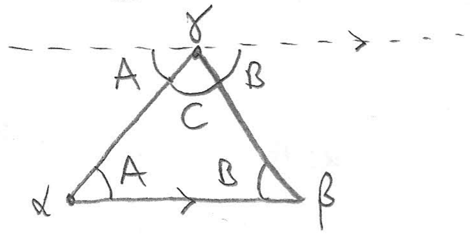
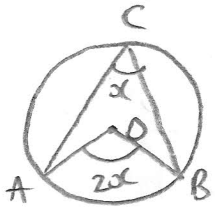
Circle Theorems

A French. May 2013.

Firstly we must prove the interior angles of a triangle sum to 180° (C1)

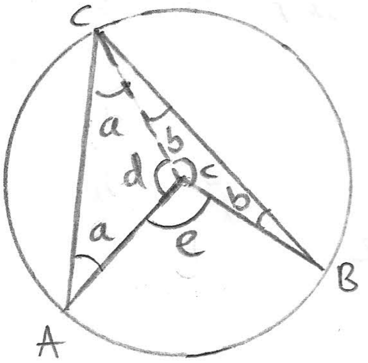


This can be achieved by drawing a line through the vertex of angle C (γ) which is $\parallel$ to line $\alpha\beta$



(C2)

Next we will prove the "Arrowhead theorem"



Split the arrowhead into two ISOCELES triangles. Using (C1)

$2a + d = 180^\circ$ ①

$2b + c = 180^\circ$ ② And also

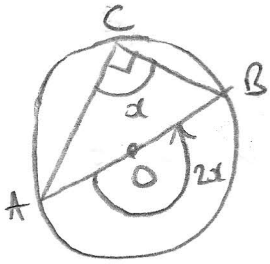
$d + c + e = 360^\circ$ ③

$\therefore$ ① + ② = ③ $\Rightarrow 2a + 2b + \cancel{d} + \cancel{c} = \cancel{d} + \cancel{c} + e$

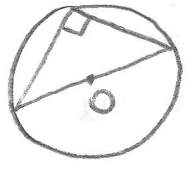
$\Rightarrow \boxed{2(a+b) = e}$

which proves the theorem

We can use this result to prove several other useful theorems

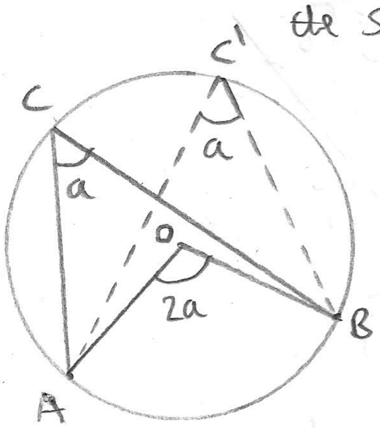


The special case of (C2) when $2x = 180^\circ$ (i.e. AB is a diameter) results in the "right angle theorem"

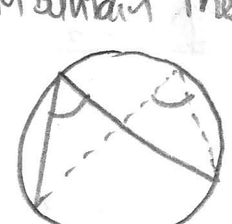


(C3)

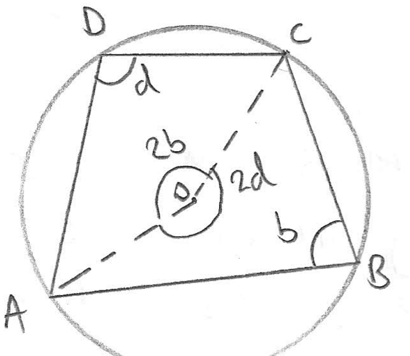
Drawing two arrowheads from the same <u>chord</u> AB implies the angular widths of 'mountains' with peaks at C and C' <u>must</u> have the same angle. Otherwise $A\hat{O}B$ would not be the same!



"Mountain Theorem"



(C4)



We can break up a cyclic quadrilateral ABCD into an arrowhead AOCB and a quadrilateral AOCD. O is the centre of the circle

$\therefore 2b+2d = 360^\circ$

$\Rightarrow \boxed{b+d = 180^\circ}$

Now AOCD is also a generalized "arrowhead" ie the "Arrowhead" theorem applies to any quadrilateral which has one vertex at the circle origin and the other three on the circumference

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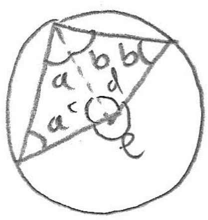

Quick recap: we can split this type of quadrilateral into two isoceles triangles as before

$2a + c = 180^\circ$ ①
$2b + d = 180^\circ$ ②
$e + c + d = 360^\circ$ ③

① + ② = ③

$\therefore \boxed{2(a+b) = e}$

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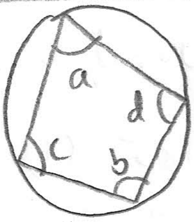
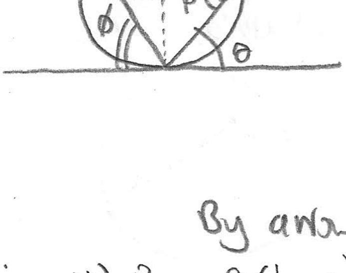
$\Rightarrow$ "opposite angles in a cyclic quadrilateral sum to 180°"

