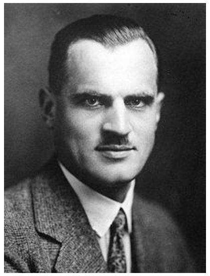
The **Compton Effect** describes a classic experiment which demonstrates the *particle nature of electromagnetic radiation*. i.e. electromagnetic radiation can be imagined as comprising spatially point-like **photons**, with **energy** $E = hf$ and **momentum** $p = h/\lambda$ (but crucially, zero mass). i.e. we use **Planck** and **de Broglie** relationships here.



Arthur Compton
1892-1962
Nobel Prize 1927

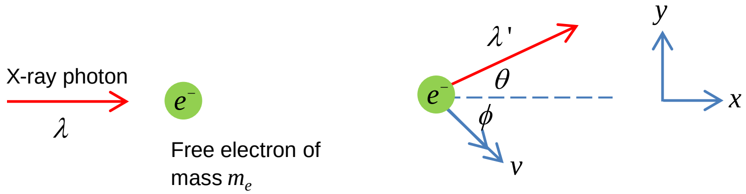
The Compton Effect refers to the scattering of **X-rays** off **free electrons**. Using the photon model of X-rays, this can be modelled as a two-dimensional collision problem. Experimentally, a beam of X-rays will be both deflected *and frequency shifted* following collision with an electron.

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$

The shift of X-ray wavelength

For simplicity, let us choose a frame of reference where free electron is initially at rest*

BEFORE **AFTER**



We must assume that the speeds involved imply a **Relativistic** treatment of dynamics is needed.

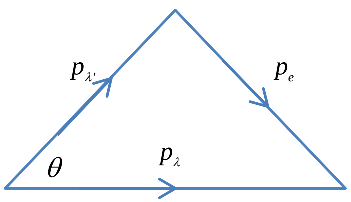
For particles which have mass

$$E = \gamma mc^2$$
$$\mathbf{p} = \gamma m\mathbf{v}$$
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

For photons

$$E = \frac{hc}{\lambda}$$
$$p = \frac{h}{\lambda}$$
$$\therefore E = pc$$

Momentum conservation: $\mathbf{p}_\lambda = \mathbf{p}_{\lambda'} + \mathbf{p}_e$



Cosine rule: $$p_e^2 = p_\lambda^2 + p_{\lambda'}^2 - 2p_\lambda p_{\lambda'}\cos\theta$$

Energy conservation: $E_\lambda + m_e c^2 = E_{\lambda'} + E_e$

From energy-momentum invariant: $E_e^2 - p_e^2 c^2 = m_e^2 c^4$ $\quad E_\lambda = p_\lambda c \quad E_{\lambda'} = p_{\lambda'} c$

Hence:

$$p_\lambda c + m_e c^2 = p_{\lambda'} c + \sqrt{m_e^2 c^4 + p_e^2 c^2}$$
$$\therefore \left((p_\lambda - p_{\lambda'})c + m_e c^2\right)^2 = m_e^2 c^4 + p_e^2 c^2$$
$$\therefore (p_\lambda - p_{\lambda'})^2 c^2 + 2(p_\lambda - p_{\lambda'})m_e c^3 + m_e^2 c^4 = m_e^2 c^4 + p_e^2 c^2$$
$$\therefore p_e^2 = (p_\lambda - p_{\lambda'})^2 + 2m_e c(p_\lambda - p_{\lambda'})$$

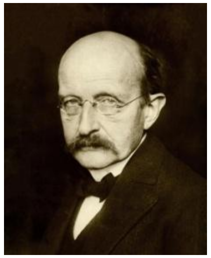
Equating these two expressions for p_e^2

$$(p_\lambda - p_{\lambda'})^2 + 2m_e c(p_\lambda - p_{\lambda'}) = p_\lambda^2 + p_{\lambda'}^2 - 2p_\lambda p_{\lambda'}\cos\theta$$
$$p_\lambda^2 + p_{\lambda'}^2 - 2p_\lambda p_{\lambda'} + 2m_e c(p_\lambda - p_{\lambda'}) = p_\lambda^2 + p_{\lambda'}^2 - 2p_\lambda p_{\lambda'}\cos\theta$$
$$m_e c(p_\lambda - p_{\lambda'}) = p_\lambda p_{\lambda'}(1-\cos\theta)$$

