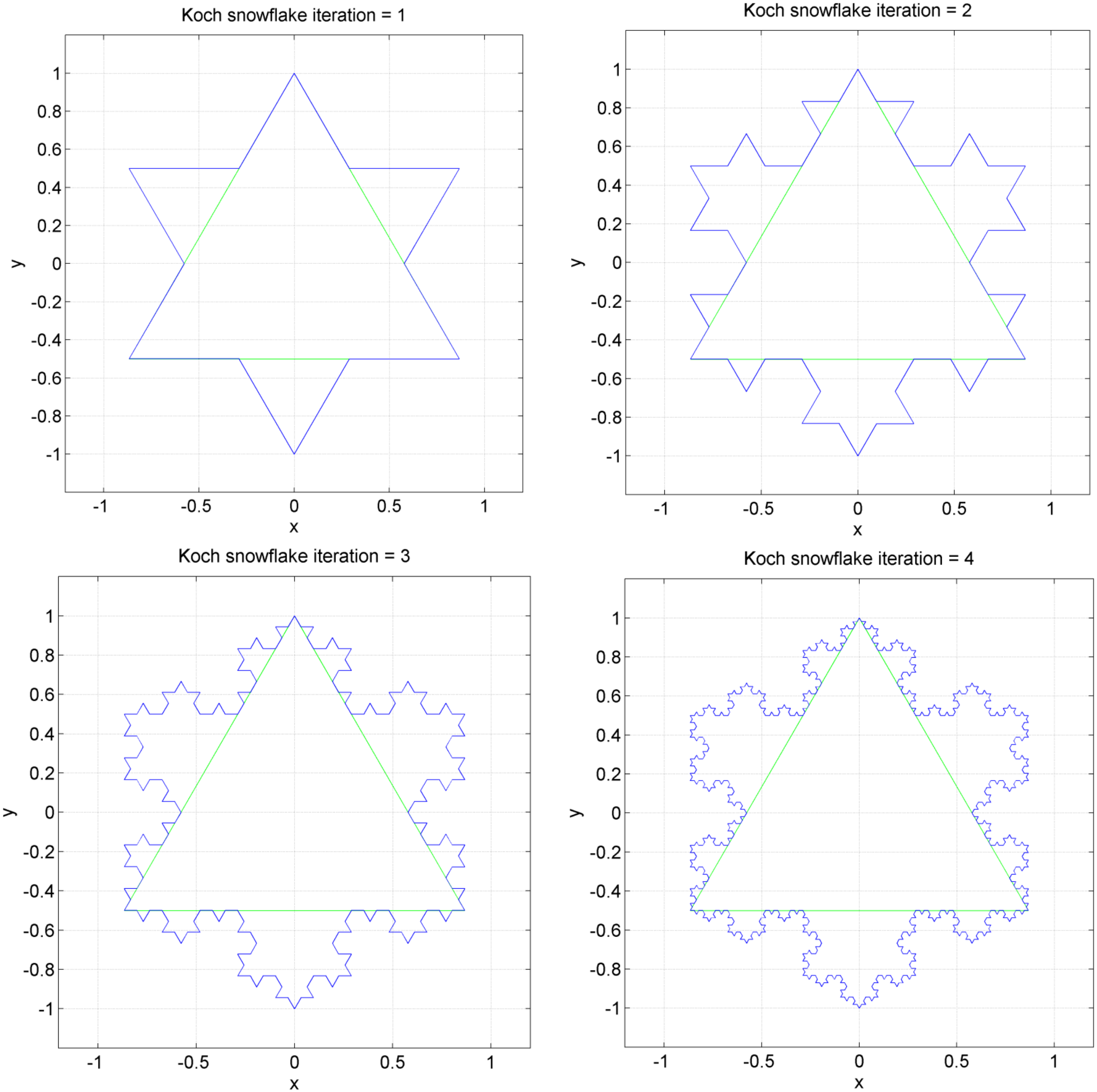
Niels Fabian Helge von Koch (1870-1924)



Perhaps the earliest example of *fractal geometry* – before I even coined the term!

Area tends to 8/5 of the area of the green triangle....
.... but the perimeter is *infinite!*

https://en.wikipedia.org/wiki/Koch_snowflake https://people.cs.clemson.edu/~goddard/handouts/math360/notes11.pdf

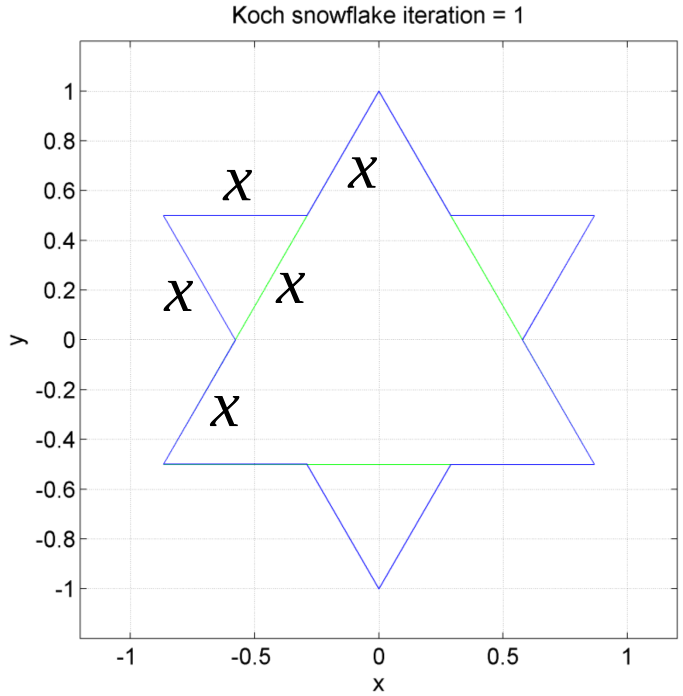


1. Start with an equilateral triangle

2. Divide each edge into thirds

3. Add another equilateral triangle to each edge with base being the central third.

Iterate from step 2 ...



For each iteration:

Every side length grows from

$3x \rightarrow 4x$ i.e. a factor of 4/3

Hence perimeter after n iterations is:

$$P_n = P_0 \left(\frac{4}{3}\right)^n$$

where P_0 is the perimeter of the original triangle.

i.e. as n becomes large, P tends to infinity!

Each triangle of edge $3x$ gains another triangle of edge size x. i.e. **gains a triangle** of **1/9 the area of previous triangles added**

Each iteration the number of sides increases by a factor of 4, so number of sides after n iterations is 3×4^n

This gives the number of extra triangles in iteration $n+1$

Hence area added in iteration k is:

$$\Delta A_k = 3 \times 4^{k-1} \times \frac{A_0}{9^k}$$

Original triangle area is A_0

Total area enclosed by Koch Snowflake is therefore:

$$\Delta A_k = 3 \times 4^{k-1} \times \frac{A_0}{9^k}$$

$$A_n = A_0 + \sum_{k=1}^{n} \Delta A_k = A_0 + 3 \times 4^{1-1} \times \frac{A_0}{9^1} + 3 \times 4^{2-1} \times \frac{A_0}{9^2} + 3 \times 4^{3-1} \times \frac{A_0}{9^3} + \ldots.$$

$$\frac{A_n}{A_0} = 1 + \tfrac{3}{4}\left(\tfrac{4}{9} + \tfrac{4^2}{9^2} + \tfrac{4^3}{9^3} + \ldots. + \tfrac{4^n}{9^n}\right)$$

$$\frac{A_n}{A_0} = 1 + \tfrac{3}{4}\tfrac{4}{9}\left(1 + \tfrac{4}{9} + \tfrac{4^2}{9^2} + \ldots. + \tfrac{4^{n-1}}{9^{n-1}}\right)$$

Geometric progression

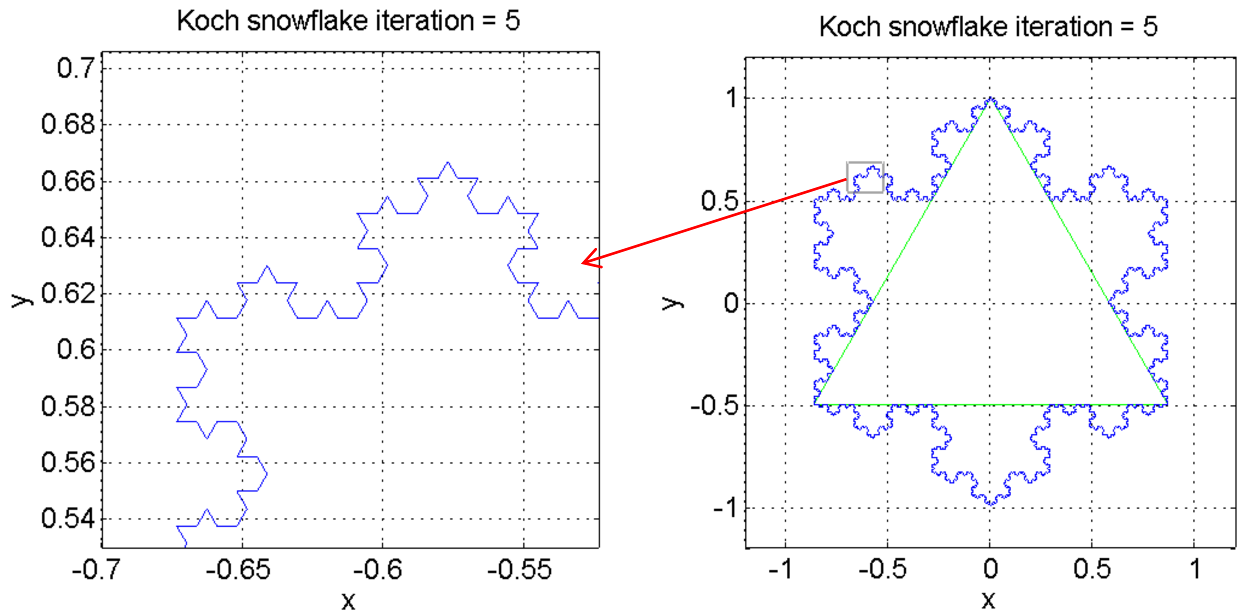
$$a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$$

$$\frac{A_n}{A_0} = 1 + \tfrac{1}{3}\frac{1-\frac{4^n}{9^n}}{1-\frac{4}{9}} = 1 + \tfrac{1}{3}\tfrac{9}{5}\left(1 - \tfrac{4^n}{9^n}\right) = \frac{5 + 3\left(1 - \frac{4^n}{9^n}\right)}{5}$$

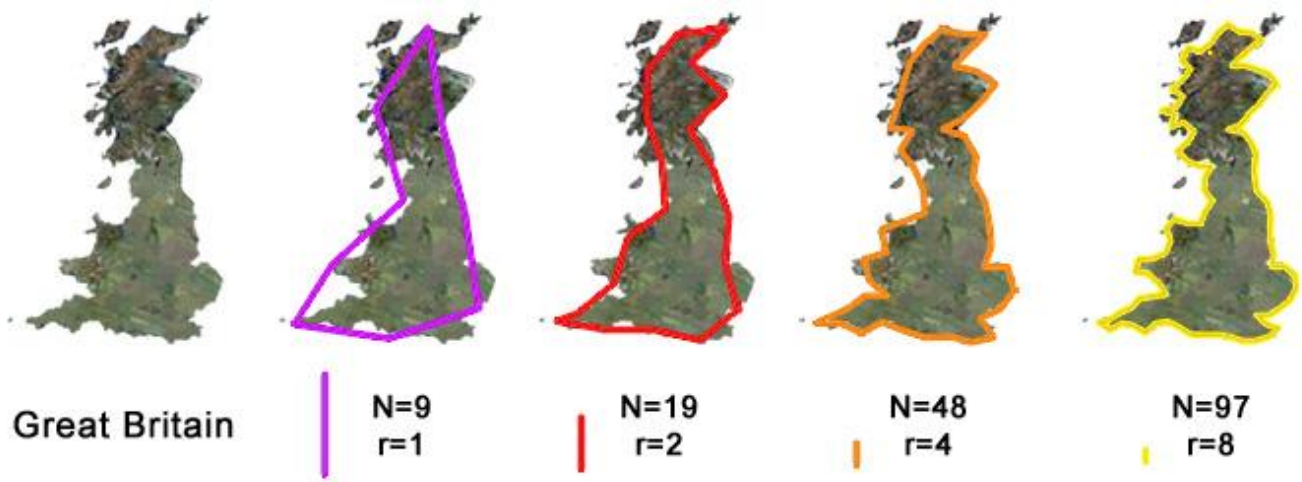
So as n becomes infinite:

$$\lim_{n\to\infty}\frac{A_n}{A_0} = \lim_{n\to\infty}\left\{\frac{5 + 3\left(1 - \frac{4^n}{9^n}\right)}{5}\right\} = \boxed{\frac{8}{5}}$$

In the limit when *n* tends to infinity, the Koch Snowflake is **self similar**, i.e. has the same structure at all magnification scales.



The Koch Snowflake has a ***fractal*** structure. A bit like the coastline of the UK. It's perimeter depends on the *lengths of our measuring sticks* which map out greater (but similarly shaped) detail as we zoom in



http://fractalfoundation.org/OFCA/uks4.jpg

Although the perimeter *is* infinite, we can calculate the number of fixed length 'sticks' which make up the perimeter. Let stick size x for iteration n be the perimeter divided by the number of sides

$$x_n = P_n / N_n = \frac{P_0 \left(\frac{4}{3}\right)^n}{3 \times 4^n} = \boxed{\tfrac{1}{3} P_0 \times 3^{-n}}$$

Define the **Fractal Dimension** D such that the **number of sticks can be defined in terms of the stick size:**

$$N_n = 3 \times \left(\tfrac{1}{3^n}\right)^{-D}$$

$$\therefore 3 \times \left(\tfrac{1}{3^n}\right)^{-D} = 3 \times 4^n$$

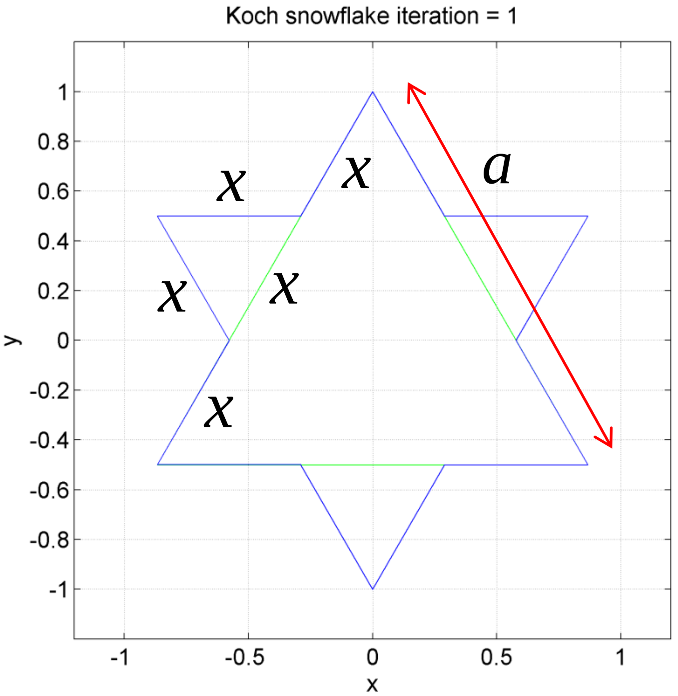
$$\left(3^{-n}\right)^{-D} = 4^n$$

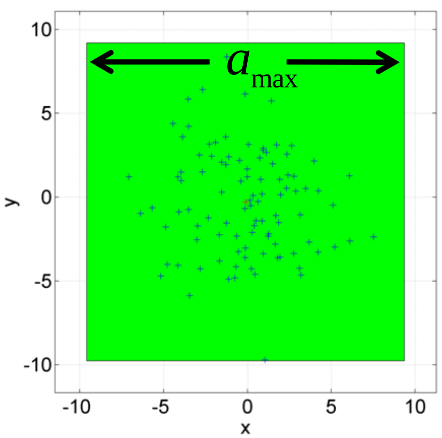
$$3^{nD} = 4^n$$

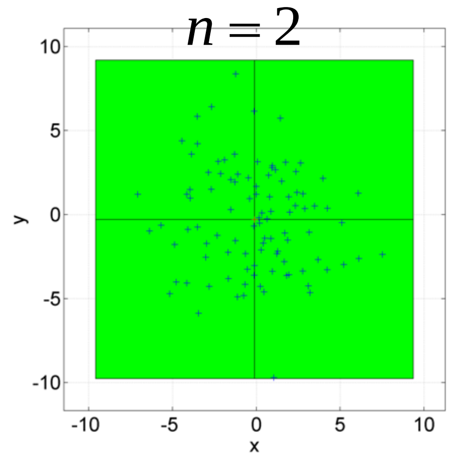
$$\therefore Dn \log 3 = n \log 4$$

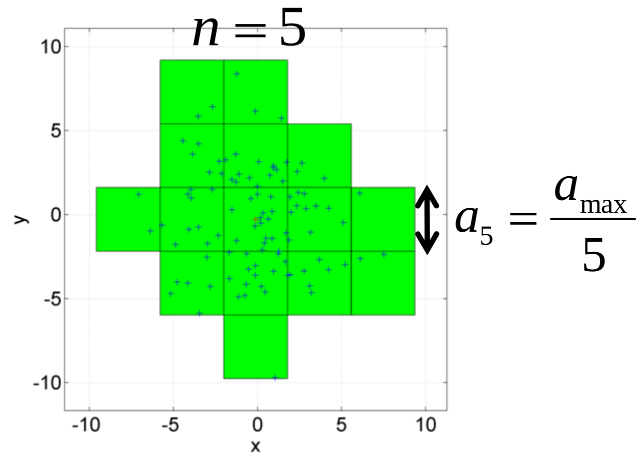
$$\boxed{D = \frac{\log 4}{\log 3} \approx 1.2619}$$

