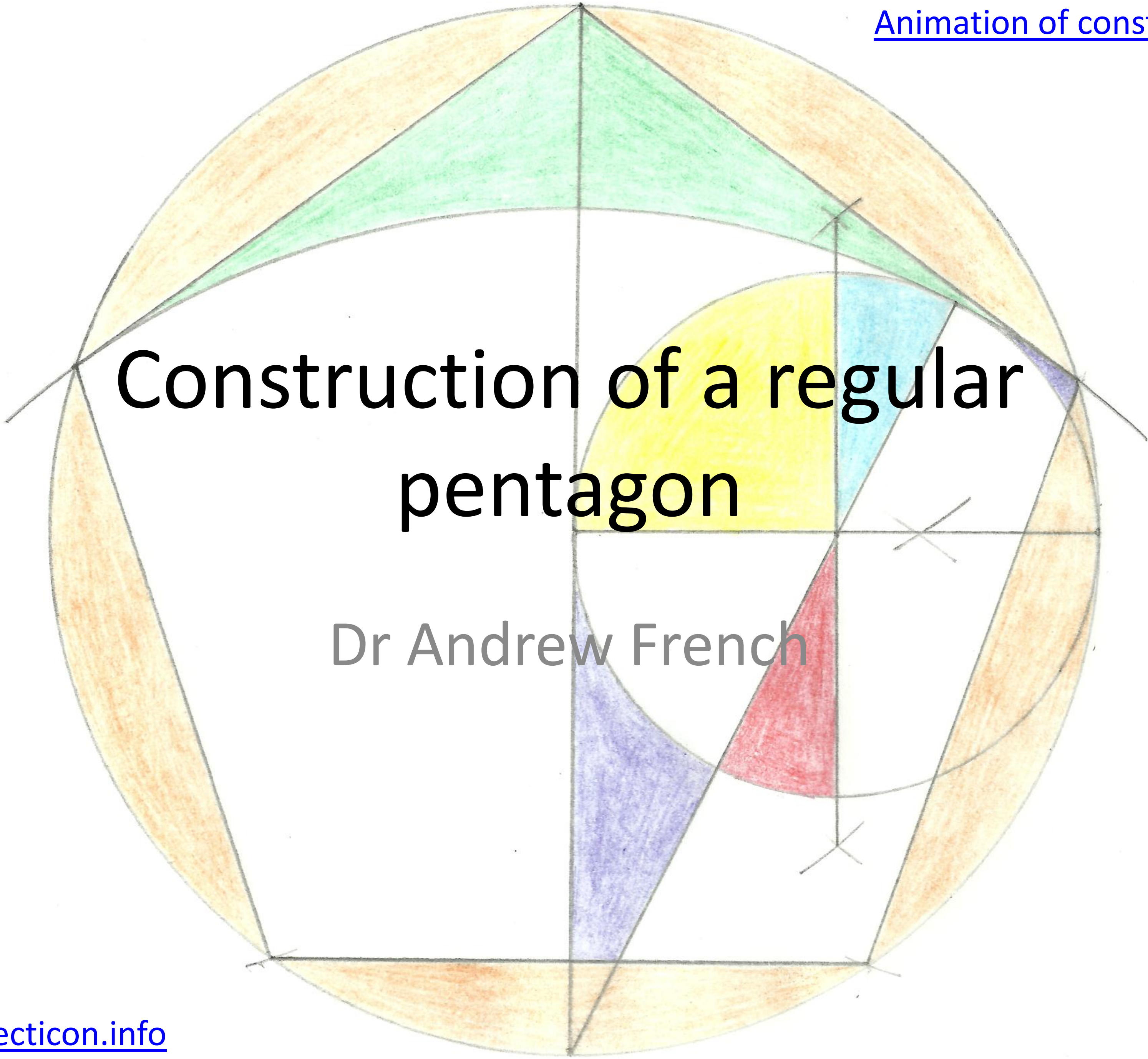