From de-Broglie relation: $p_\lambda = h/\lambda \quad p_{\lambda'} = h/\lambda'$

Hence: $$m_e ch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h^2}{\lambda\lambda'}(1-\cos\theta)$$
$$\therefore m_e c(\lambda' - \lambda) = h(1-\cos\theta)$$

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$



Max Planck
1858 – 1947
Nobel Prize 1918

Maximum wavelength shift is when: $\cos\theta = 1 \Rightarrow \theta = 180^\circ$

i.e. when X-ray photon recoils backwards. In this case:

$$\Delta\lambda_{\max} = \frac{2h}{m_e c}$$

Louis de Broglie
1892 – 1987
Nobel Prize 1929

$$E_\lambda = hc/\lambda \Rightarrow \lambda = hc/E_\lambda$$
$$\therefore \frac{\Delta\lambda_{\max}}{\lambda} = \frac{2h}{m_e c}\frac{E_\lambda}{hc}$$

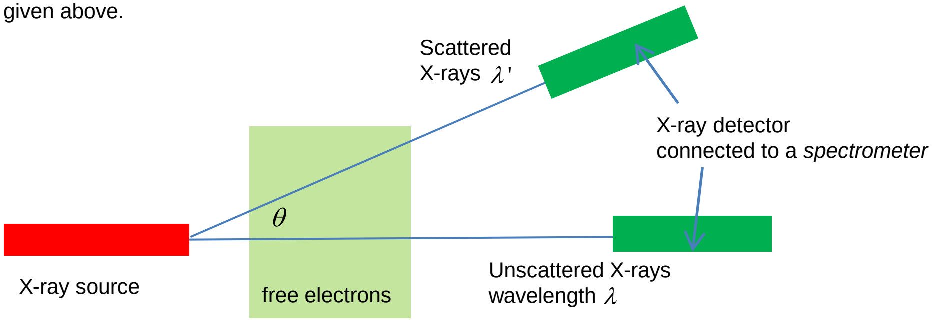
$$E_\lambda = \tfrac{1}{2} m_e c^2 \frac{\Delta\lambda_{\max}}{\lambda}$$

$$e = 1.6021766208(98)\times 10^{-19}\,\mathrm{C}$$
$$c = 2.99792458\times 10^{8}\,\mathrm{ms^{-1}}$$
$$h = 6.626070040(81)\times 10^{-34}\,\mathrm{kgm^2s^{-1}}$$
$$m_e = 9.10938356(11)\times 10^{-31}\,\mathrm{kg}$$
$$m_e = 511\,\mathrm{keV}/c^2$$

*For a real system there will likely be a distribution of electron and X-ray momenta, so the Compton effect will have to be analysed based upon statistical variations

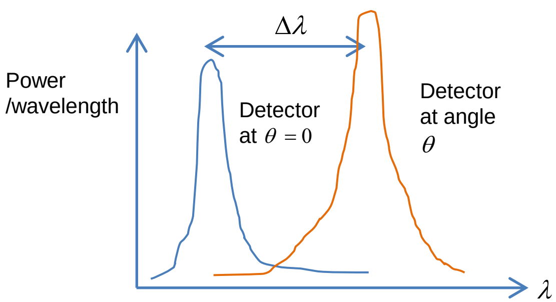
If one places an X-ray detector at angle θ from an incident beam, one should detect scattered X-rays which are wavelength shifted by the amount given above.

(Note X-ray intensity will diminish with range due to absorption and dispersion. Hence detectors should be at the same range, i.e. on a movable arc).