The Koch curve has a 'fractional dimension' of about 1.2619



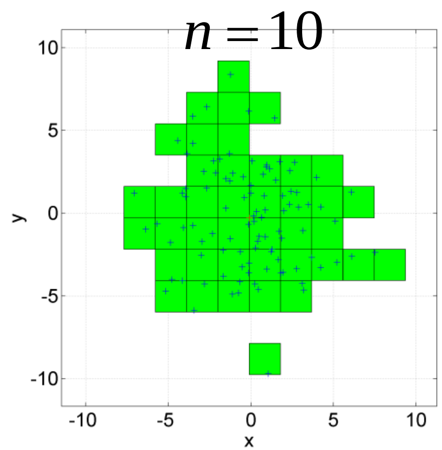




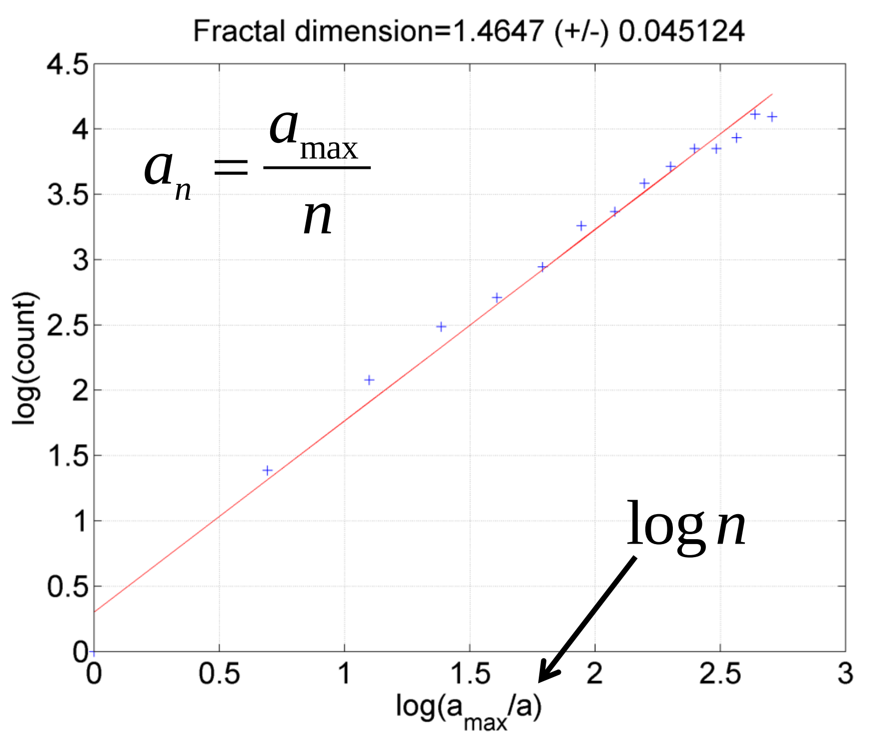


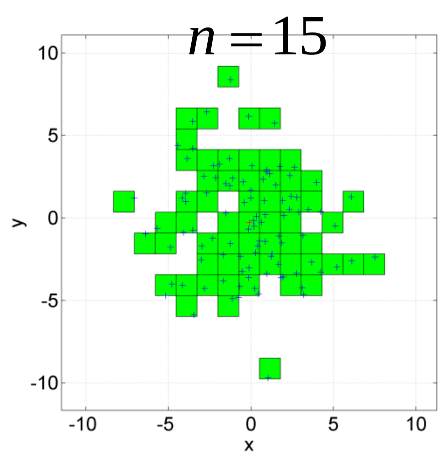
Fractal dimension box counting method

This is better for areas or volumes



Count the green squares that contain the points →





Barnsley fern

Intriguingly, fractal structures like the Koch curve can be generated using an **iterated random process.** This is called the *'Chaos Game'*

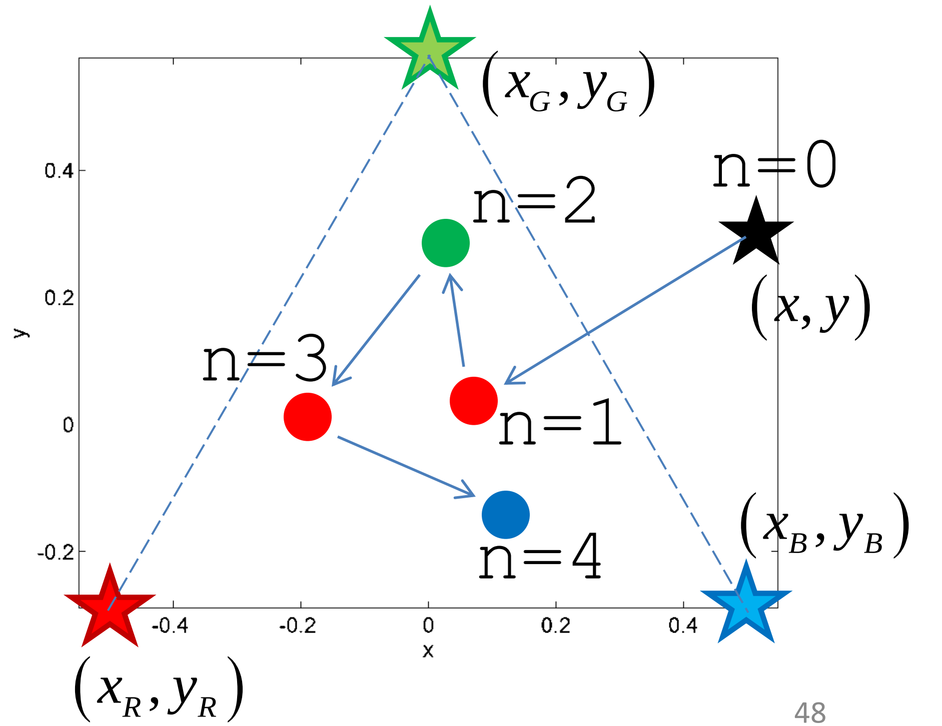
```
function fern

%Define number of iterations
N = 1e5;

%Pixel size
psize = 0.1;

%Start x,y coordinates
x =0;
y = 0;
xx = 0;
yy = 0;

%Generate Barnsley fractal
for n=1:N
    r = rand;
    if r<=0.02
        %Stem
        xxyy = [0,0;0,0.16]*[xx;yy];
        xx = xxyy(1); yy = xxyy(2);
        x = [x,xx];
        y = [y,yy];
    elseif (r>0.01) && (r<=0.85)
        %Smaller leaflets
        xxyy = [0.85,0.04;-0.04,0.85]*[xx;yy] + [0;1.60] ;
        xx = xxyy(1); yy = xxyy(2);
        x = [x,xx];
        y = [y,yy];
    elseif (r>0.85) && (r<=0.92)
        %Largest left-hand leaflet
        xxyy = [0.20,-0.26;0.23,0.22]*[xx;yy] + [0;1.60] ;
        xx = xxyy(1); yy = xxyy(2);
        x = [x,xx];
        y = [y,yy];
    else
        %Largest right hand leaflet
        xxyy = [-0.15,0.28;0.26,0.24]*[xx;yy] + [0;0.44] ;
        xx = xxyy(1); yy = xxyy(2);
        x = [x,xx];
        y = [y,yy];
    end
end

%Plot fractal
figure('color',[1 1 1],'name','Barnsley fern','renderer','opengl');
plot(x,y,'g.','markersize',psize);
axis equal
axis off

%End of code
print(gcf,'barnsley fern.png','-dpng','-r300');
```

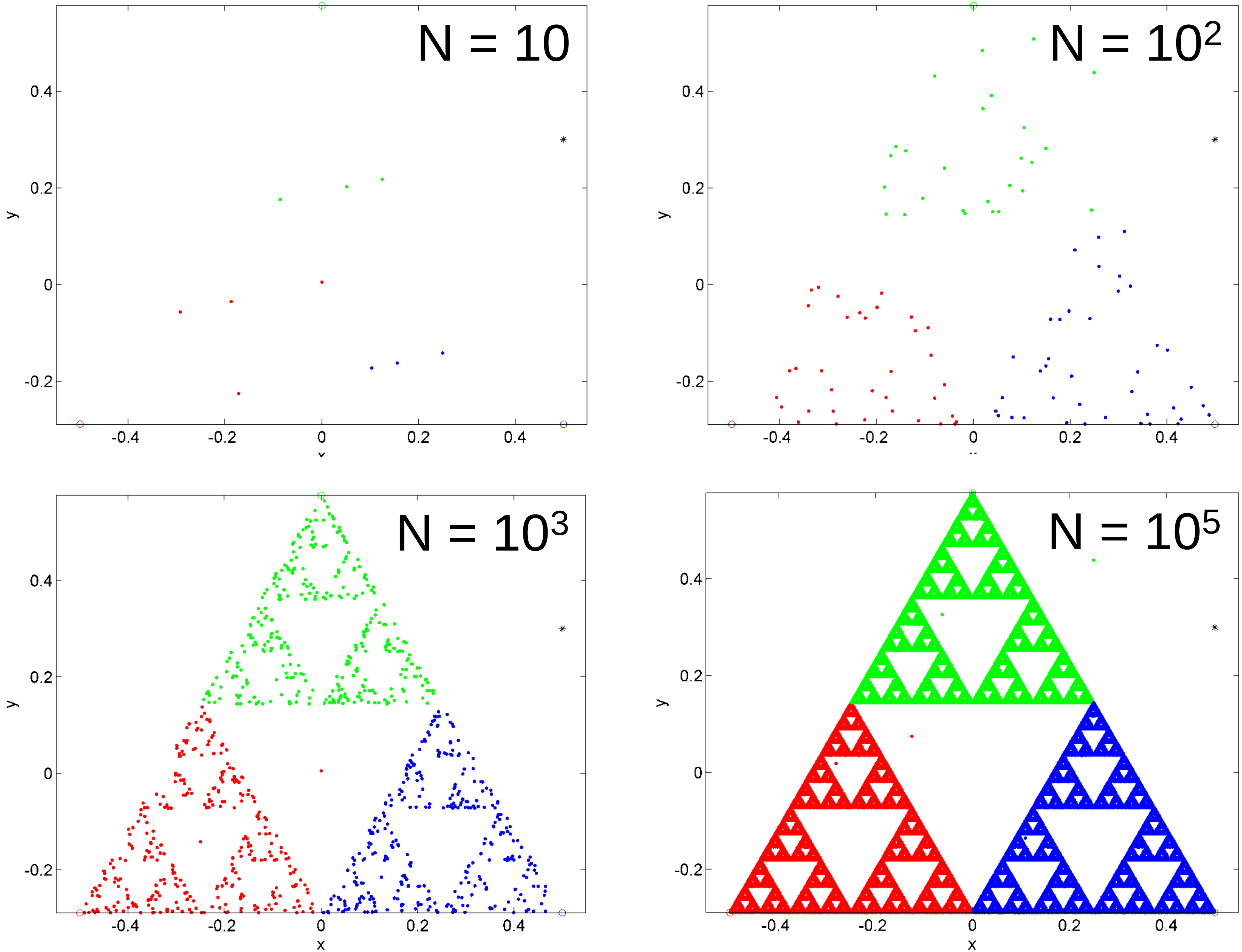
The **Barnsley Fern** is a fractal named after the British mathematician **Michael Barnsley** who first described it in his book *Fractals Everywhere*. He made it to resemble the Black Spleenwort, [*Asplenium adiantum-nigrum*].

[https://en.wikipedia.org/wiki/Barnsley_fern]

The Sierpinski Triangle

```
for n=1:N
    r = rand; %Generate a random number
    if ( r <= 1/3 )
        %Move half way towards red star
        x = 0.5*( xR + x );
        y = 0.5*( yR + y );
        %Plot a red dot
        plot( x,y, 'r.' );
    elseif ( r > 1/3 ) && ( r <=2/3 )
        %Move ... blue star
        x = 0.5*( xB + x );
        y = 0.5*( yB + y );
        %Plot a blue dot
        plot( x,y, 'b.' );
    else
        %Move ... green star
        x = 0.5*( xG + x );
        y = 0.5*( yG + y );
        %Plot a green dot
        plot( x,y, 'g.' );
    end
end
```



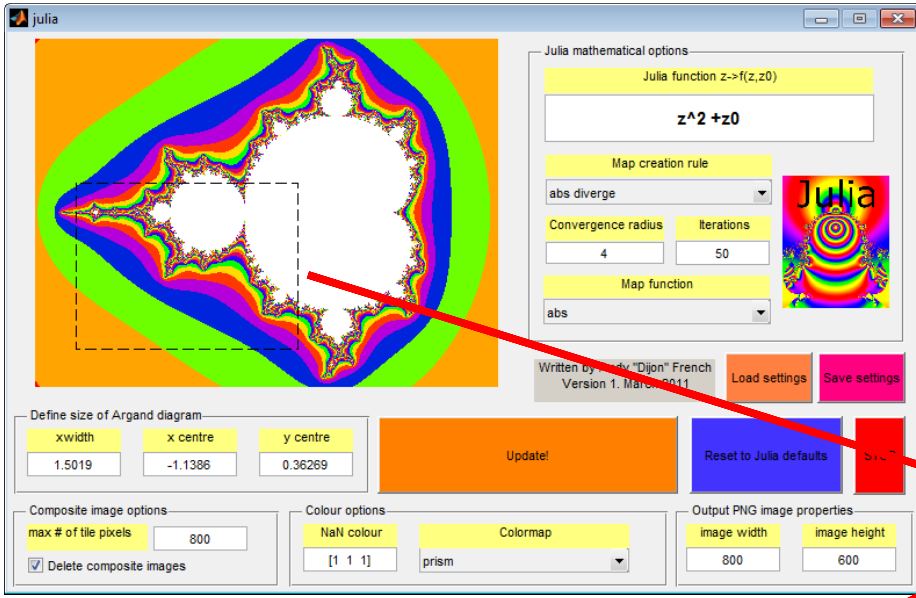


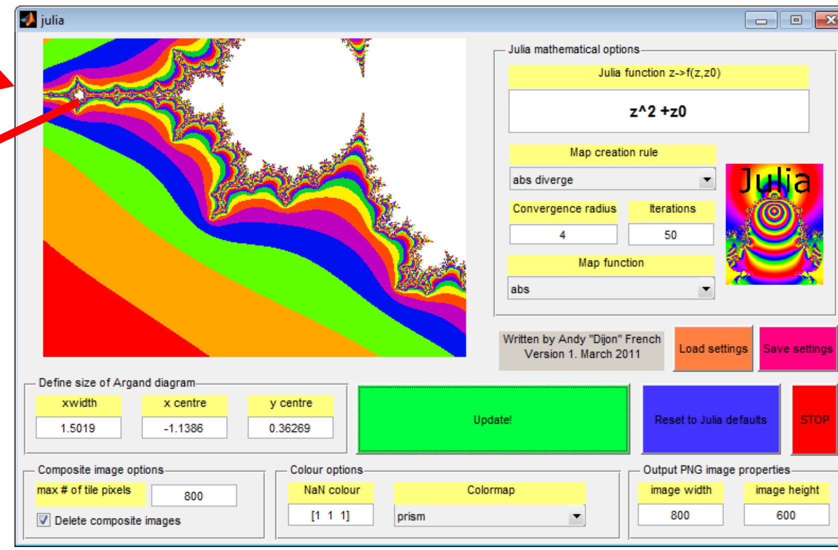
N = 10
$N = 10^2$
$N = 10^3$
$N = 10^5$
y
x
0.4
0.2
0
-0.2
-0.4

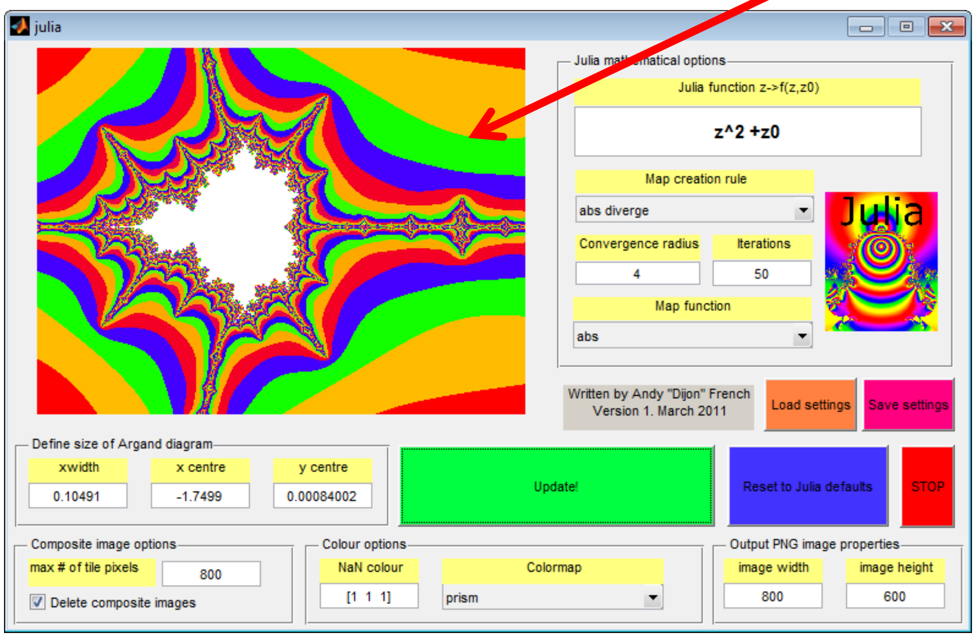
Mandlebrot, complex numbers and iteration

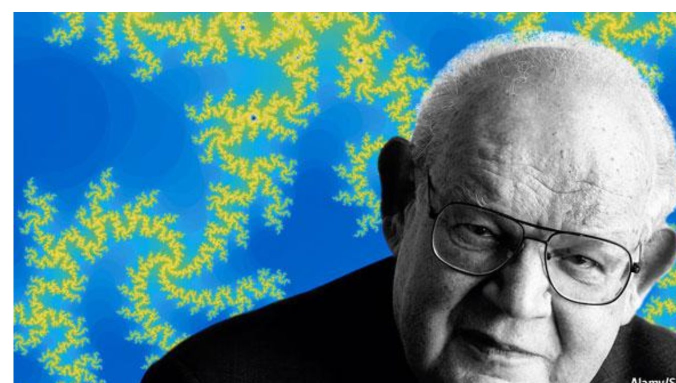
The *Mandlebrot Set* has infinite complexity!