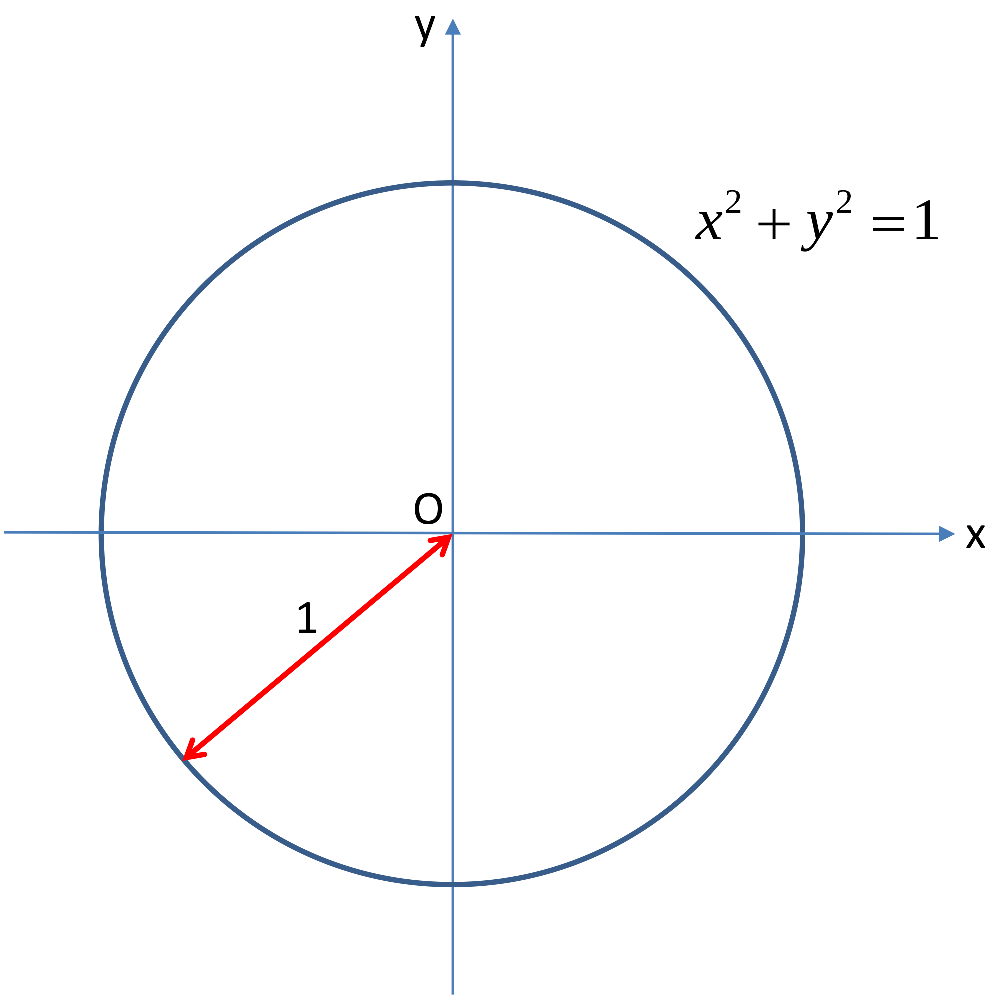
[Animation of construction]