Typical X-rays have a wavelength of 0.01 to 10nm, so this shift may be a small percentage change. Since the shift is absolute, rather than proportional to incident wavelength like a Doppler effect, it makes sense to use *harder* (i.e. smaller wavelength) X-rays in order to see a more pronounced effect.

The spectra for X-rays might look something like this: $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$



What is the recoil speed of the electron?

Conservation of energy:

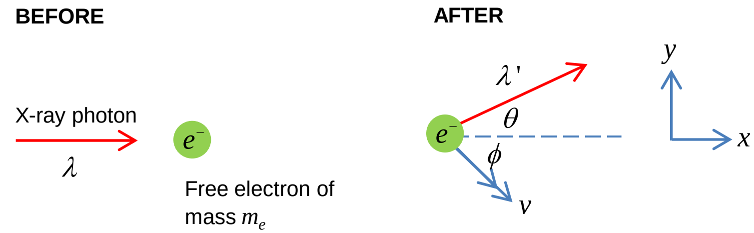
$$hc/\lambda + m_e c^2 = hc/\lambda' + \left(1-\tfrac{v^2}{c^2}\right)^{-\frac{1}{2}} m_e c^2$$

$$\therefore \left(\frac{hc/\lambda - hc/\lambda' + m_e c^2}{m_e c^2}\right)^{-2} = 1-\tfrac{v^2}{c^2}$$

$$\therefore v = c\sqrt{1-\left(\frac{m_e c^2}{hc/\lambda - hc/\lambda' + m_e c^2}\right)^2}$$

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta)$$

We can also find the **electron recoil angle** using conservation of momentum:



Conservation of momentum:

$$x:\quad p_\lambda = p_{\lambda'}\cos\theta + p_e\cos\phi$$

$$y:\quad 0 = p_{\lambda'}\sin\theta - p_e\sin\phi$$

Hence:

$$p_e\sin\phi = p_{\lambda'}\sin\theta$$

$$p_e\cos\phi = p_\lambda - p_{\lambda'}\cos\theta$$

$$\therefore \tan\phi = \frac{p_{\lambda'}\sin\theta}{p_\lambda - p_{\lambda'}\cos\theta}$$

$$\therefore \tan\phi = \frac{\sin\theta}{p_\lambda / p_{\lambda'} - \cos\theta}$$

$$\frac{p_\lambda}{p_{\lambda'}} = \frac{h}{\lambda}\frac{\lambda'}{h} = \frac{\lambda'}{\lambda}$$

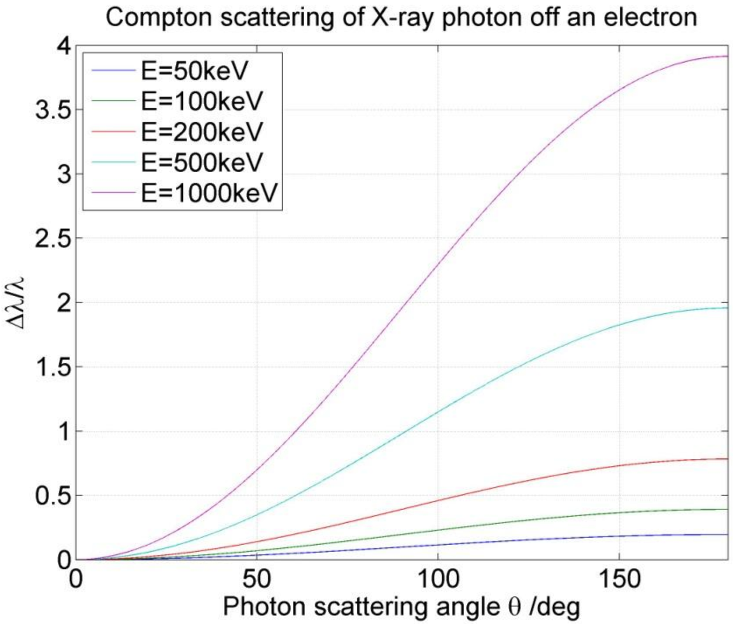
$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$