... But a recursive *fractal* geometry









Benoit Mandlebrot (1924-2010)

Mandlebrot transformations of **complex numbers**

$$i^2 = -1$$

$$z = x + iy$$

$$x = \mathrm{Re}(z)$$

$$y = \mathrm{Im}(z)$$

$$|z| = \sqrt{x^2 + y^2}$$

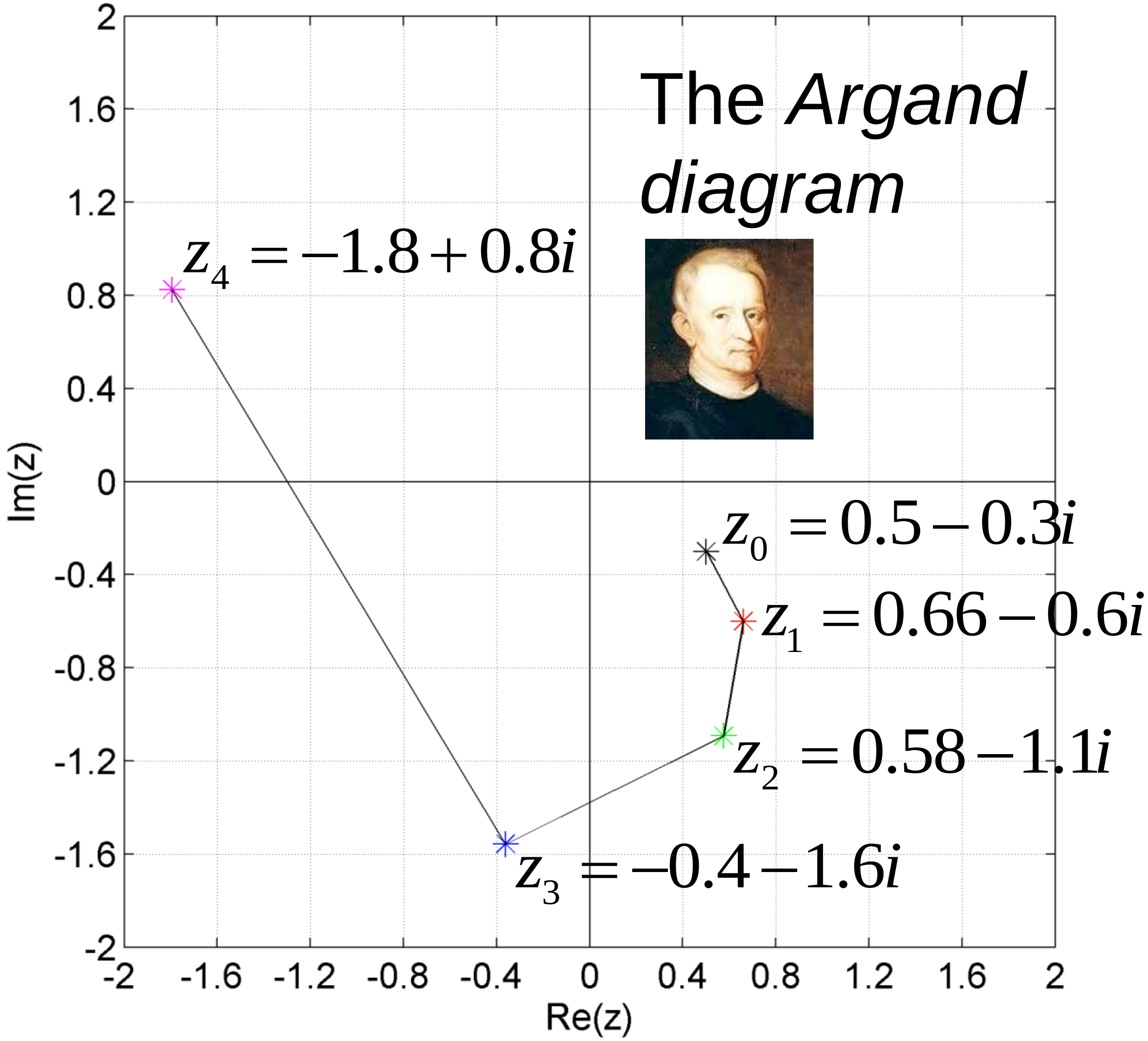
$$(1+i)(1+i)$$

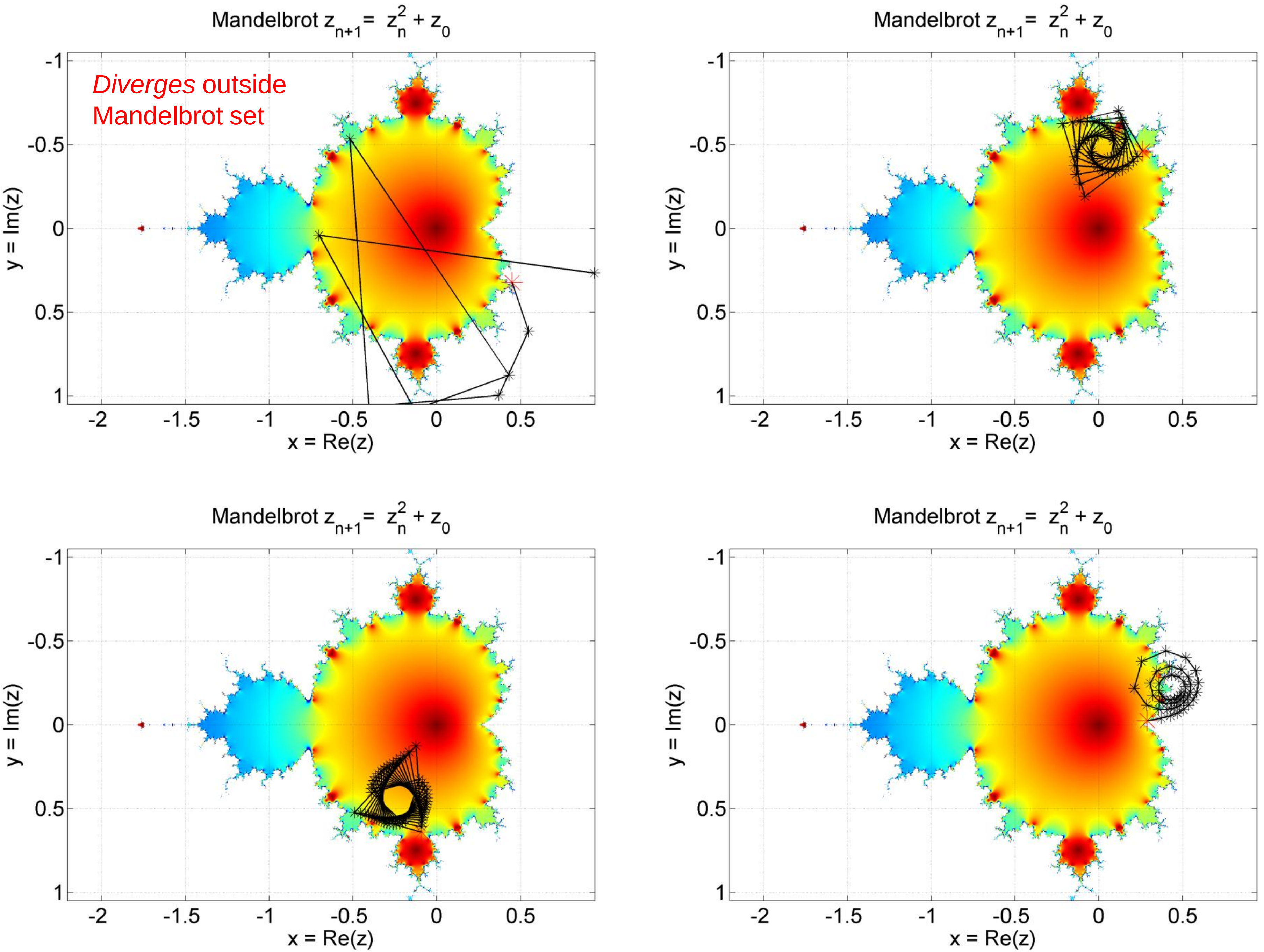
$$= 1 + 2i + i^2$$

$$= 1 + 2i - 1$$

$$= 2i$$

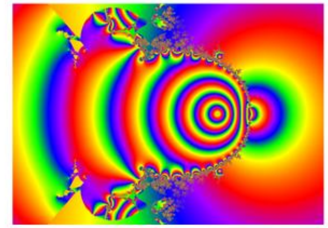
$$z_{n+1} = z_n^2 + z_0$$

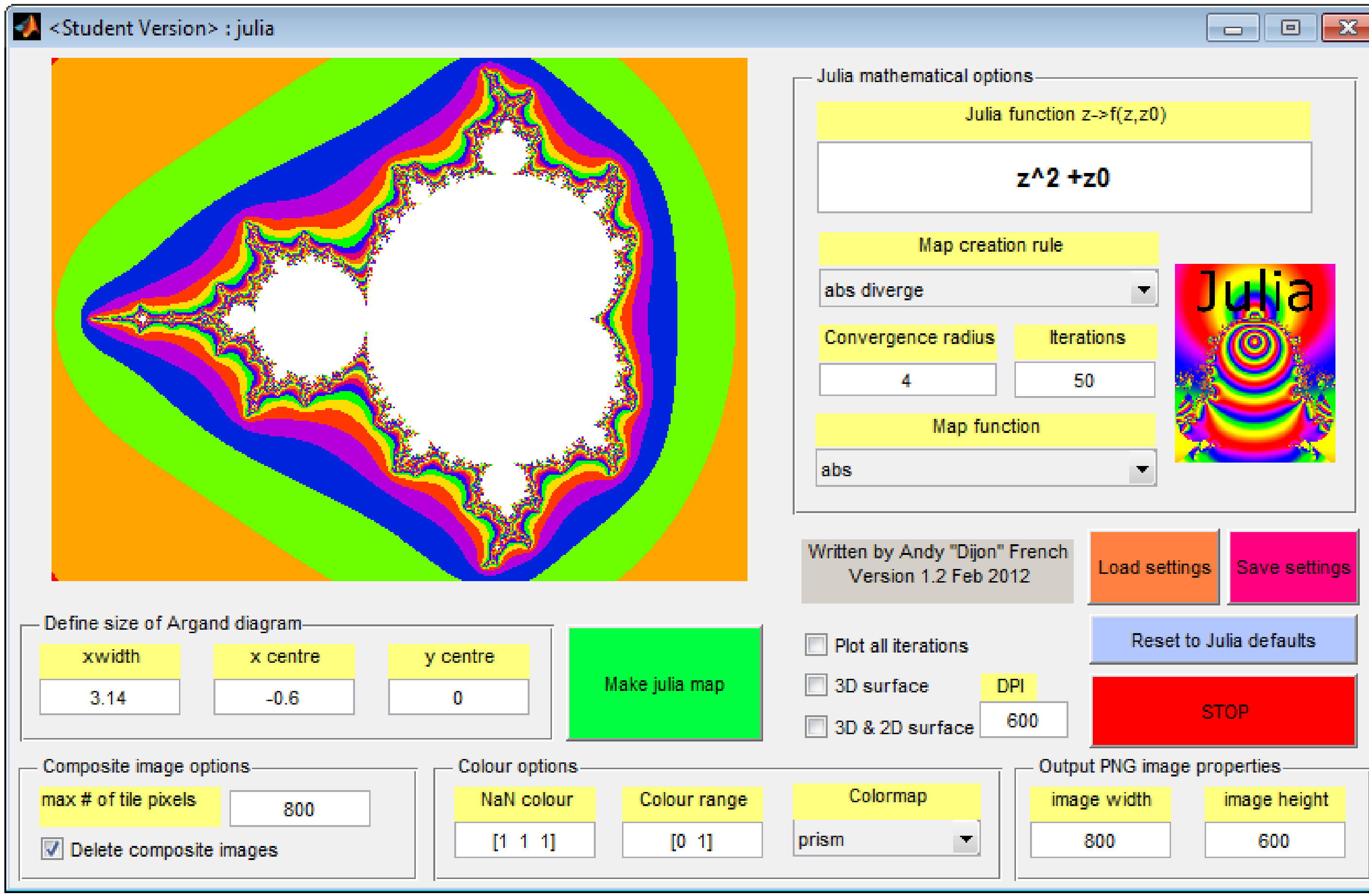


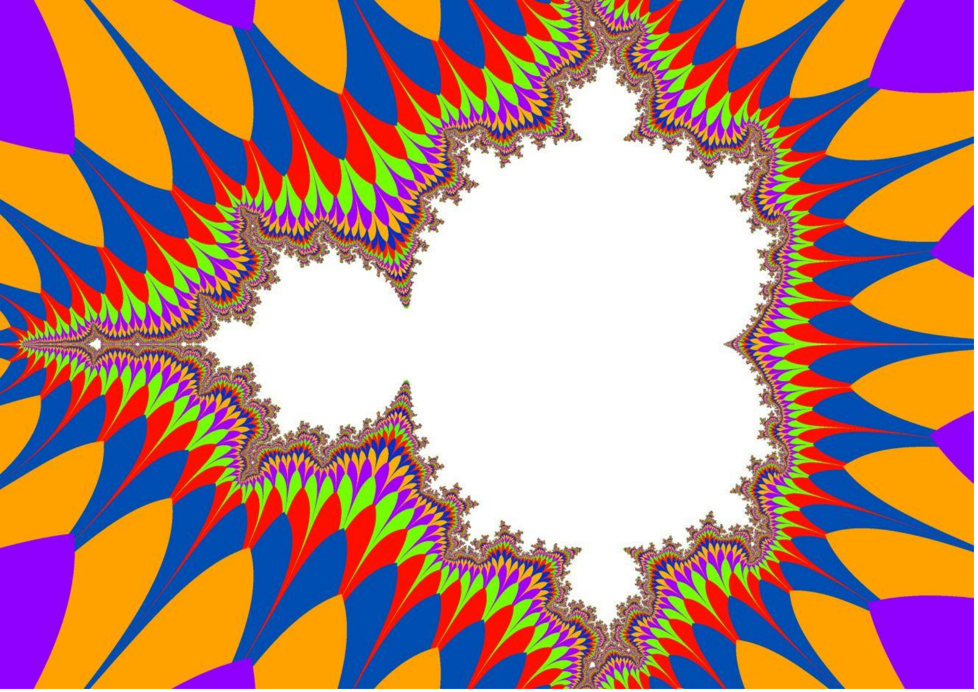

Mandelbrot $z_{n+1} = z_n^2 + z_0$
Diverges outside Mandelbrot set
y = Im(z)
x = Re(z)
-1
-0.5
0
0.5
1
-2
-1.5
-1
-0.5
0
0.5
Mandelbrot $z_{n+1} = z_n^2 + z_0$
y = Im(z)
x = Re(z)
Mandelbrot $z_{n+1} = z_n^2 + z_0$
y = Im(z)
x = Re(z)
Mandelbrot $z_{n+1} = z_n^2 + z_0$
y = Im(z)
x = Re(z)

Gaston Julia
(1893-1978)

julia



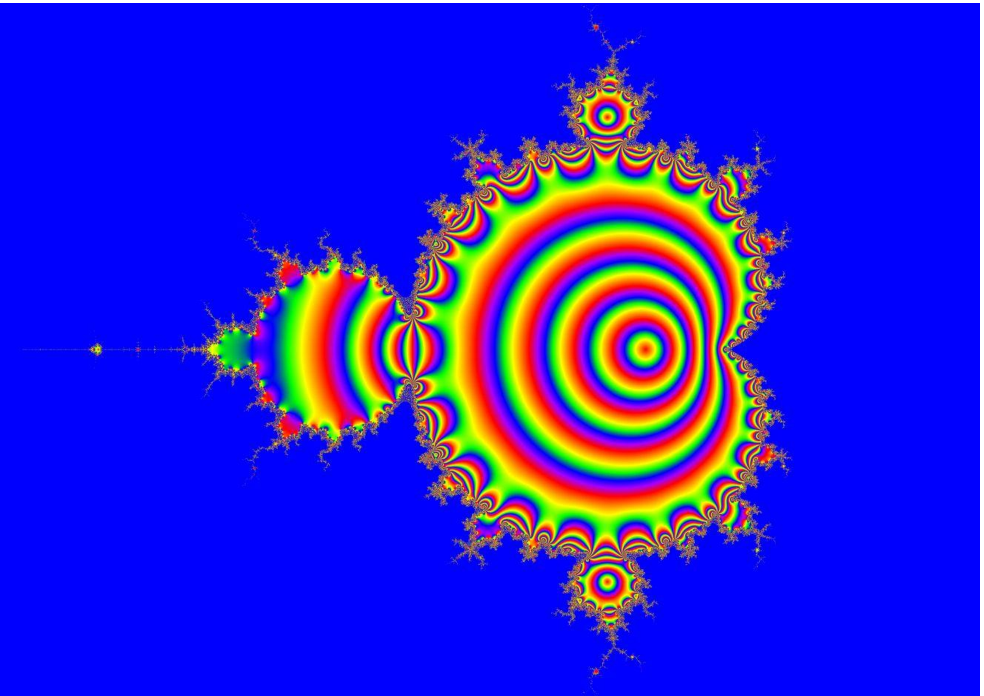




`julia.m` plot option `abs diverge`

Plot a surface with height *h(x,y)*. This is the *iteration number* when $|z|$ exceeds a certain value e.g. 4

In this case *colours* indicate height *h(x,y)*. It is a 'colour-map'.

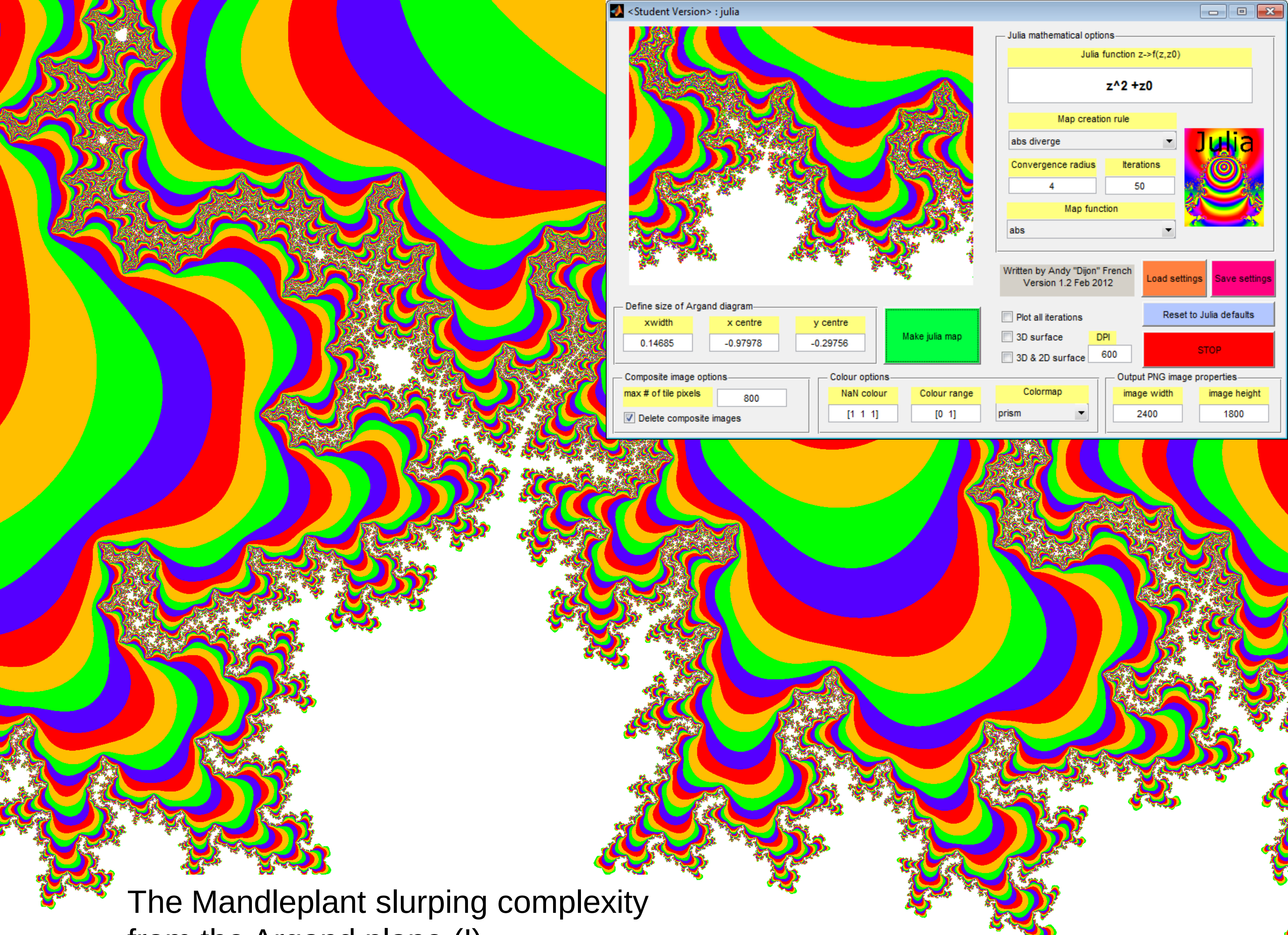


`julia.m` plot option `plot z`

Plot a surface with height *h(x,y)*

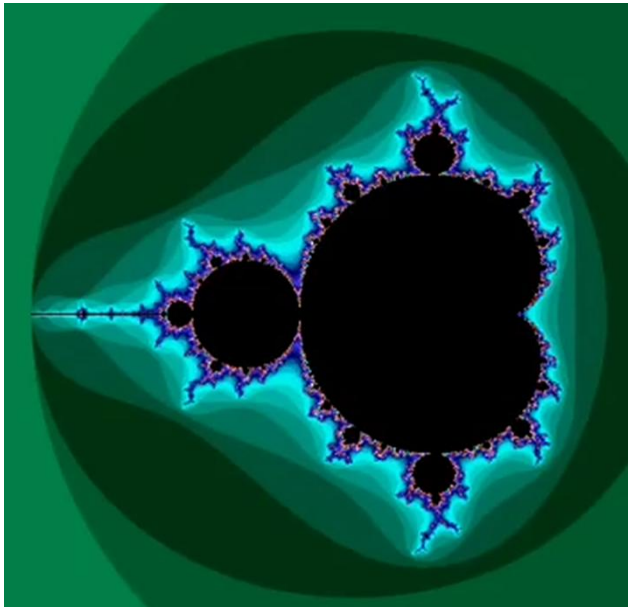
$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