Construction of a regular pentagon

Dr Andrew French



'Construction' means one can only draw lines using the following equipment

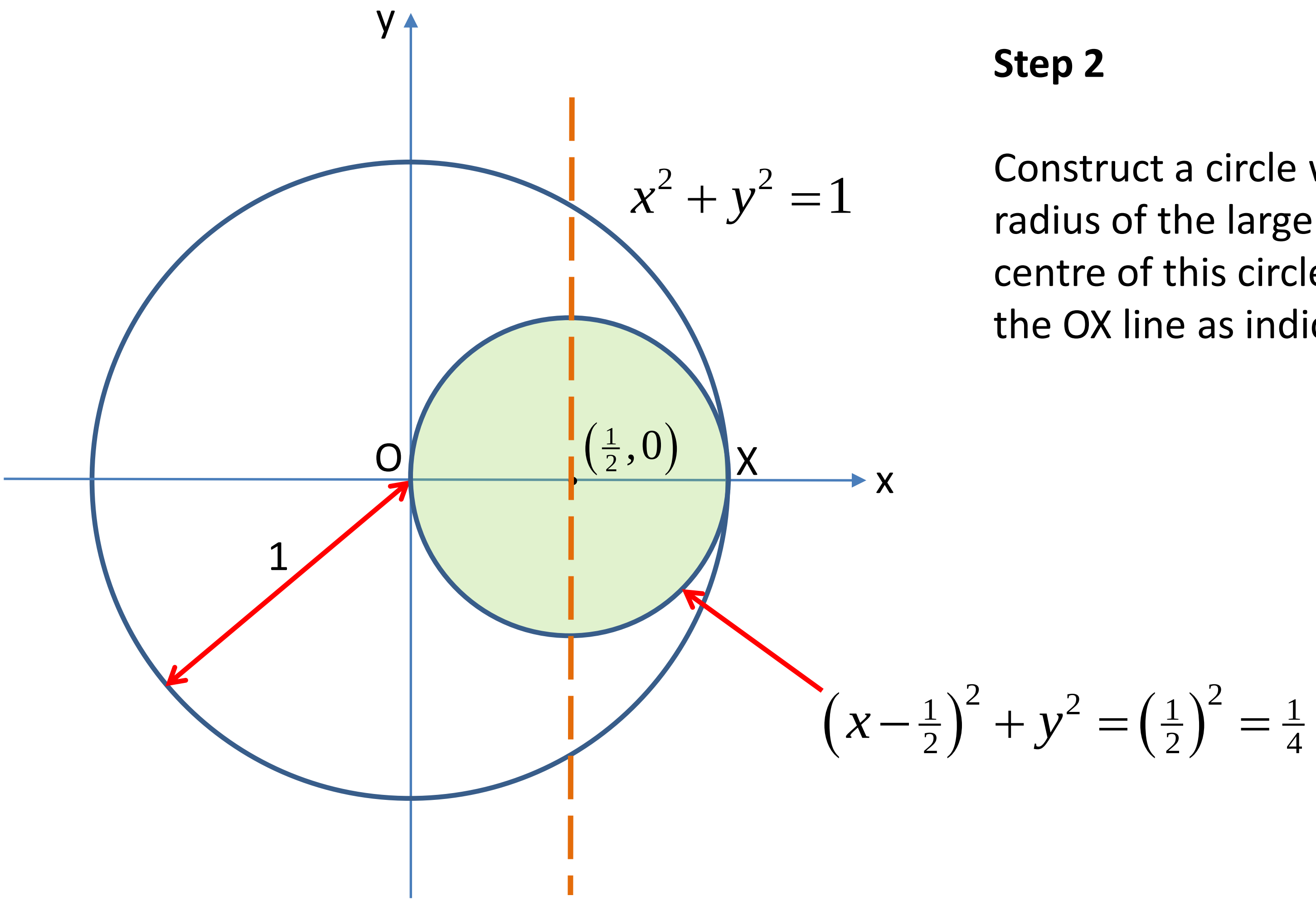
- A straight edge
- A compass

Step 1

Draw a circle and divide it vertically and horizontally to form the y and x axes.