(C5)



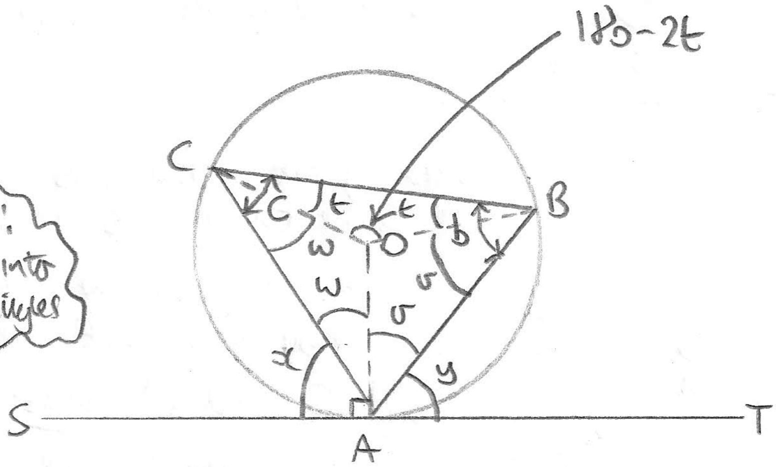
$a+b = 180^\circ$
$c+d = 180^\circ$

Alternate Segment theorem



(C6)

Proof: Split ABC into three triangles



Define: $b = t + v$ ① $\Rightarrow t = b - v$
$c = t + w$ ② $\Rightarrow t = c - w$

By arrowhead theorem (C2) $180^\circ - 2t = 2(w+v)$ ③

$\therefore 180^\circ = 2(b-v) + 2w + 2v = 2b + 2w$ $\therefore \boxed{w = 90^\circ - b}$

$\therefore 180^\circ = 2(c-w) + 2w + 2v = 2c + 2v$ $\therefore \boxed{v = 90^\circ - c}$

SAT is a tangent to the circle (⊥ to radius)

So: $w = 90^\circ - b$

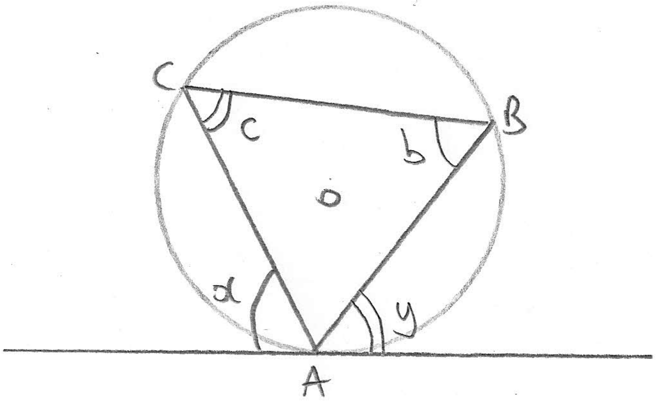
$v = 90^\circ - c$

Now $w + x = 90^\circ \Rightarrow w = 90^\circ - x$

and $v + y = 90^\circ \Rightarrow v = 90^\circ - y$

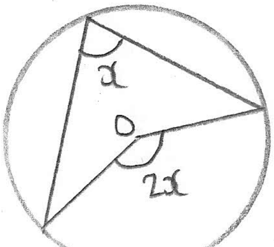
$\therefore$ $90^\circ - x = 90^\circ - b \Rightarrow \boxed{x = b}$

$90^\circ - y = 90^\circ - c \Rightarrow \boxed{y = c}$

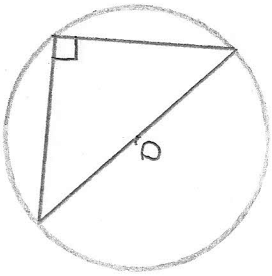


Alternate segment theorem (C6)

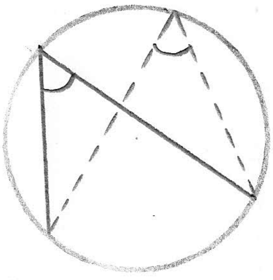
Summary of circle theorems



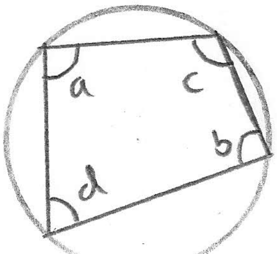
"Arrowhead"



"90° diameter triangle"



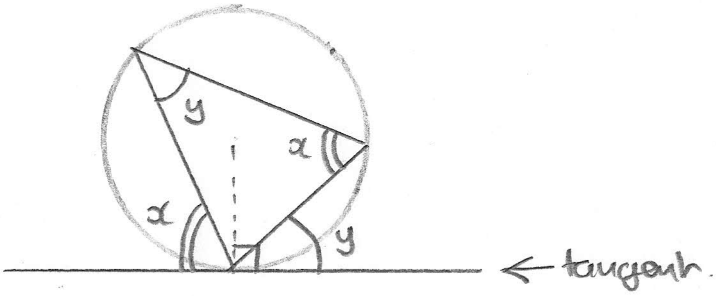
"Mountain"



$a + b = 180^\circ$

$d + c = 180^\circ$

"Cyclic quadrilateral"



"Alternate segment"