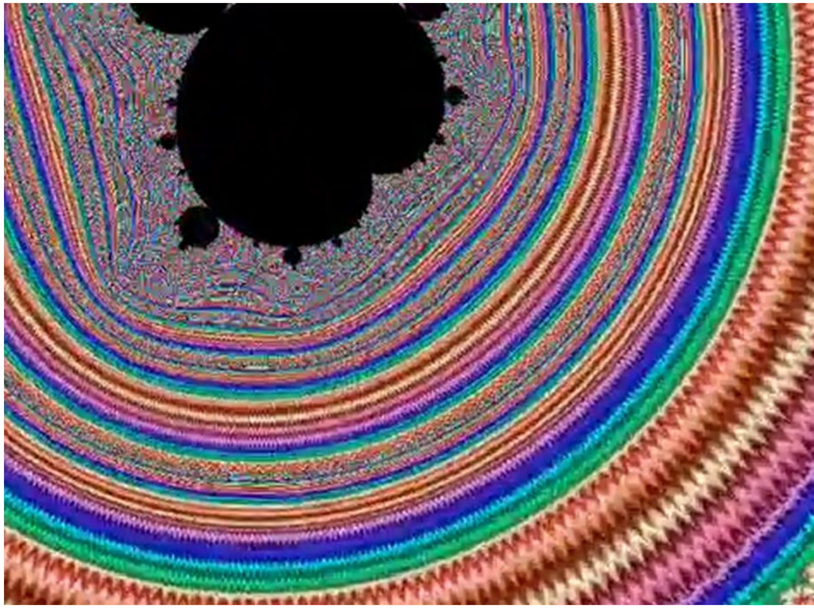
$$h(x,y) = e^{-\sqrt{x^2+y^2}}$$

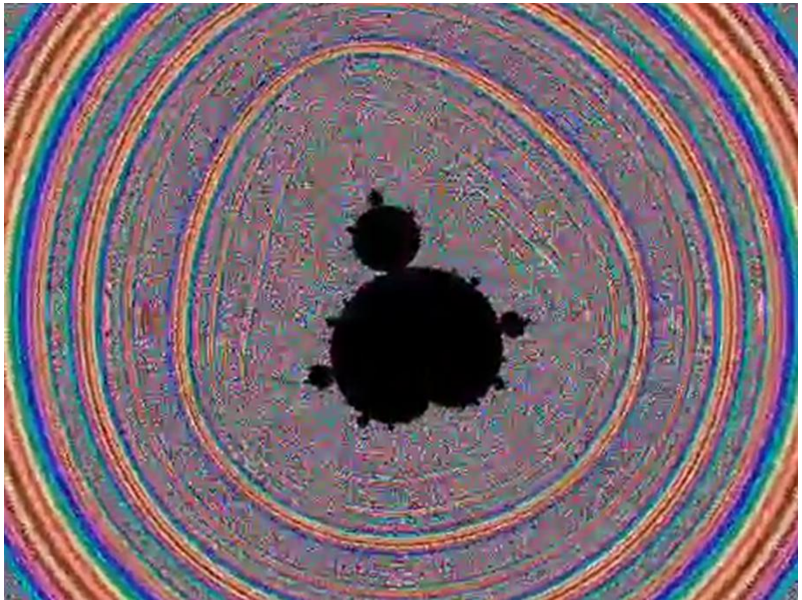


The Mandleplant slurping complexity from the Argand plane (!)

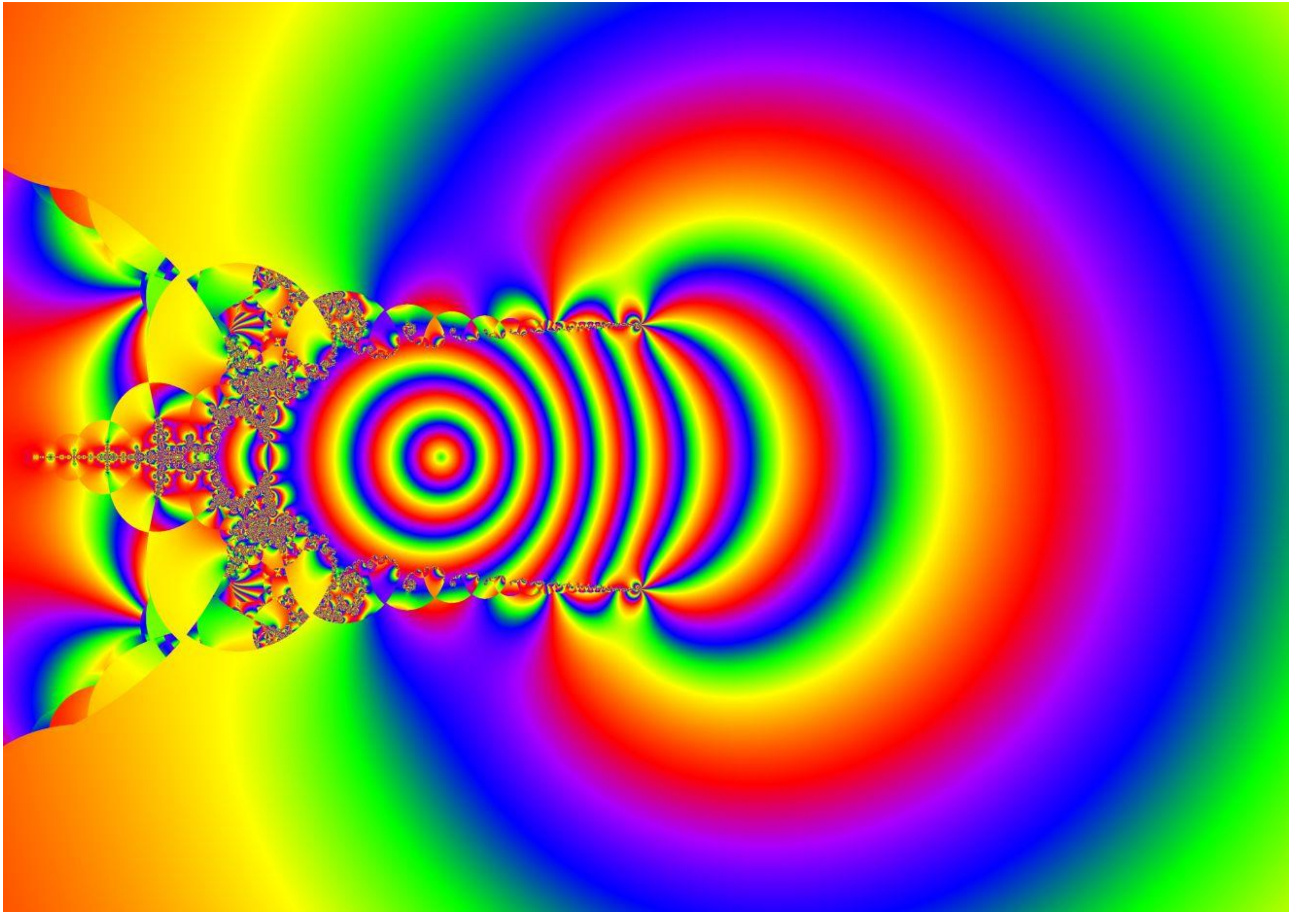
Mandelbrot Deep Zoom







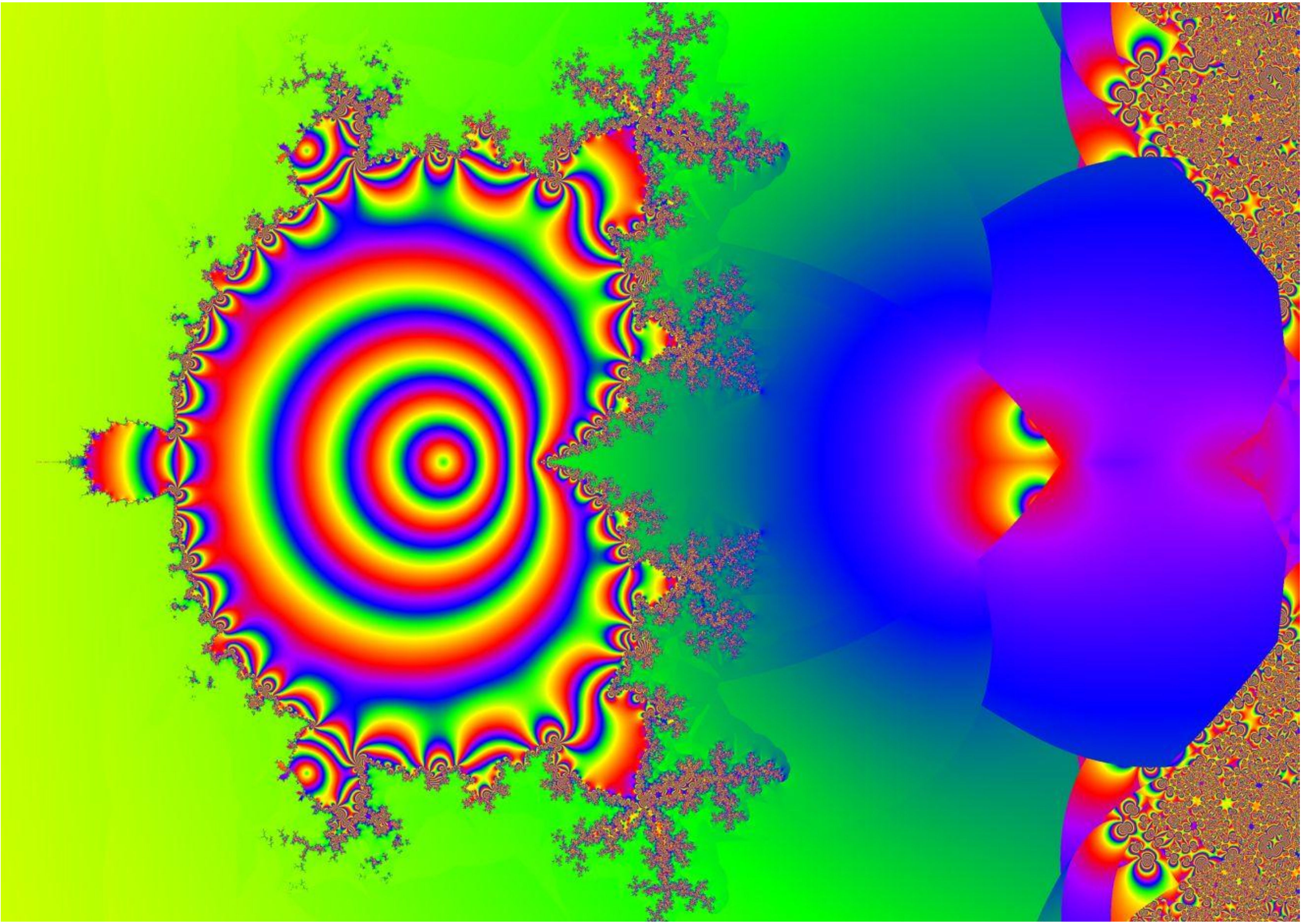

The
Mandlebrot
Variations



The light bulb

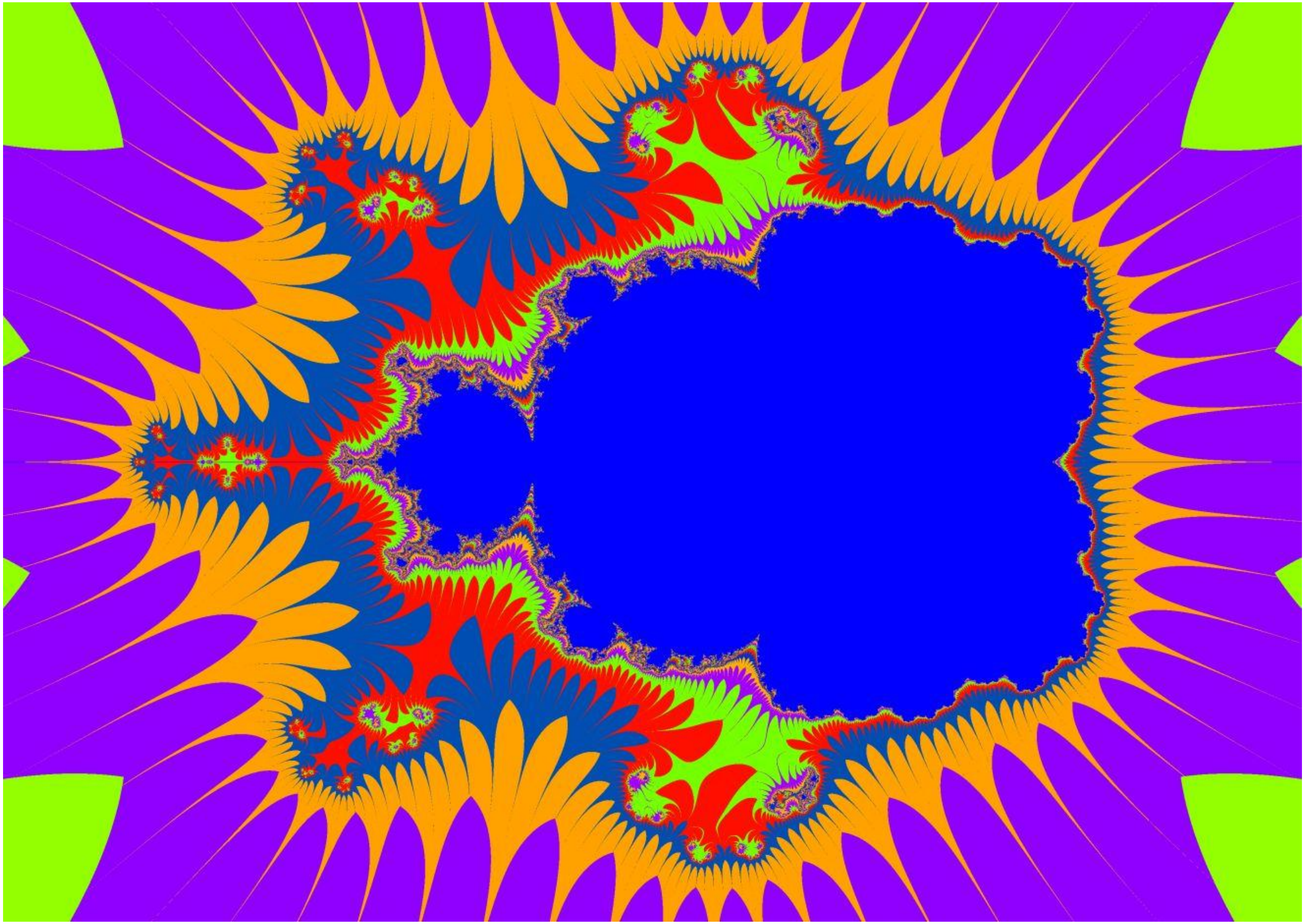
$$z_{n+1} = \log\left(z_n^2 + z_0\right)$$

7 steps to enlightenment $z_{n+1} = \tan^{-1}\left(z_n^2 + z_0\right)$



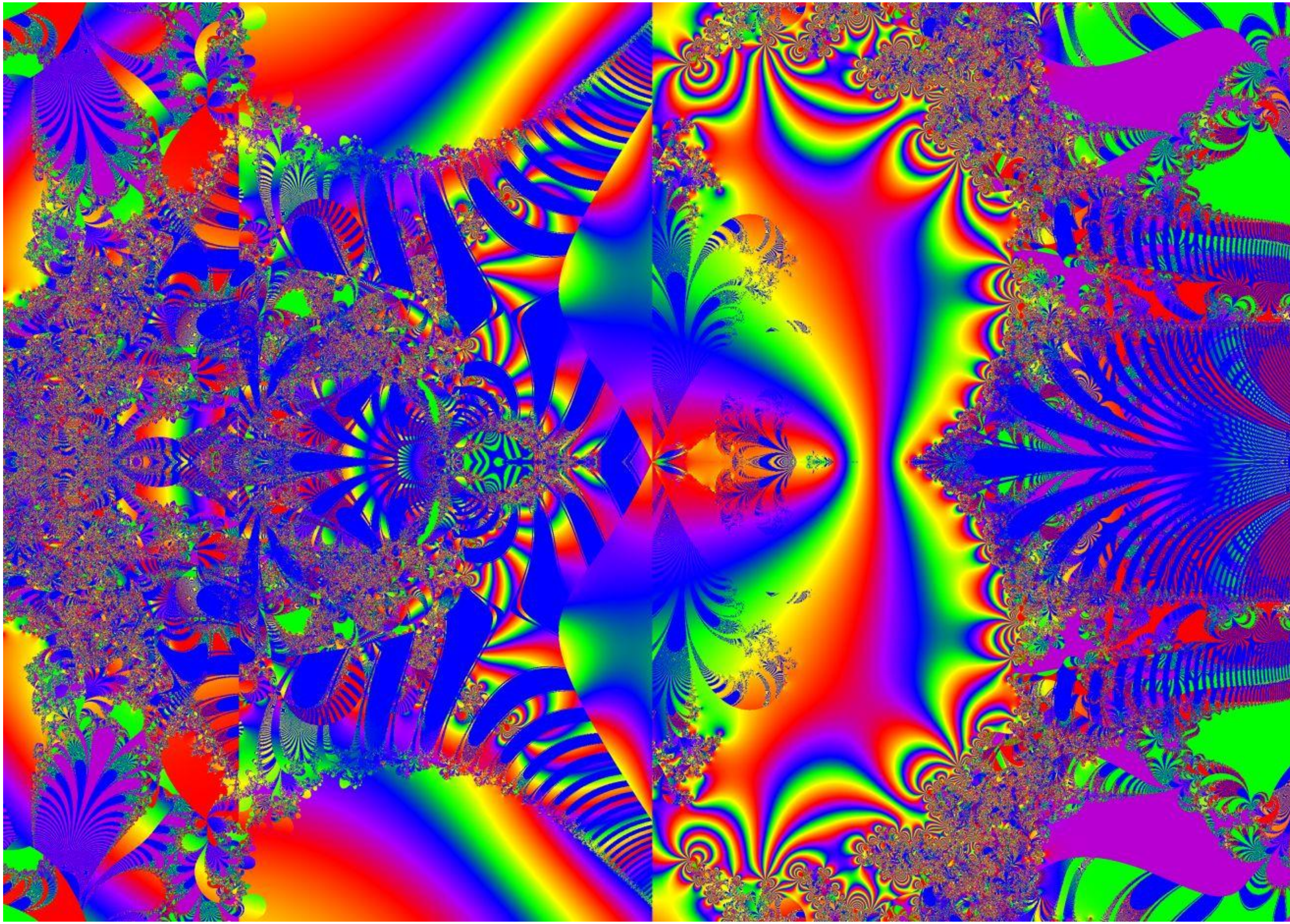
The Mandlerocket!