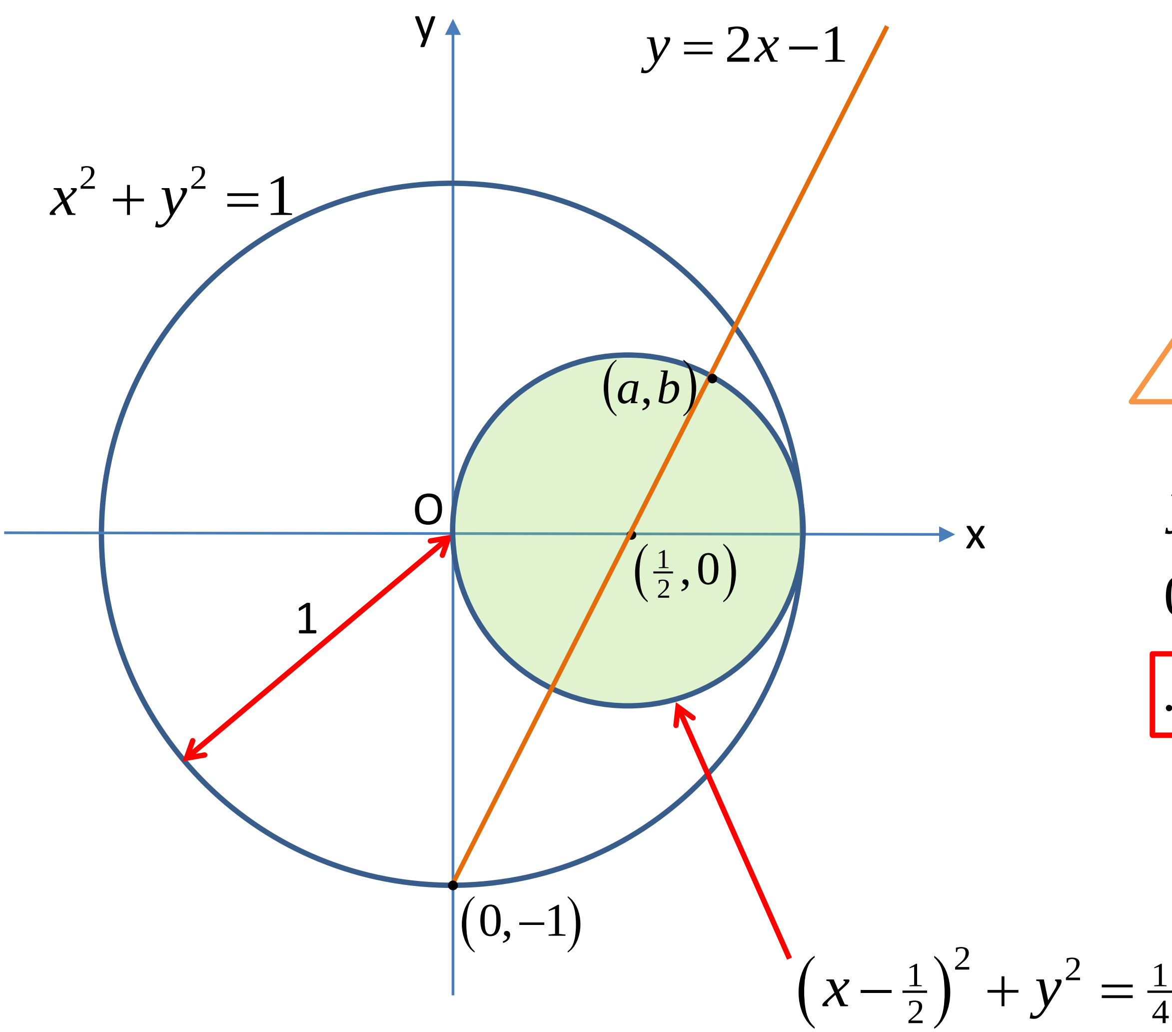
Use the line bisection method to find the x axis, given the y axis.

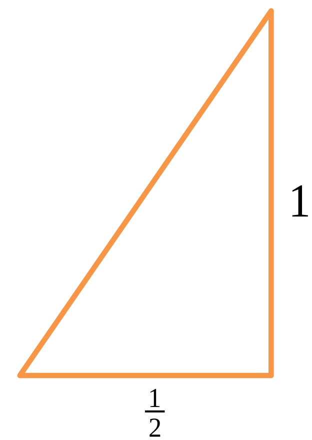
Without loss of generality we can consider the **unit circle.** In practical terms this means one unit of drawing scale corresponds to the radius of this circle.



Step 2

Construct a circle with half the radius of the larger circle. Find the centre of this circle by bisecting the OX line as indicated.