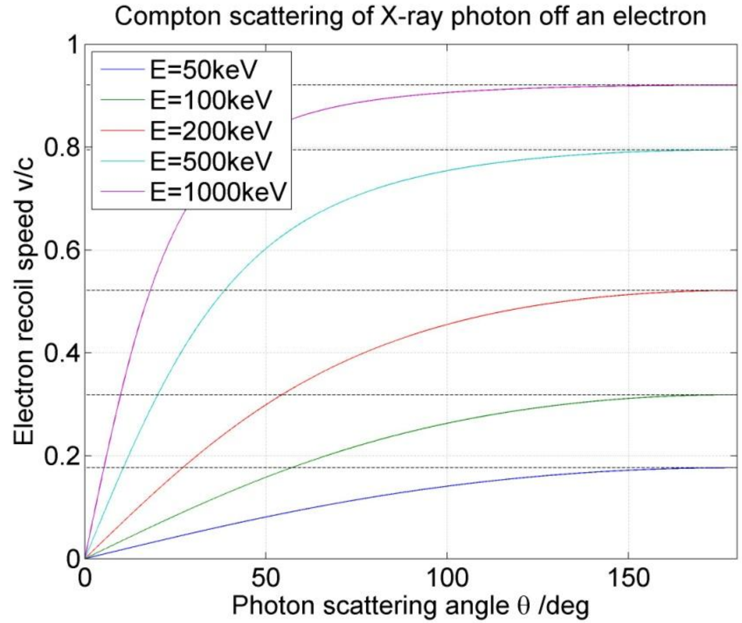
$$\therefore \frac{\lambda'}{\lambda} - 1 = \frac{h}{m_e c\lambda}(1-\cos\theta)$$

$$\therefore \frac{\lambda'}{\lambda} = \frac{h}{m_e c\lambda}(1-\cos\theta) + 1$$

Hence:

$$\tan\phi = \frac{\sin\theta}{1 + \frac{h}{m_e c\lambda}(1-\cos\theta) - \cos\theta}$$





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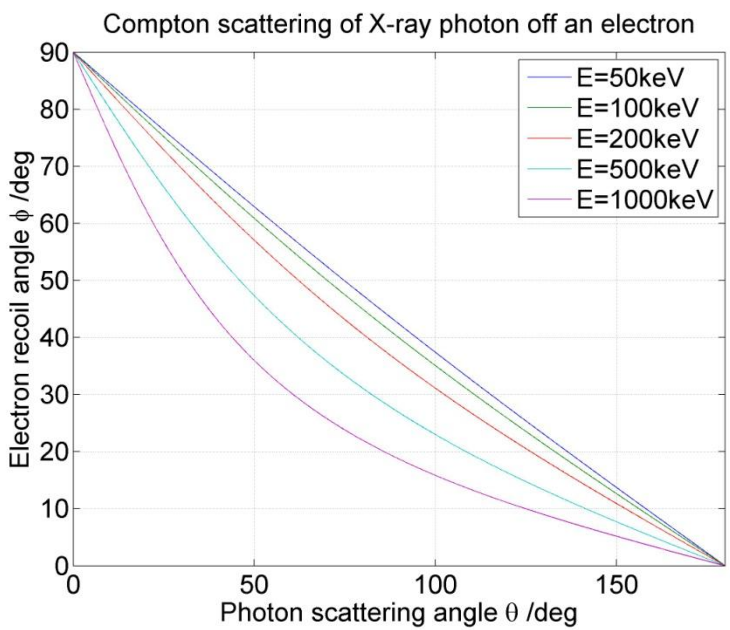

Compton scattering summary:

$$\lambda = hc/E_\lambda$$

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta)$$

$$\tan\phi = \frac{\sin\theta}{1+\frac{h}{m_e c\lambda}(1-\cos\theta)-\cos\theta}$$

$$v = c\sqrt{1-\left(\frac{m_e c^2}{hc/\lambda - hc/\lambda' + m_e c^2}\right)^2}$$