$$z_{n+1} = \sin^{-1}\left(z_n^2 + z_0\right)$$



Micro mandlebeast

$$z_{n+1} = \left(z_n^2 + z_0 \right)^2$$

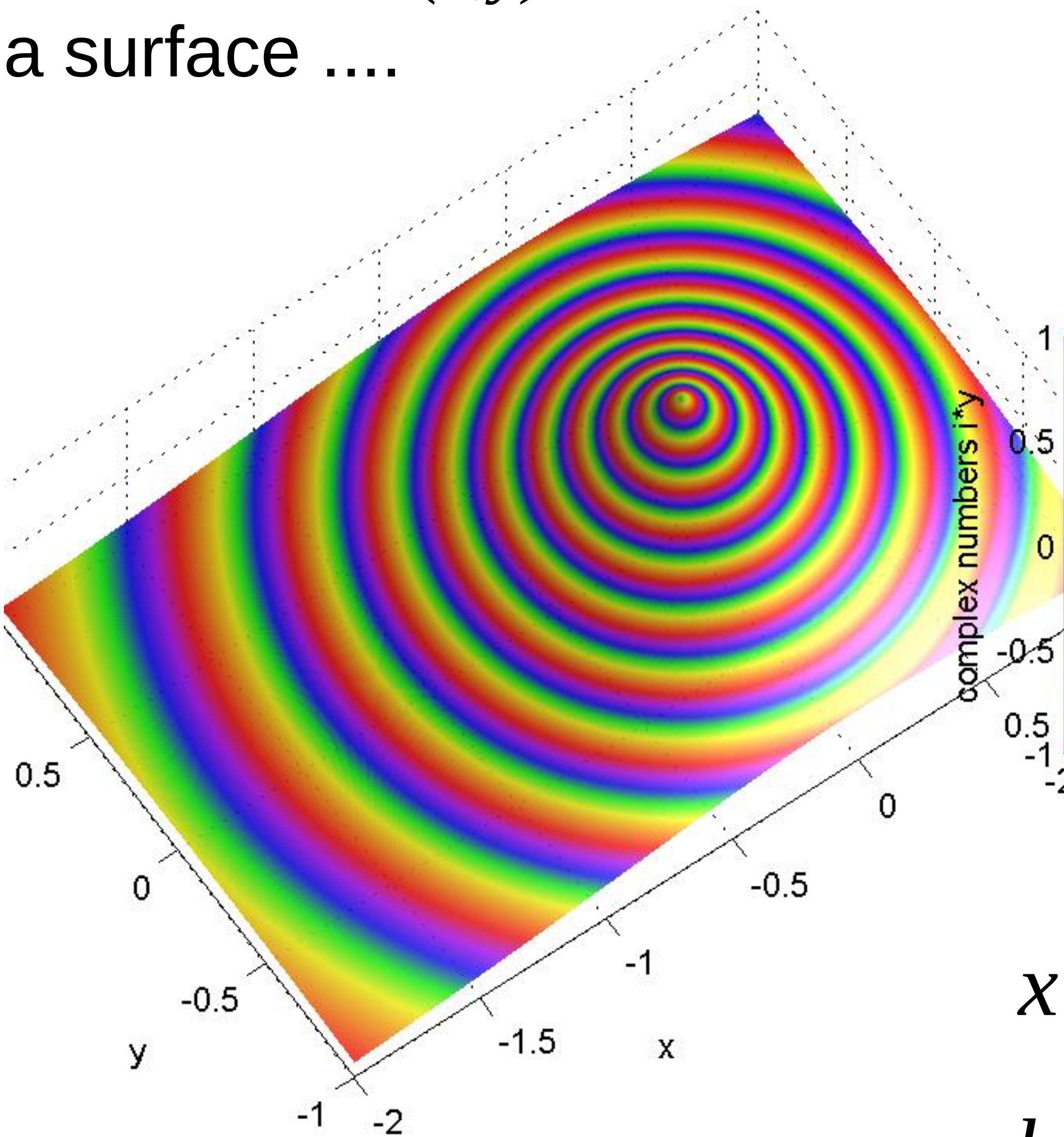


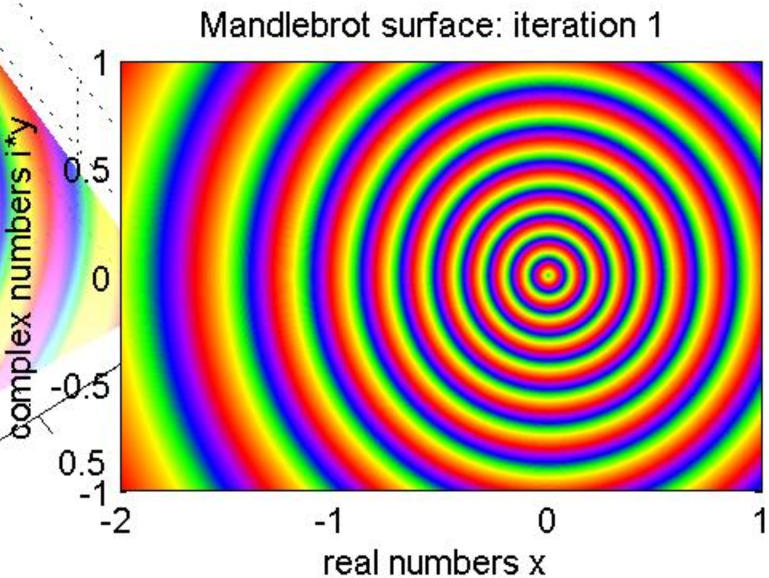
The profusion of power

$$z_{n+1} = \left(z_n^2 + z_0 \right)^{z_n}$$

Remember *h(x,y)* is
a surface ....