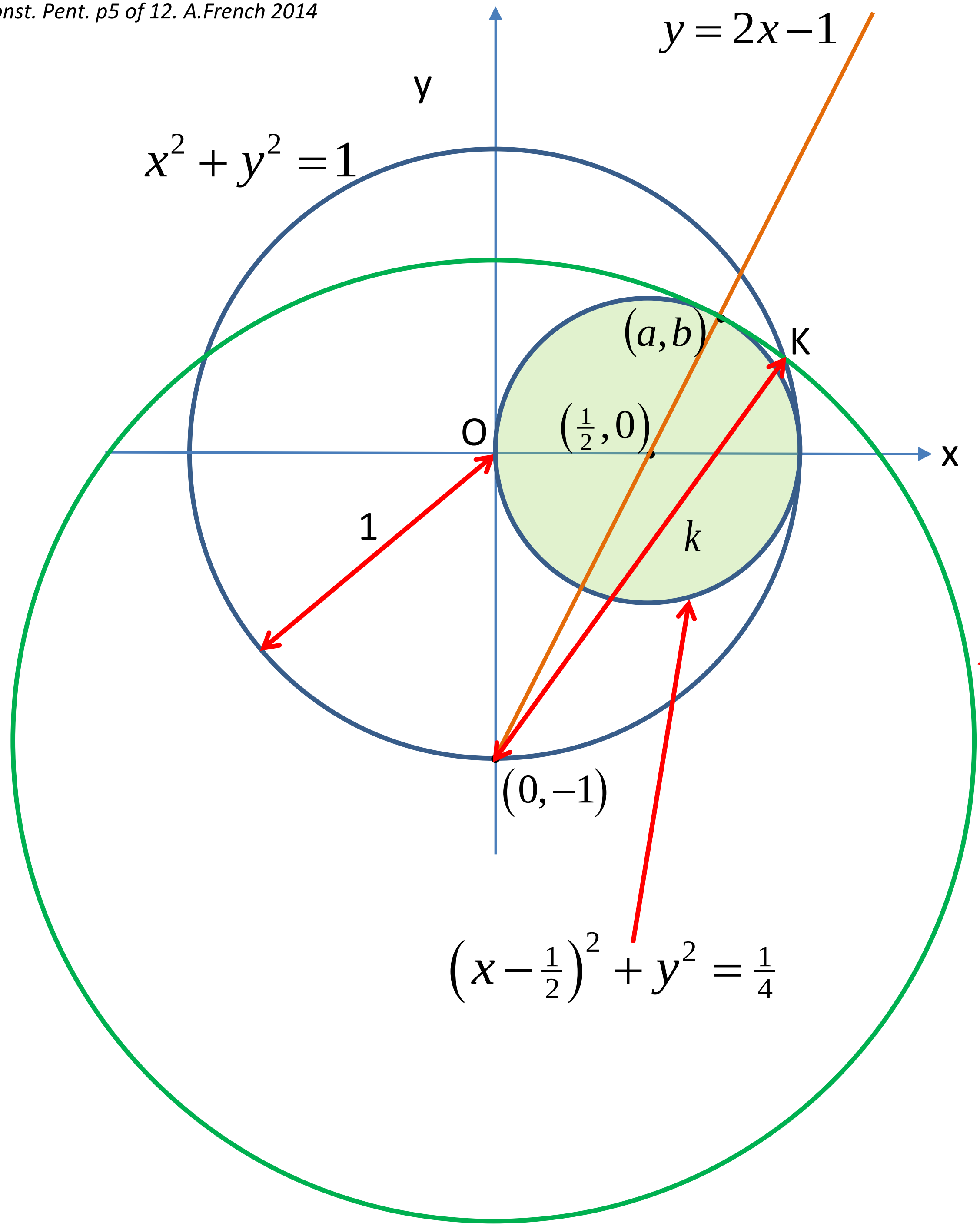


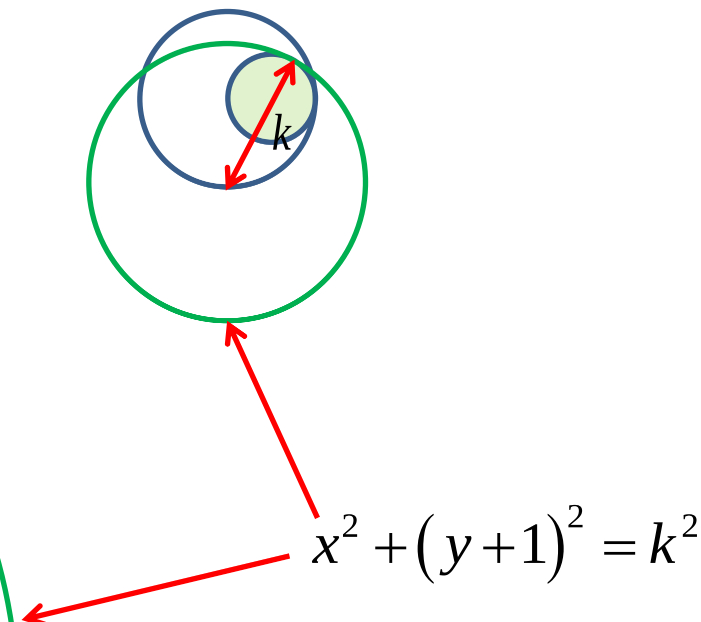
$$y = 2x + c$$

$$0 = 2\left(\tfrac{1}{2}\right) + c \Rightarrow c = -1$$

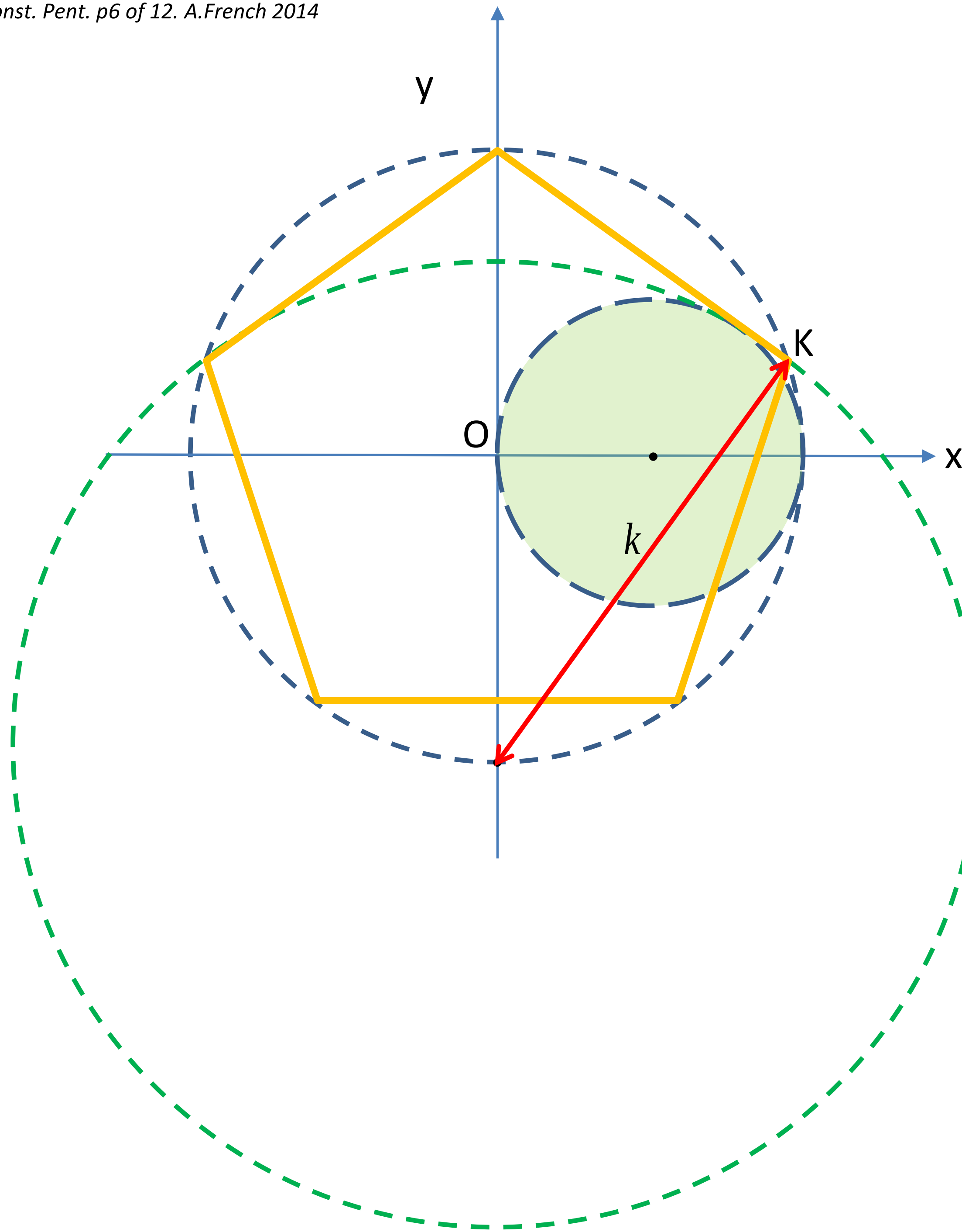
$$\boxed{\therefore y = 2x - 1}$$

Step 3. Draw a line from the base of the larger circle and through the centre of the smaller circle. Find where this crosses the upper edge of the smaller circle.





Step 4. Draw an arc of a large circle with radius from the base of the first circle to the small circle crossing point found in Step 3. Find position K where this crosses the original circle.

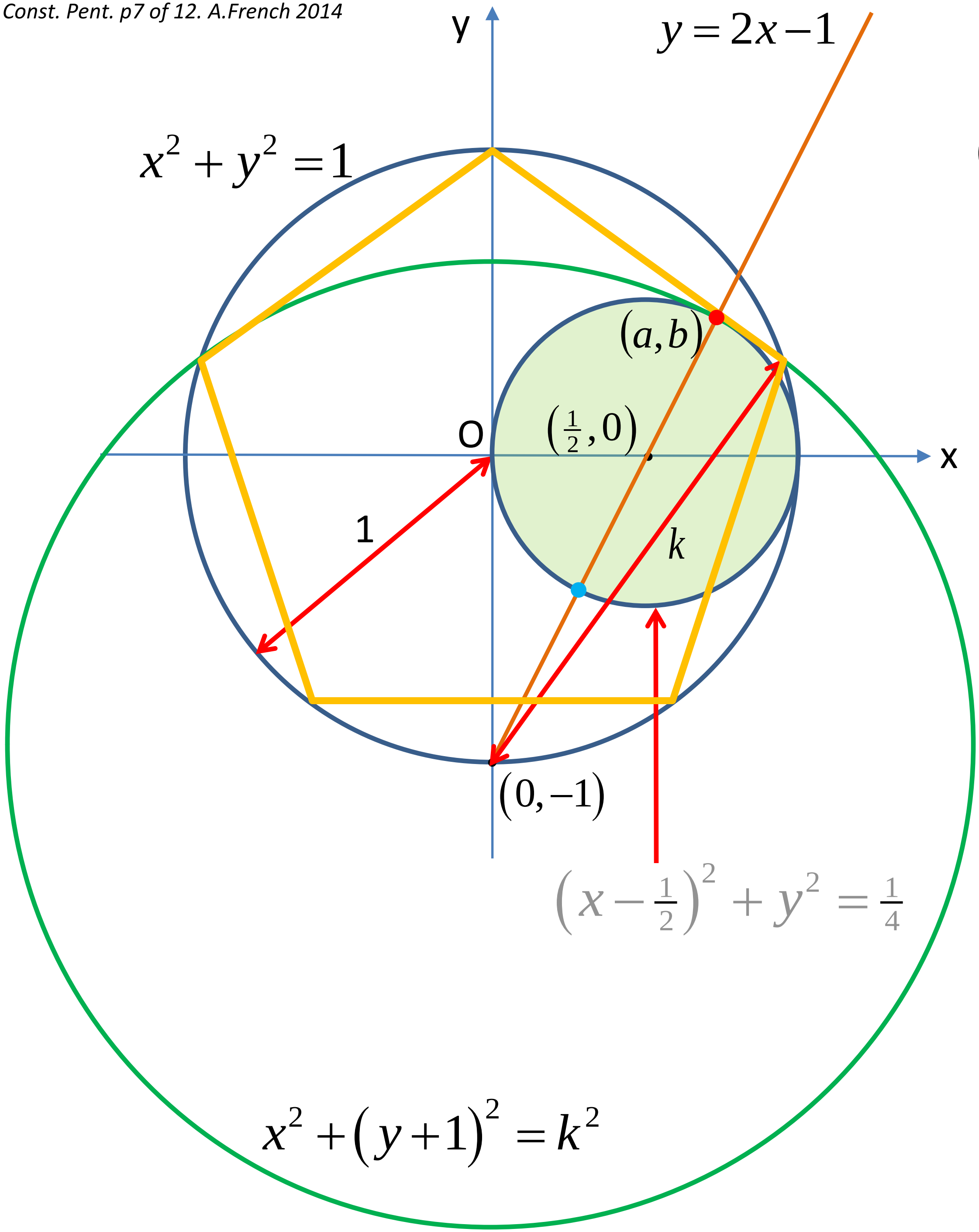


Step 5

Draw a straight line from the intersection of the y axis and the original circle and the crossing point K found in Step 4. This is an edge of a regular pentagon!