$$z_{n+1} = z_n^2 + z_0$$

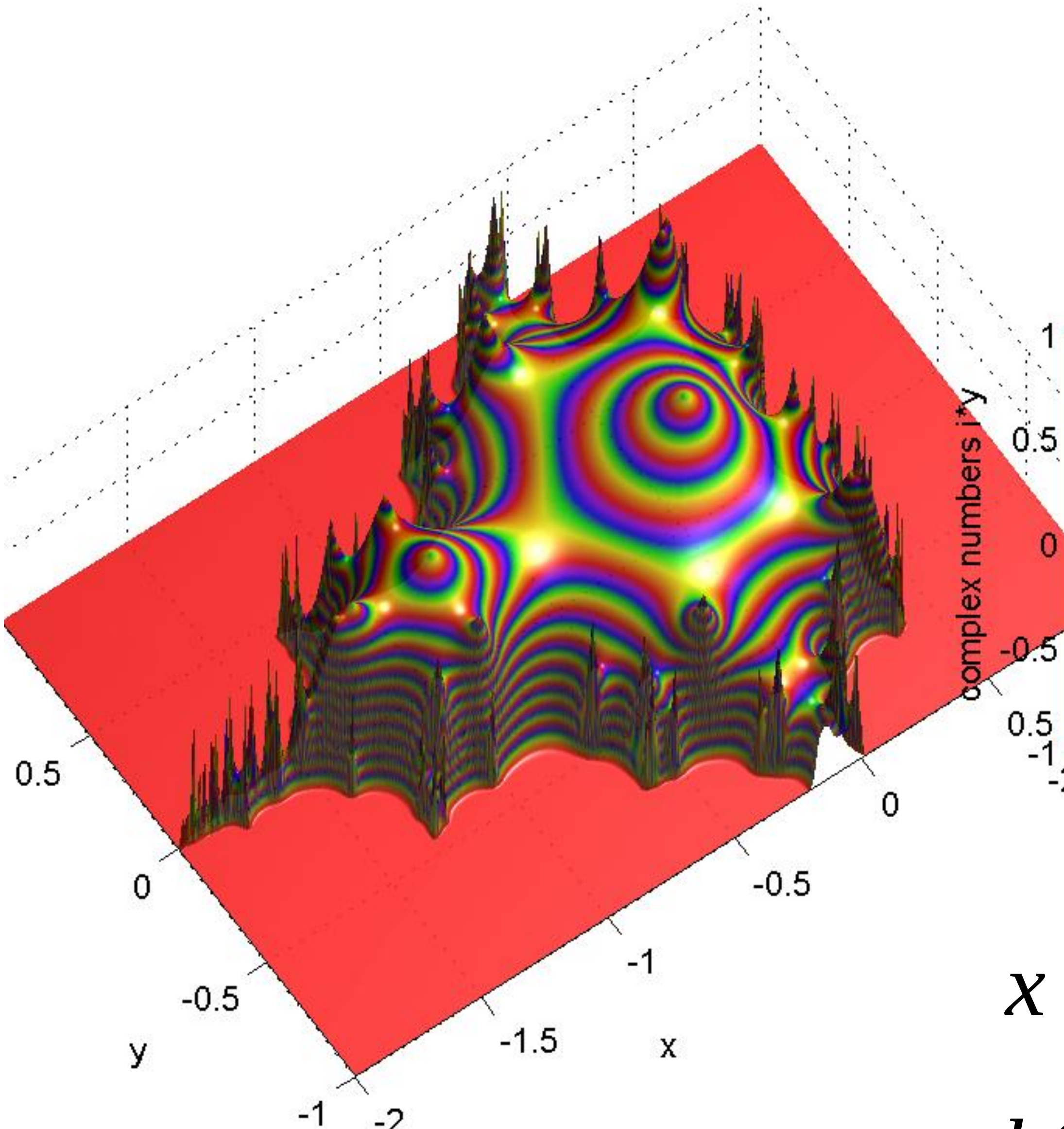


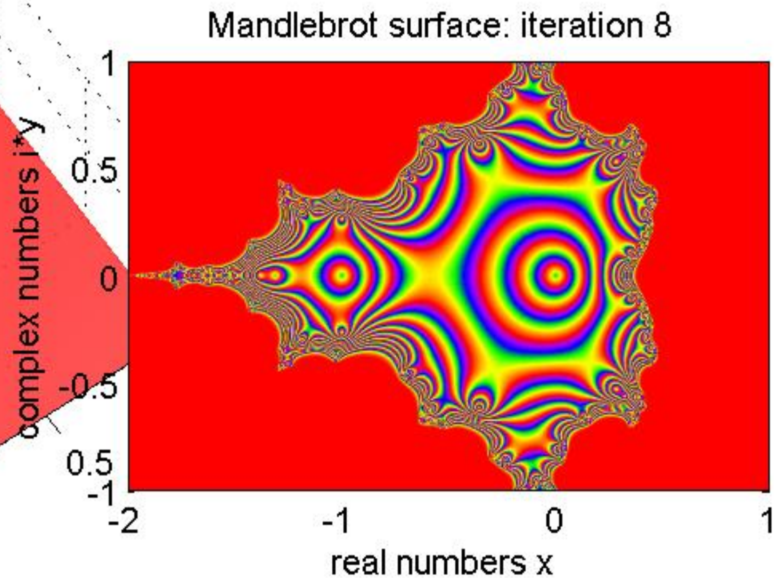


$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

$$h(x,y) = e^{-\sqrt{x^2+y^2}}$$

$$z_{n+1} = z_n^2 + z_0$$

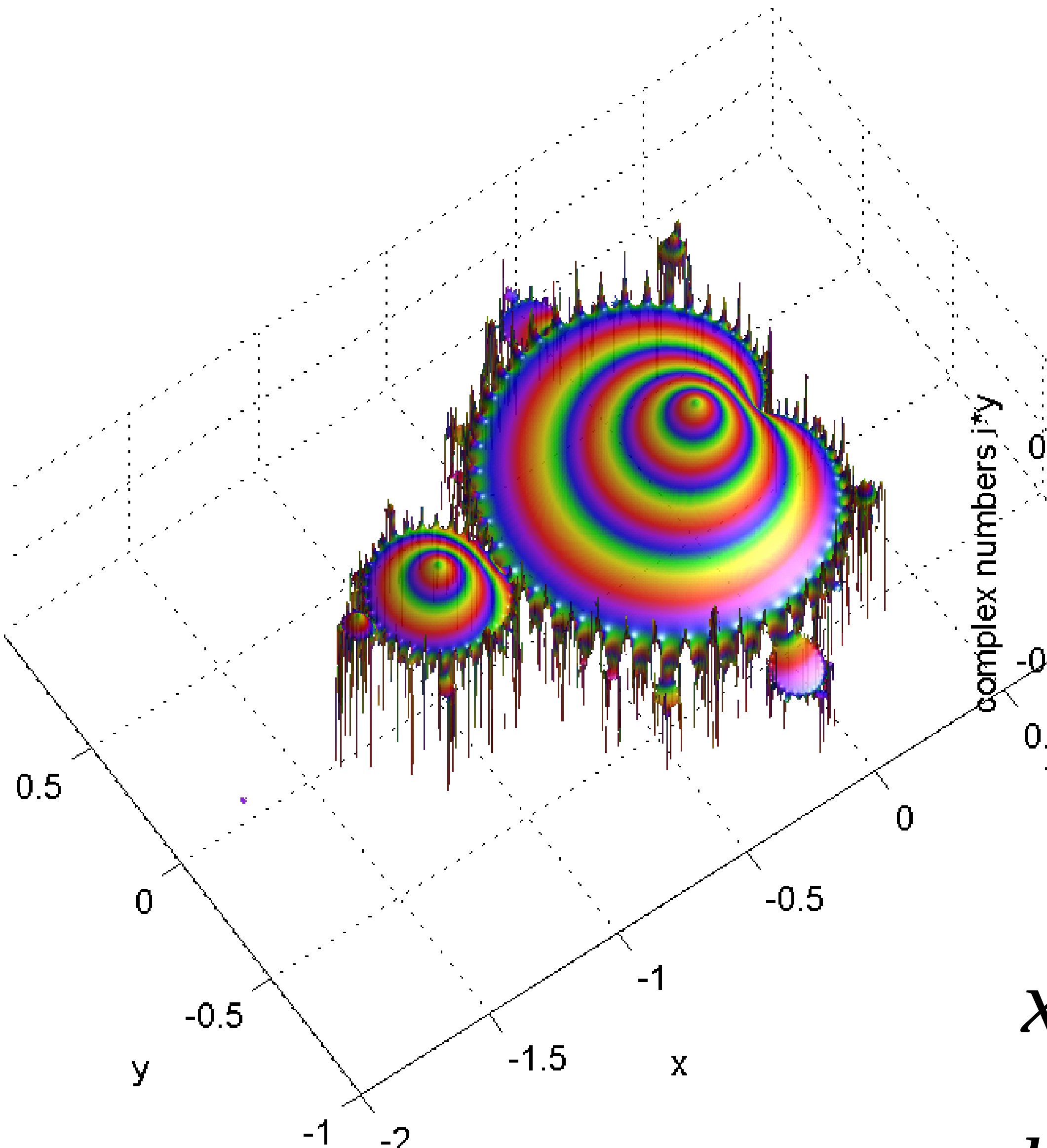


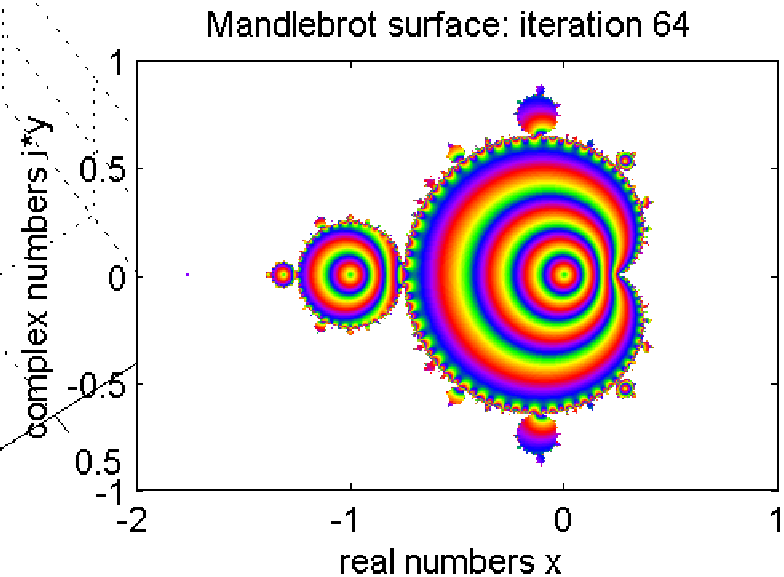


$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

$$h(x, y) = e^{-\sqrt{x^2 + y^2}}$$

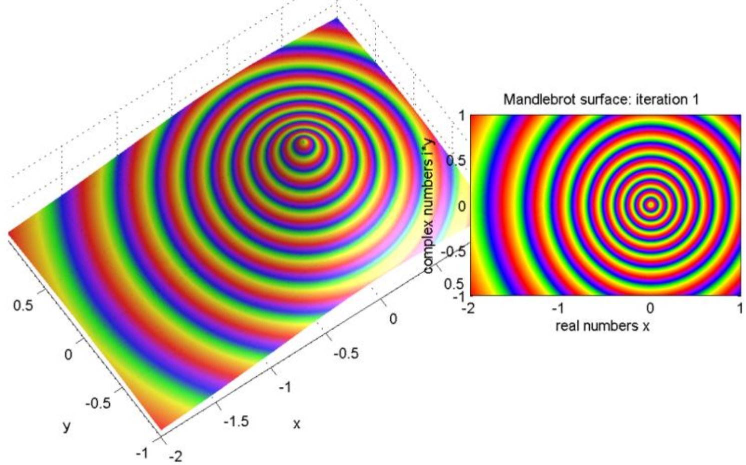
$$z_{n+1} = z_n^2 + z_0$$

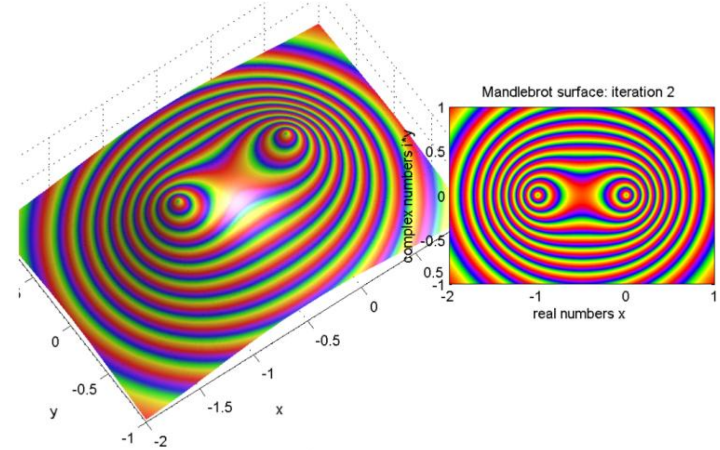


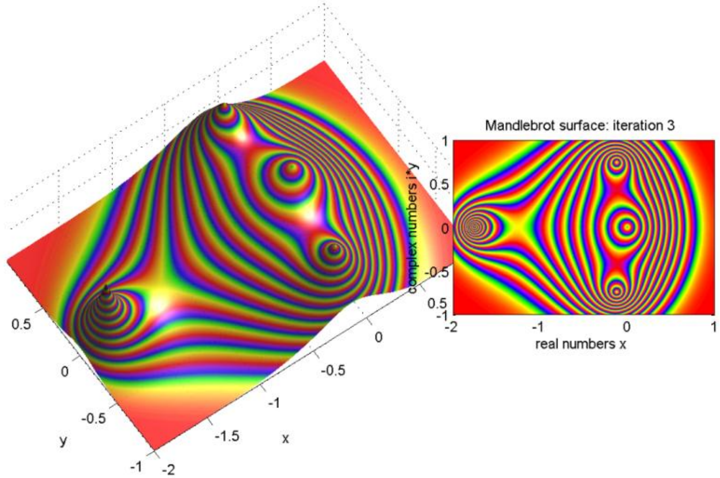


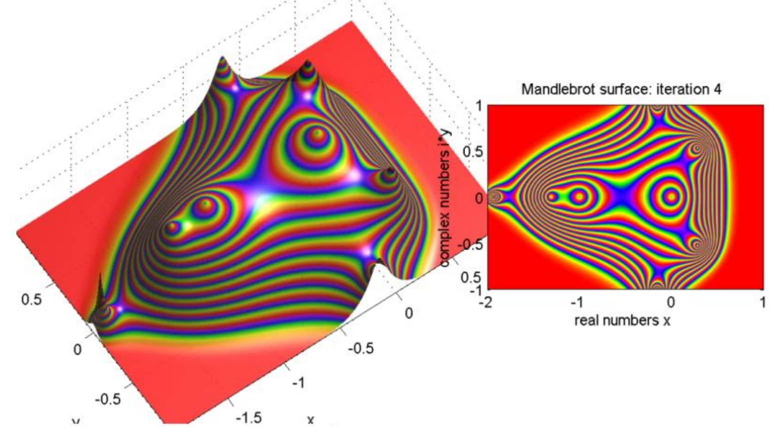
$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

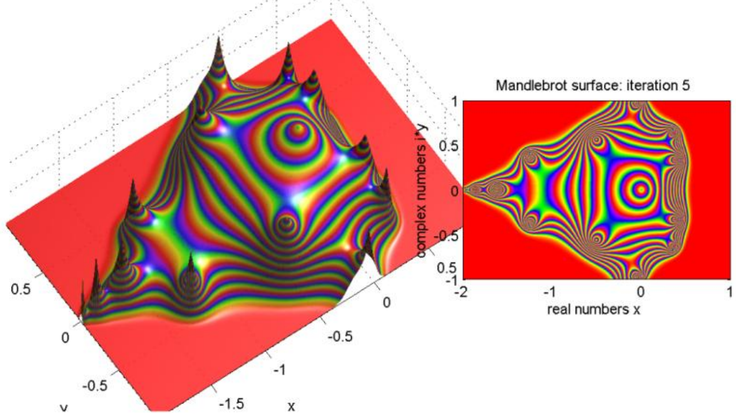
$$h(x, y) = e^{-\sqrt{x^2 + y^2}}$$

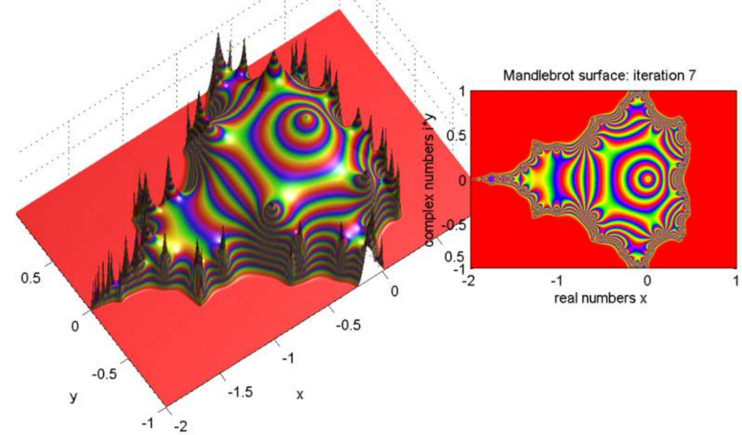

Mandlebrot surface: iteration 1
complex numbers i*y
real numbers x
x
y

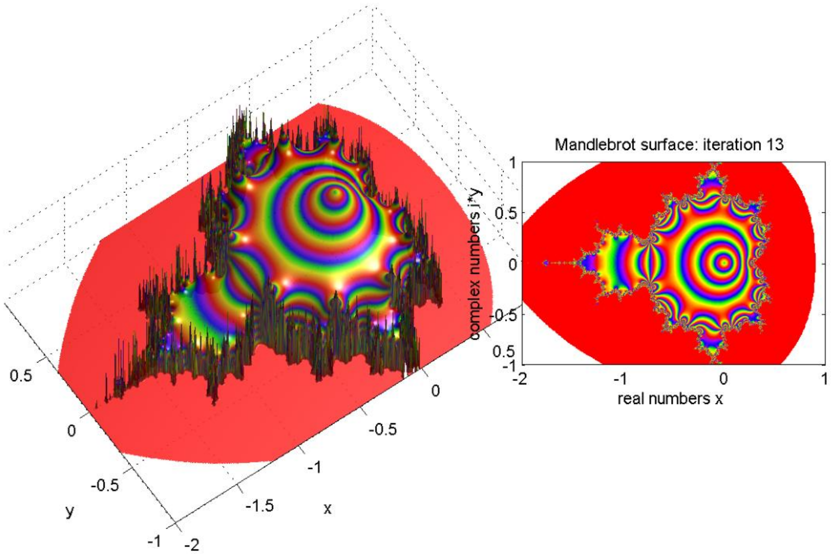

Mandlebrot surface: iteration 2
complex numbers i*y
real numbers x
x
y

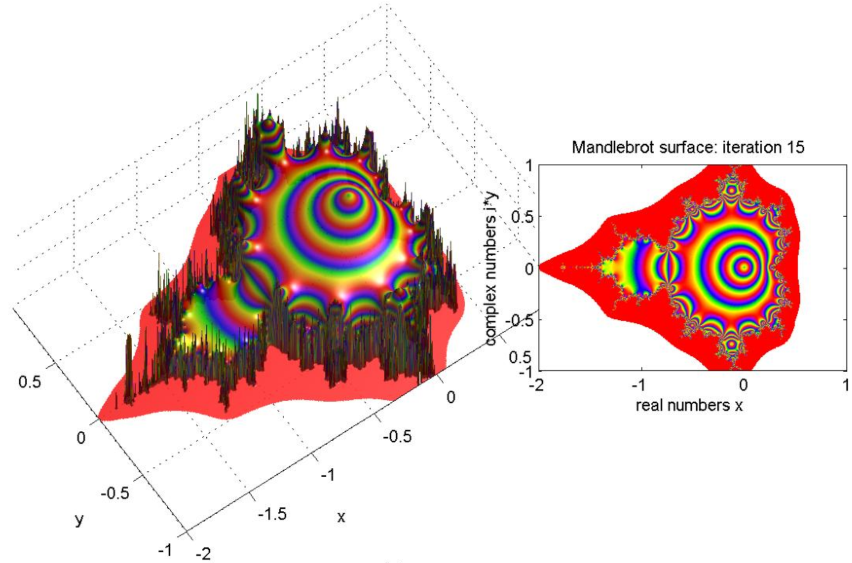

Mandlebrot surface: iteration 3
complex numbers i*y
real numbers x
x
y

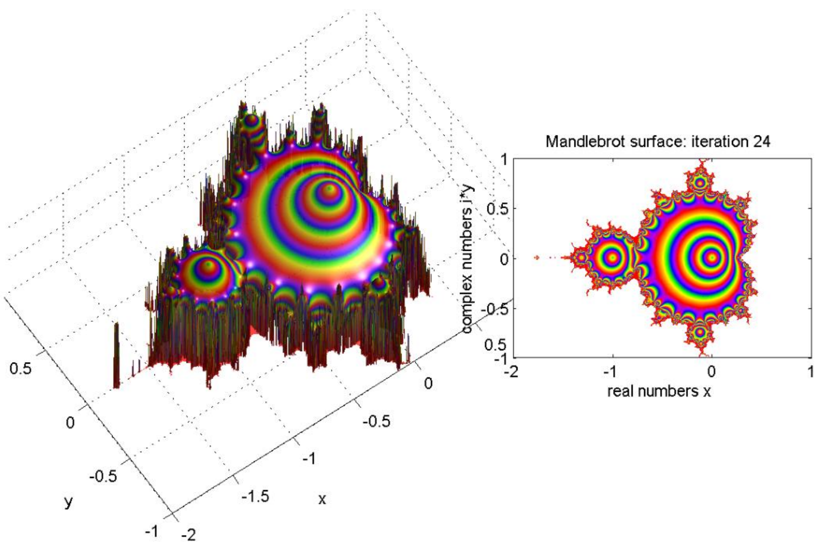

Mandlebrot surface: iteration 4
complex numbers i*y
real numbers x
x

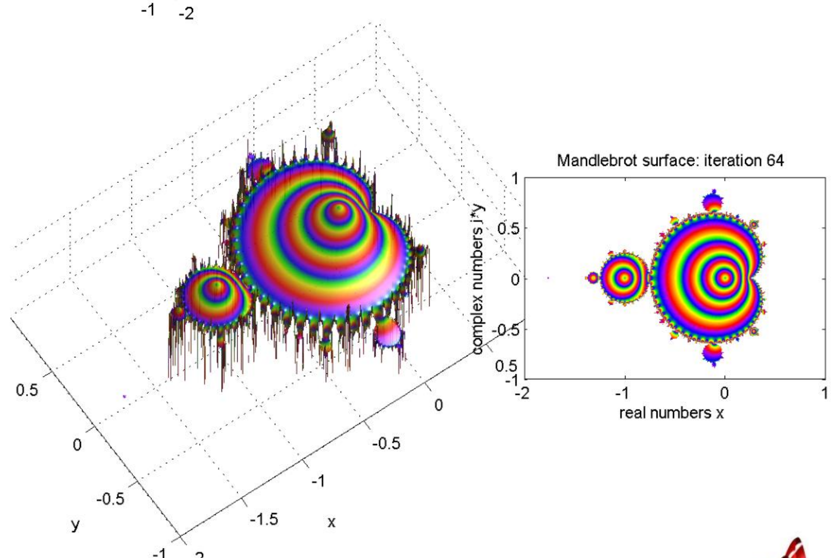

Mandlebrot surface: iteration 5
complex numbers i*y
real numbers x
x


Mandlebrot surface: iteration 7
complex numbers i*y
real numbers x
x
y







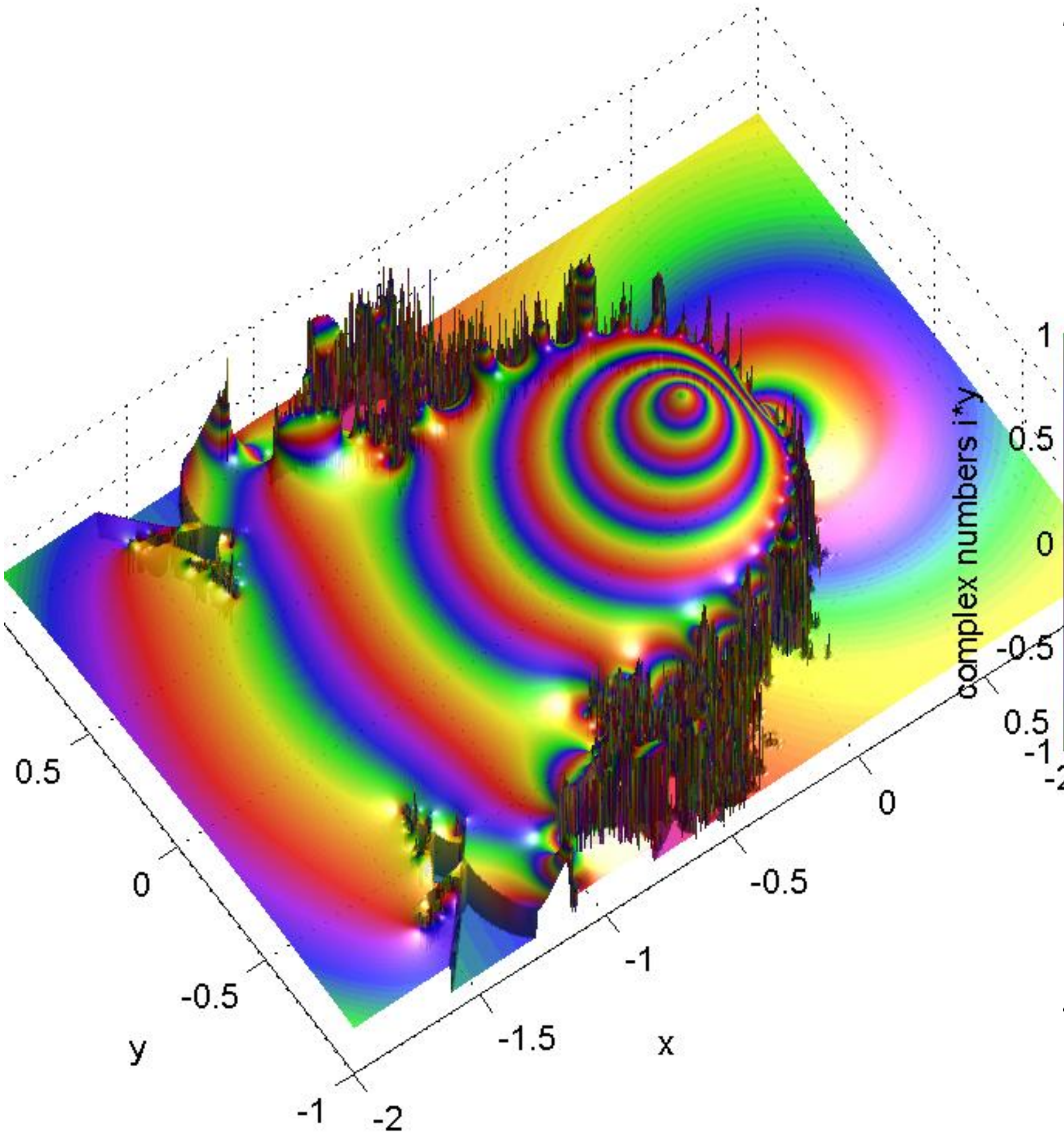


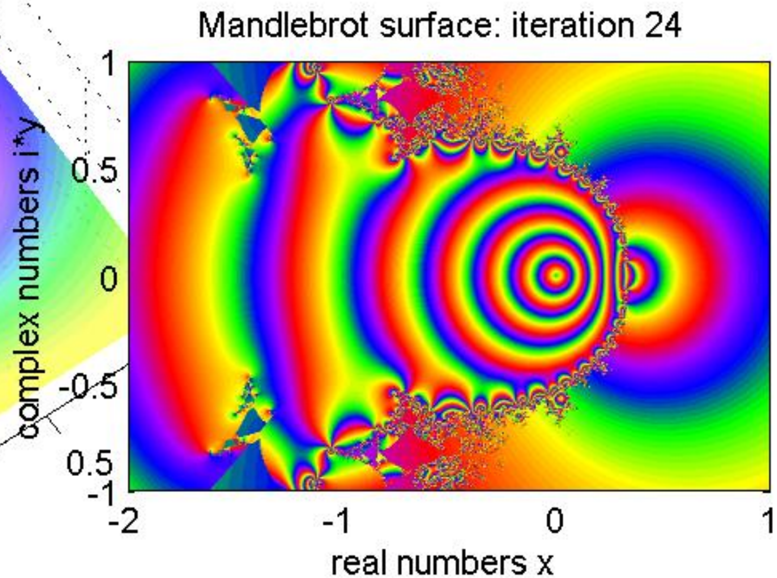
Selection from *Day of Julia.*
Mathematicon Exhibition, 2014

μathematicon

7 steps to enlightenment

$$z_{n+1} = \tan^{-1}\left(z_n^2 + z_0\right)$$



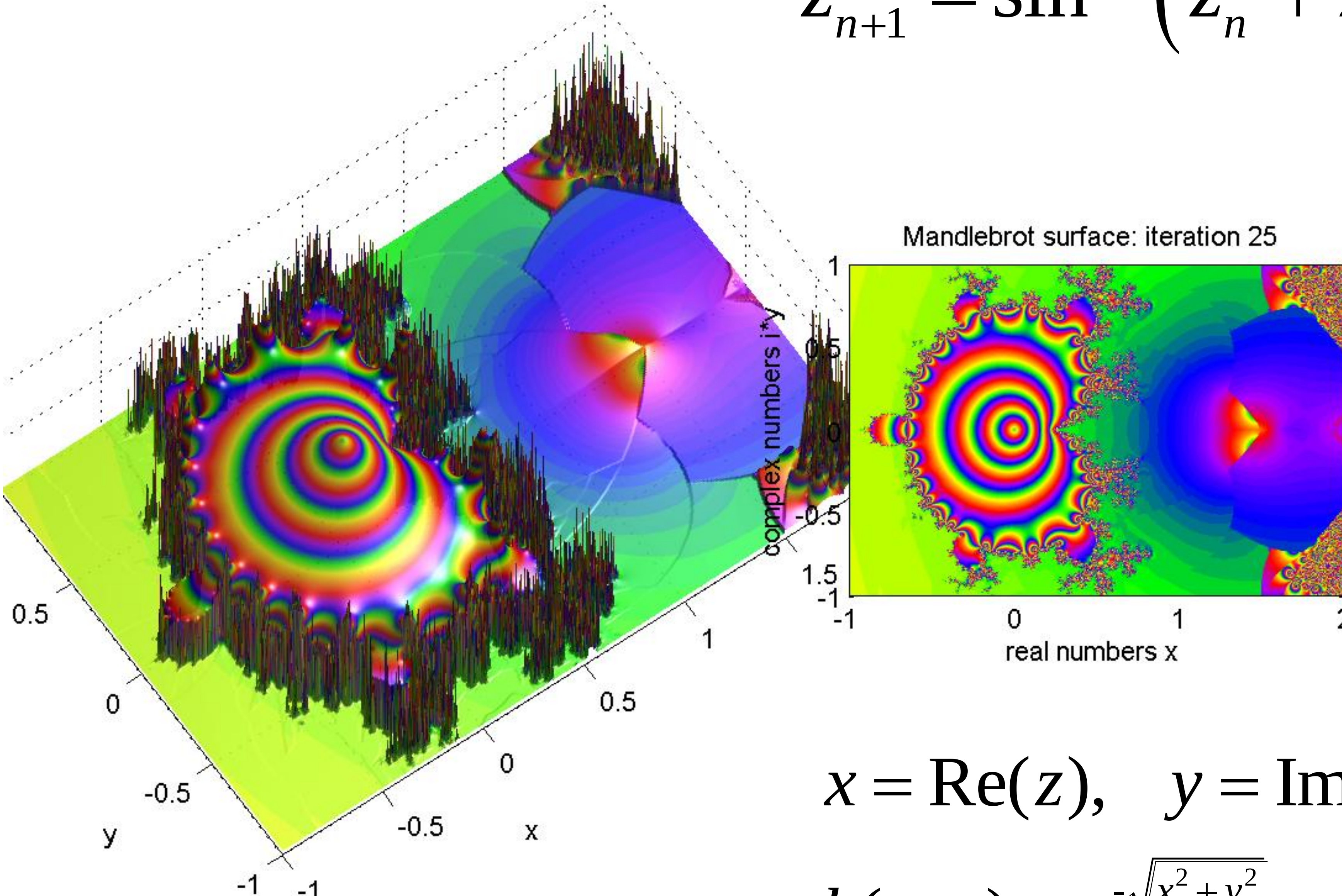


$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

$$h(x, y) = e^{-\sqrt{x^2 + y^2}}$$

The Mandlerocket

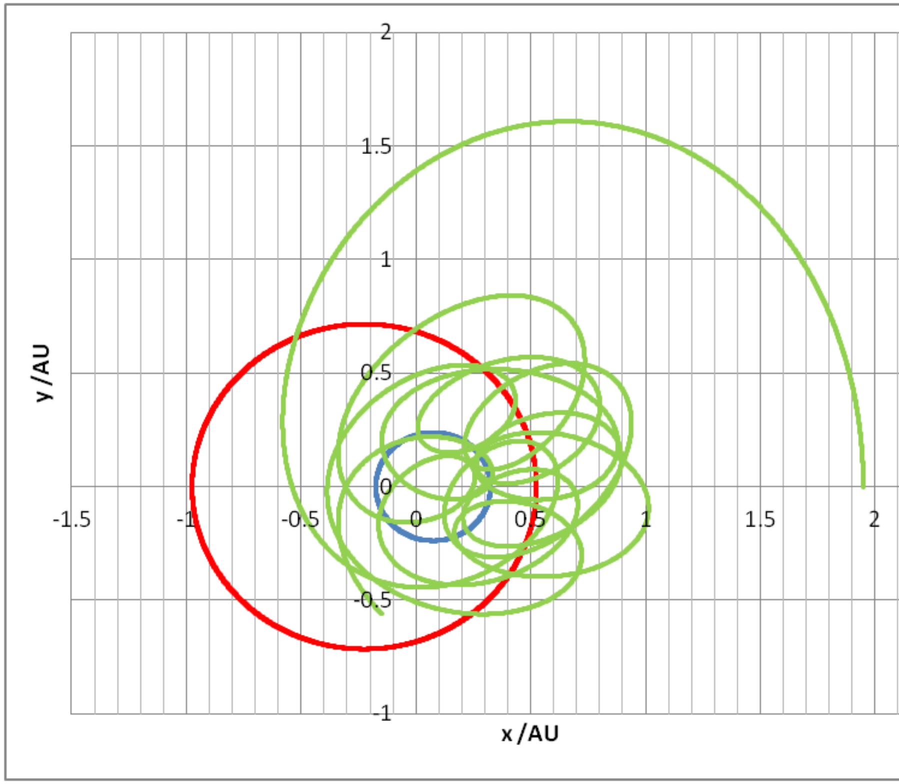
$$z_{n+1} = \sin^{-1}\left(z_n^2 + z_0\right)$$



$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

$$h(x,y) = e^{-\sqrt{x^2+y^2}}$$

Chaos in planetary systems



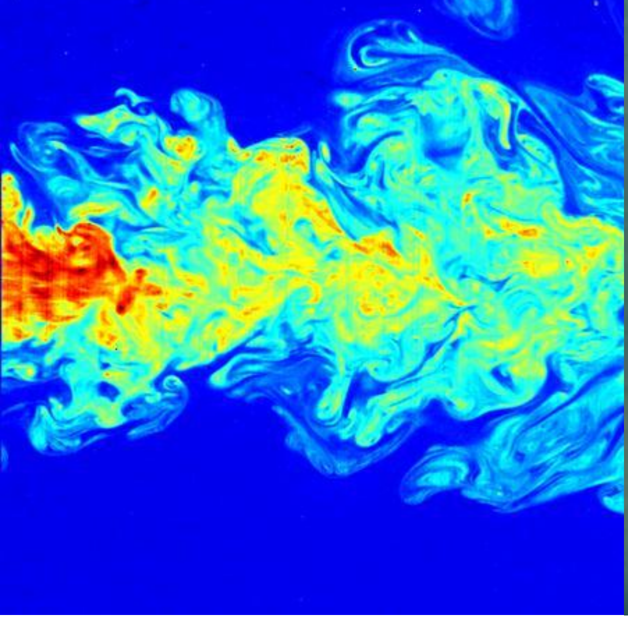
Mass of star1 (solar masses)	6
Mass of star2 (solar masses)	2
Mass of planet (Earth masses)	1
Semi-major axis of initial star orbit /AU	1
Semi-major axis of initial planet orbit /AU about star1	2.12345
Orbital eccentricity of initial star orbit	0.3
Orbital eccentricity of initial planet orbit about star 1	0
Initial polar angle /degrees of initial star orbit	0
Initial polar angle /degrees of initial planet orbit about star 1	0
timestep /years	0.000072

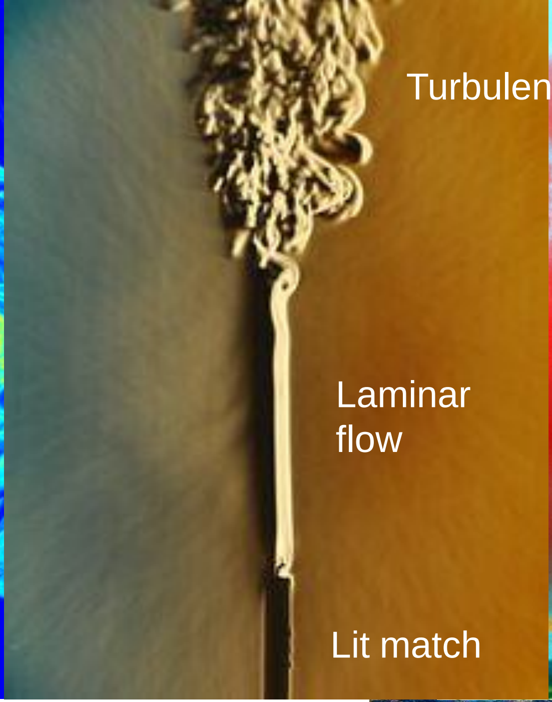
The motion of a planet in a close binary star system can be chaotic

The ‘three body problem’ has no closed form solution!

The small moons of **Pluto** (Nix, Hydra, Styx, and Kerberos) *rotate chaotically* →

Chaos in fluid flow

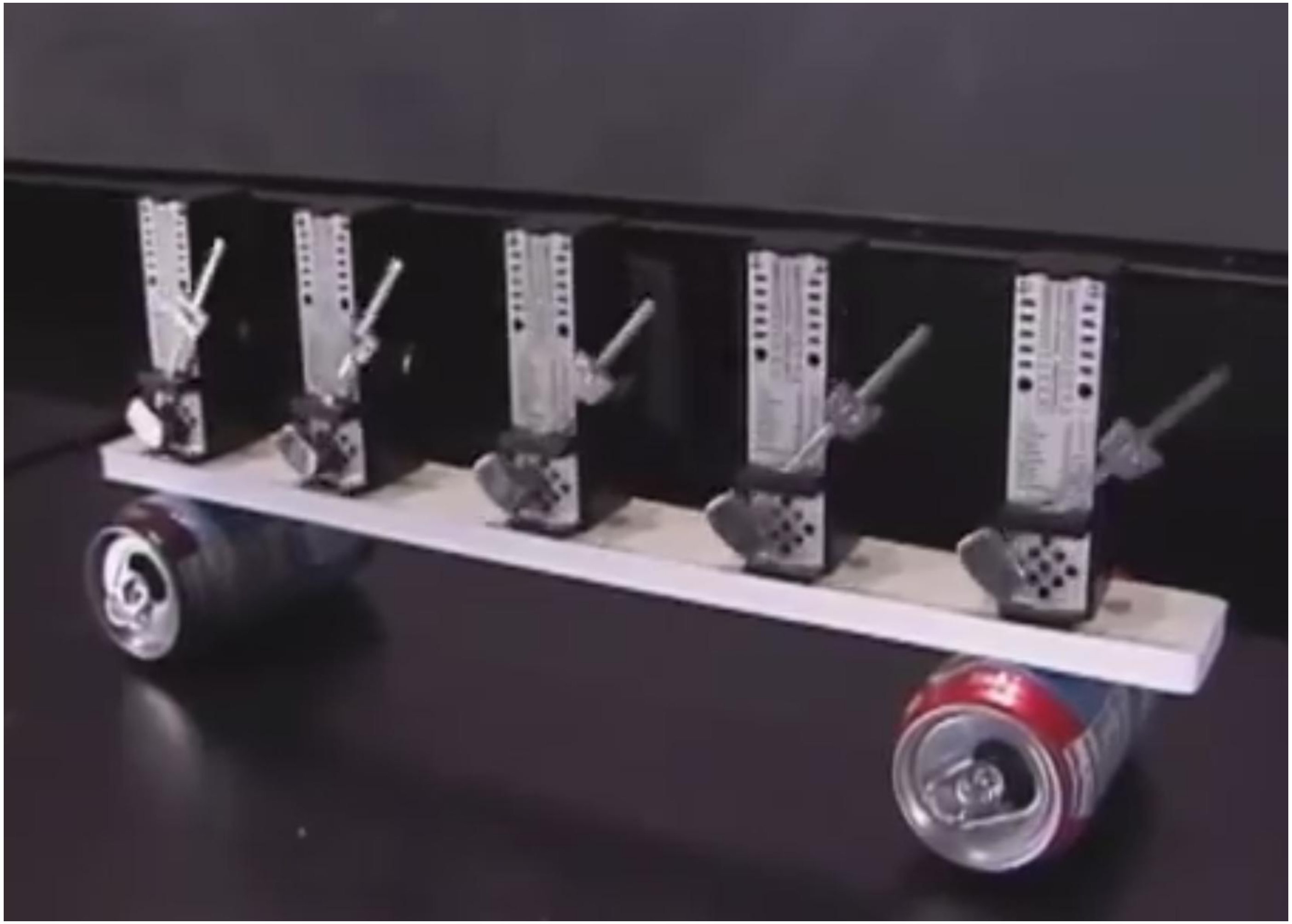




Leonardo da Vinci

Vincent Van Gogh *Starry Night* (1889)

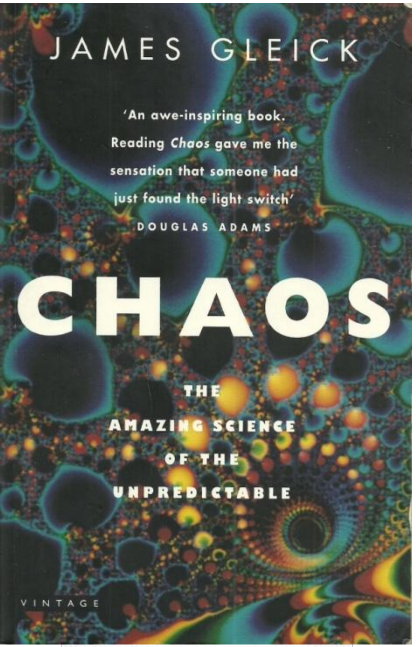
Phase locking - spontaneous ***order*** **from chaos** due to 'nonlinear feedback'

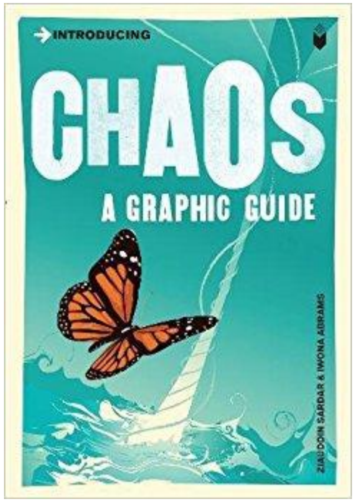


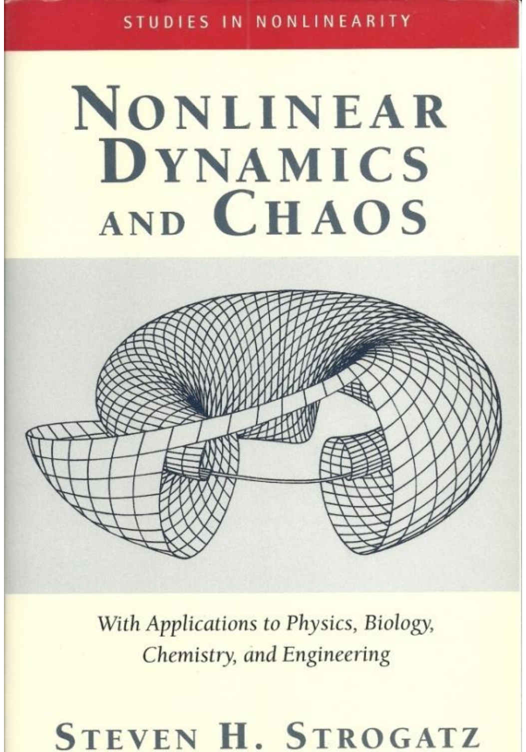
https://www.youtube.com/watch?v=Aaxw4zbULMs

Steven Strogatz **TED** talk–*The Science of Sync*

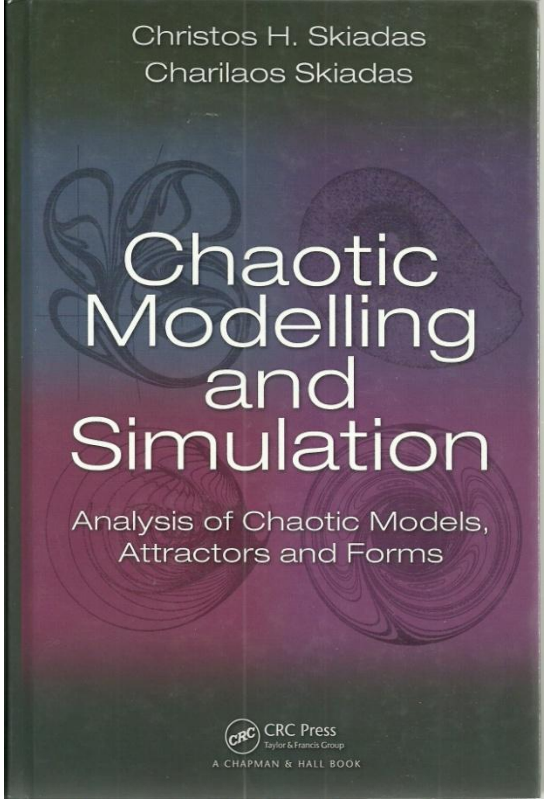
Further reading

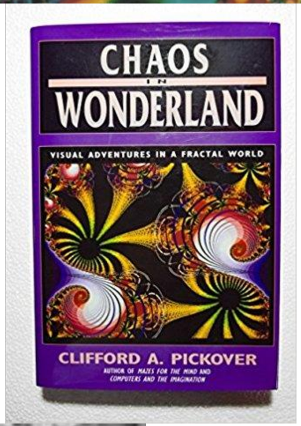


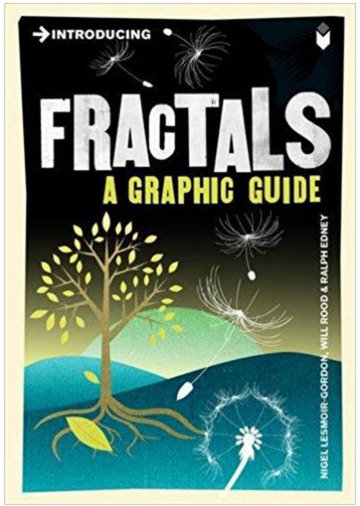


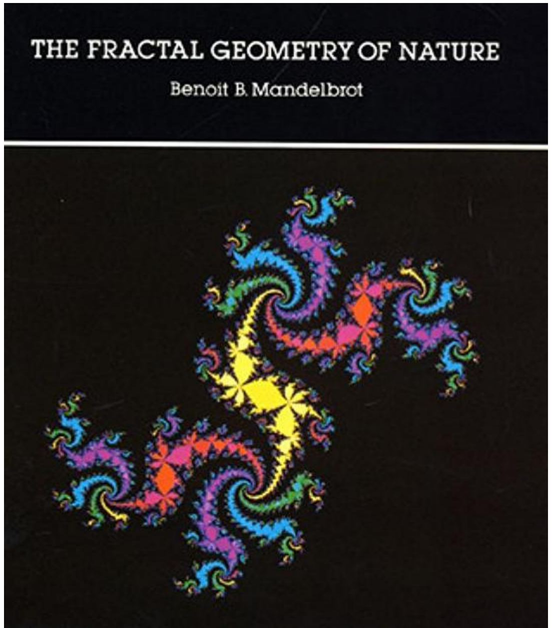


[TED talk–*The Science of Sync*]()





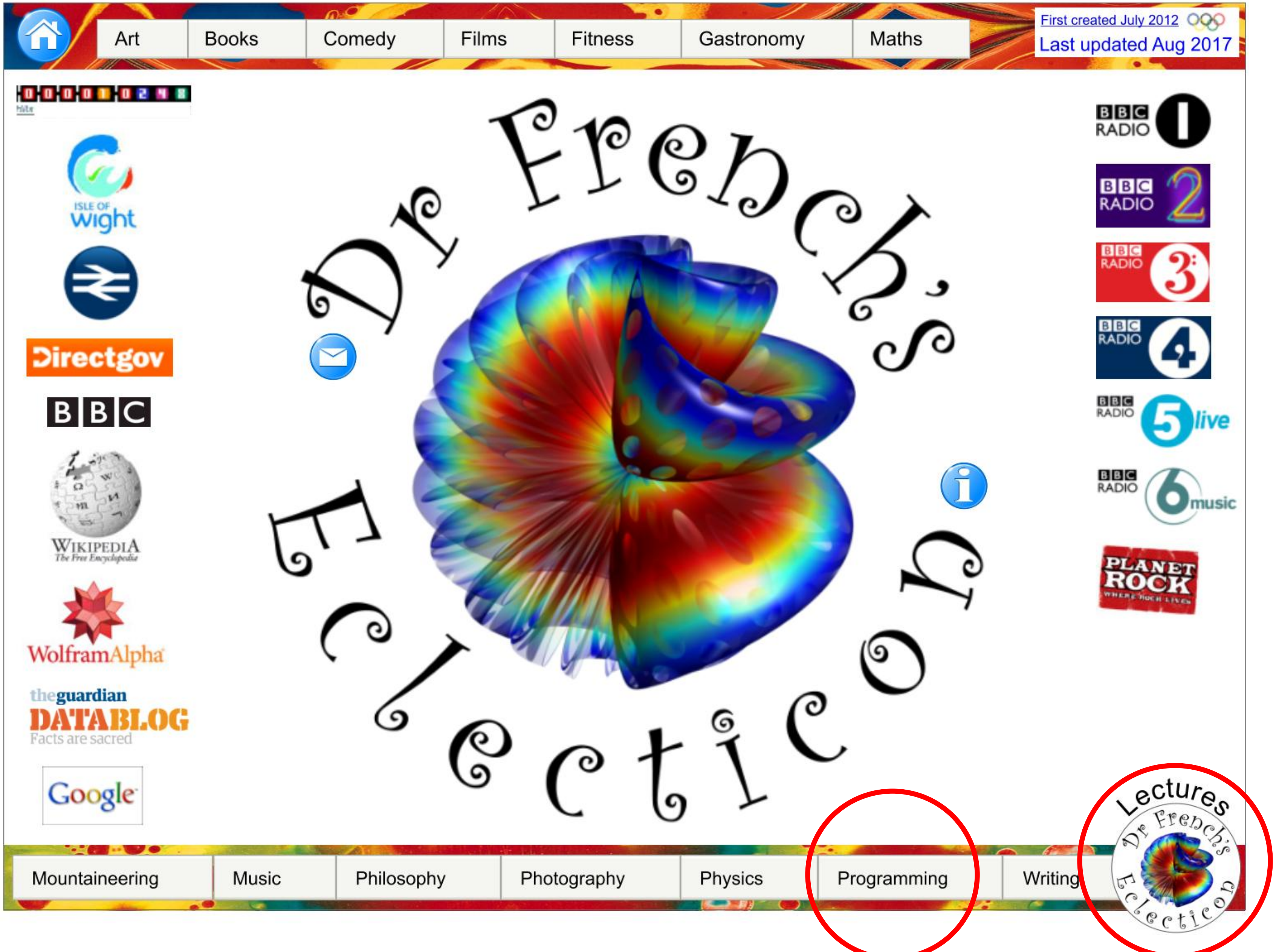




Dr French's Eclecticon

www.eclecticon.info

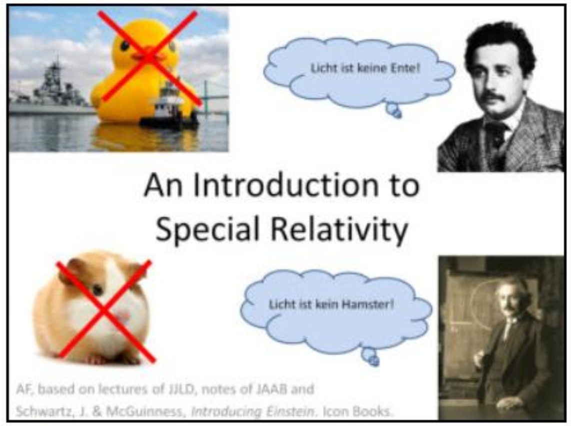
Shaw *et al*; "Chaos", Scientific American 54:12 (1986) 46-57



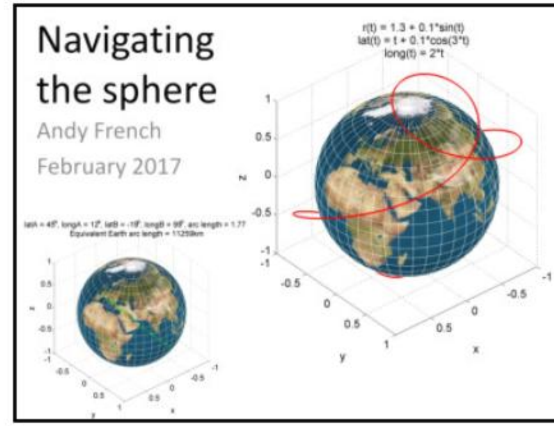
www.eclecticon.info

Lectures (3)

ppt synopsis



ppt



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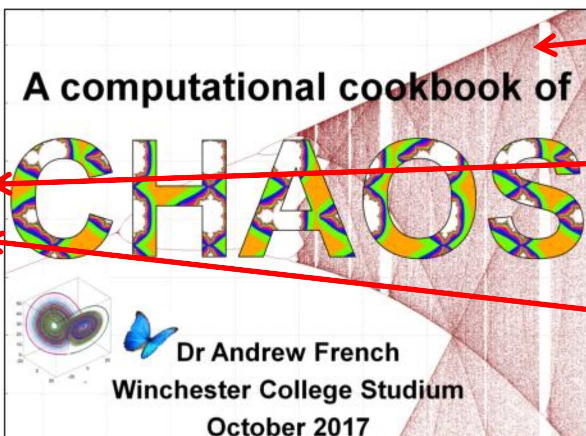


ppt

Click this to download a PDF of this presentation

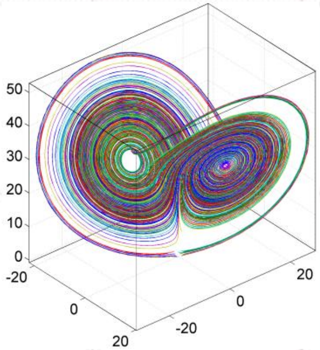
Click this to download all the MATLAB code used to generate the figures

Click this to download the Presentation + the linked movies, synopsis and references

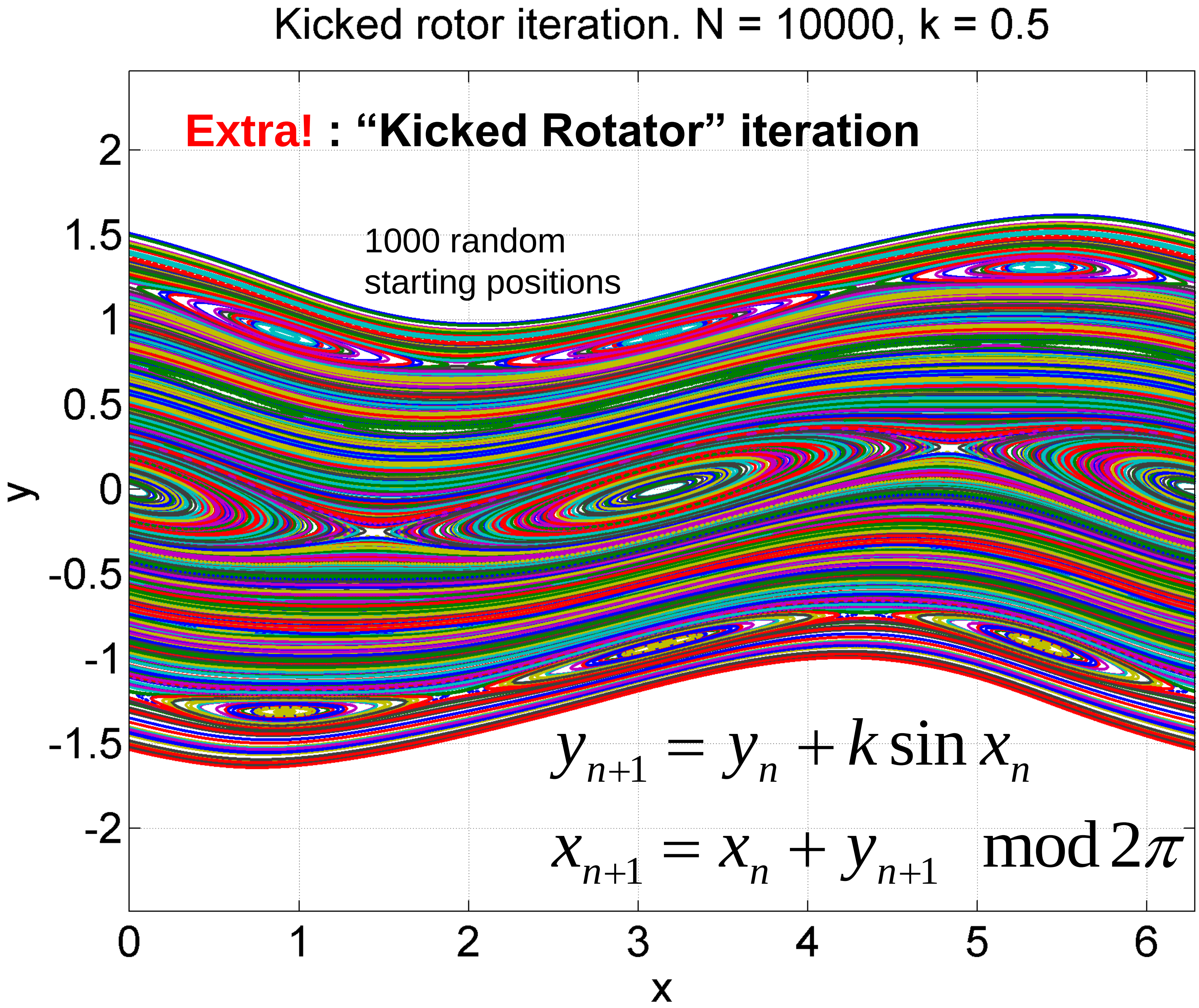
Alternative location in the `eclecticon` under `Programming/Teaching`

www.eclecticon.info

Questions

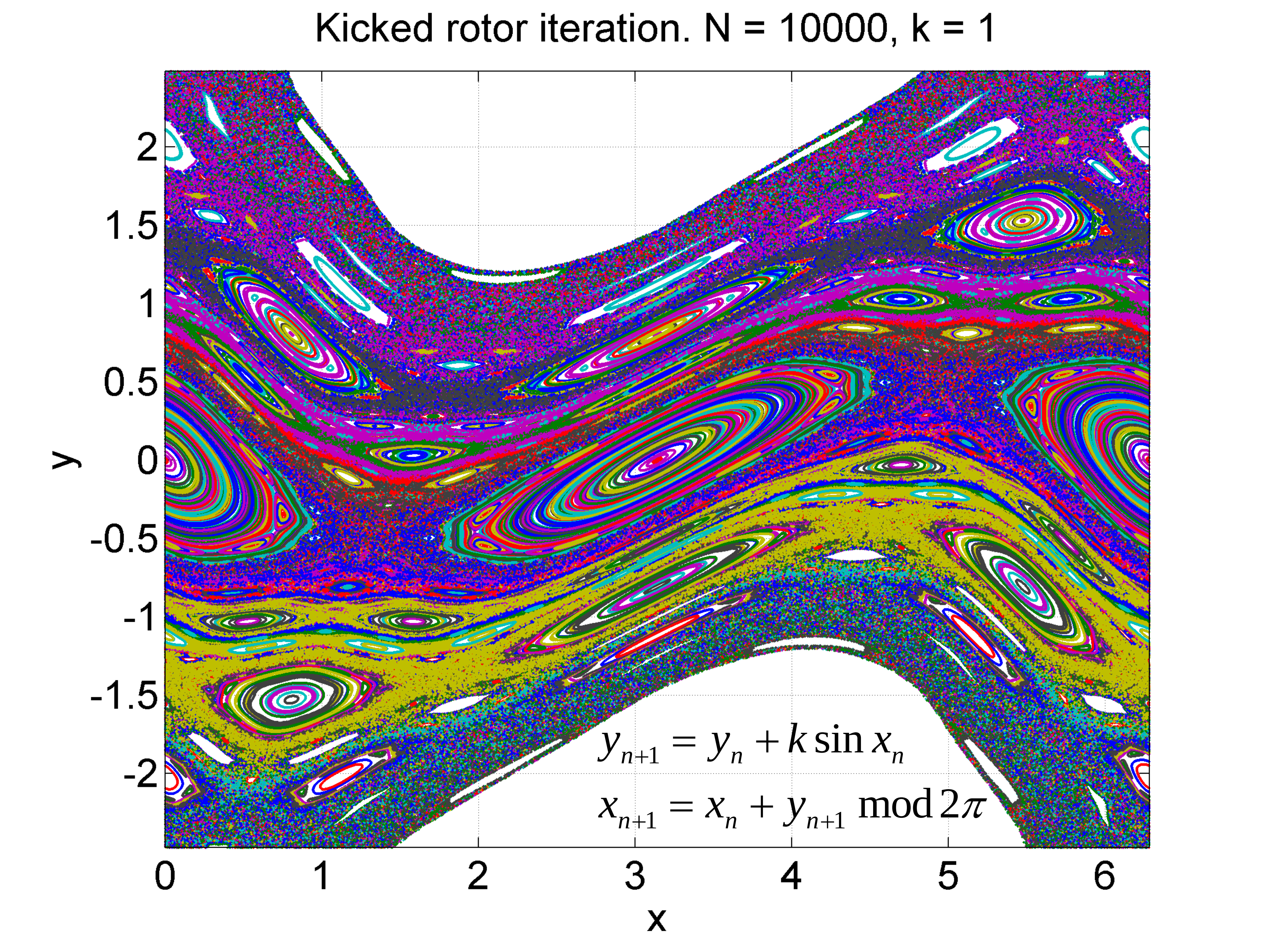


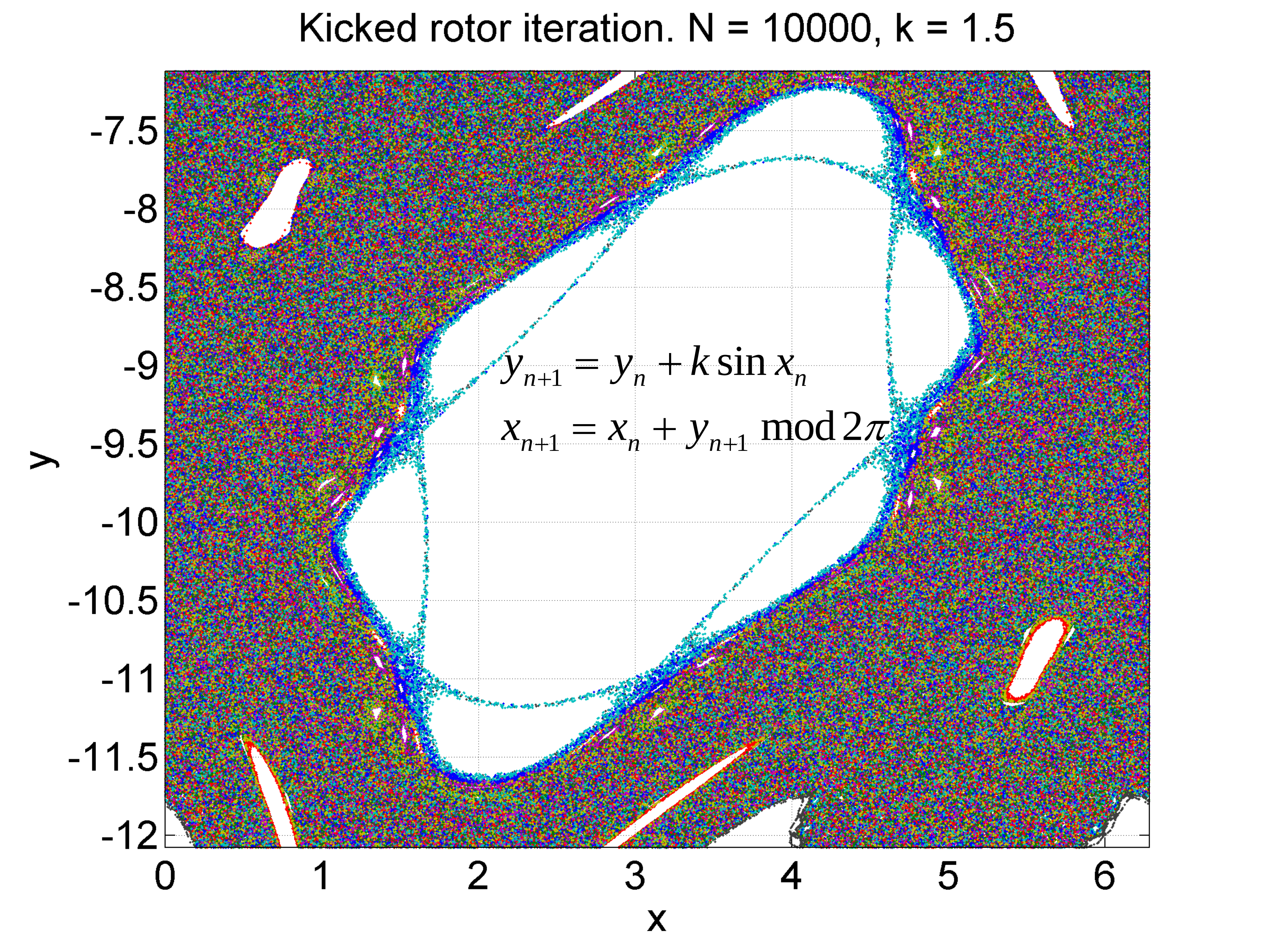
Extra! : “Kicked Rotator” iteration



$$y_{n+1} = y_n + k \sin x_n$$

$$x_{n+1} = x_n + y_{n+1} \quad \mathrm{mod}\, 2\pi$$


Kicked rotor iteration. N = 10000, k = 1
y
x
0
1
2
3
4
5
6
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
$y_{n+1} = y_n + k \sin x_n$
$x_{n+1} = x_n + y_{n+1} \mod 2\pi$



Kicked rotor iteration. N = 10000, k = 1.5
$y_{n+1} = y_n + k\sin x_n$
$x_{n+1} = x_n + y_{n+1} \bmod 2\pi$
y
x
-7.5
-8
-8.5
-9
-9.5
-10
-10.5
-11
-11.5
-12
0
1
2
3
4
5
6