Use a compass to step round the original circle to find the other three points.

To minimize the effects of drawing errors, work out the lower vertices from the left and right large circular arc intersections rather than step round using a compass from one side only.



$$(a,b) \quad \begin{cases} a^2 + (b+1)^2 = k^2 \\ b = 2a - 1 \\ \left(a - \tfrac{1}{2}\right)^2 + b^2 = \tfrac{1}{4} \end{cases}$$

$$a^2 + (2a)^2 = k^2$$

$$\Rightarrow \boxed{k = a\sqrt{5}}$$

$$\left(a - \tfrac{1}{2}\right)^2 + (2a-1)^2 = \tfrac{1}{4}$$

$$a^2 - a + \tfrac{1}{4} + 4a^2 - 4a + 1 = \tfrac{1}{4}$$

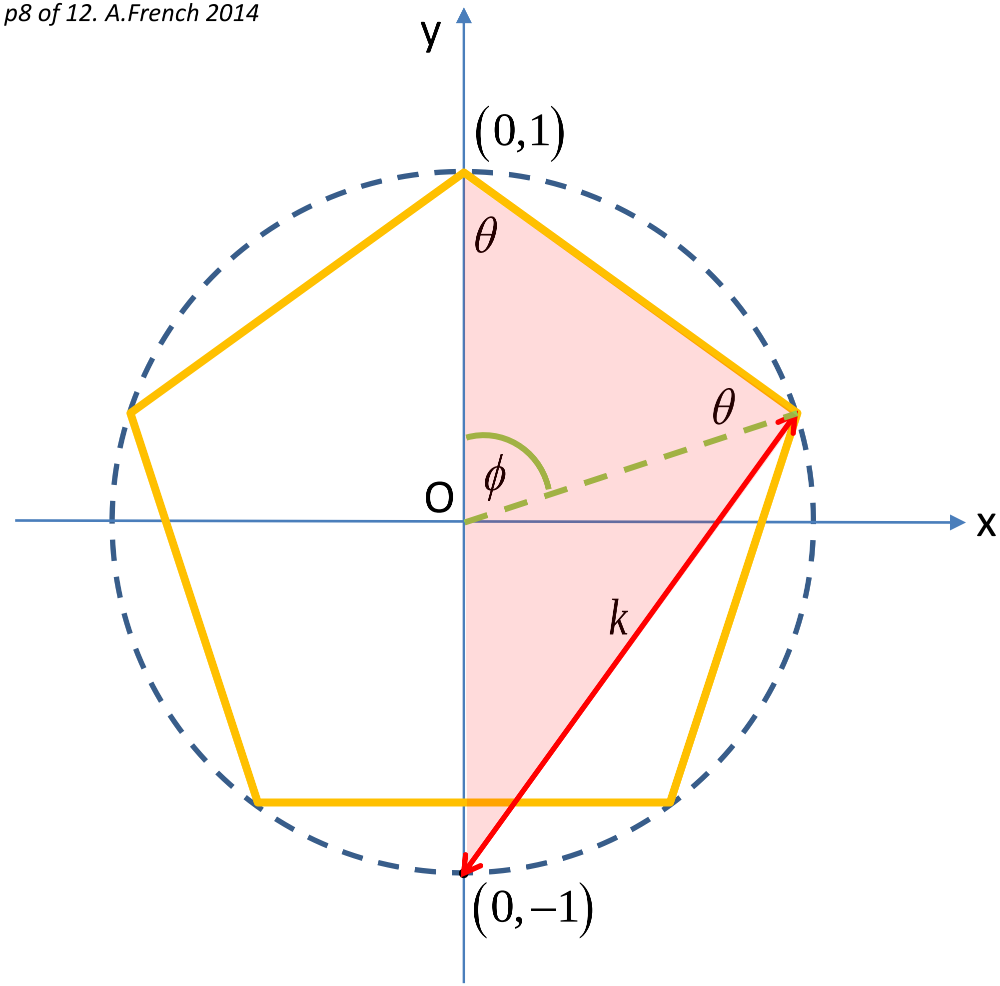
$$5a^2 - 5a + 1 = 0$$

$$\therefore a = \frac{5 \pm \sqrt{25-20}}{10}$$

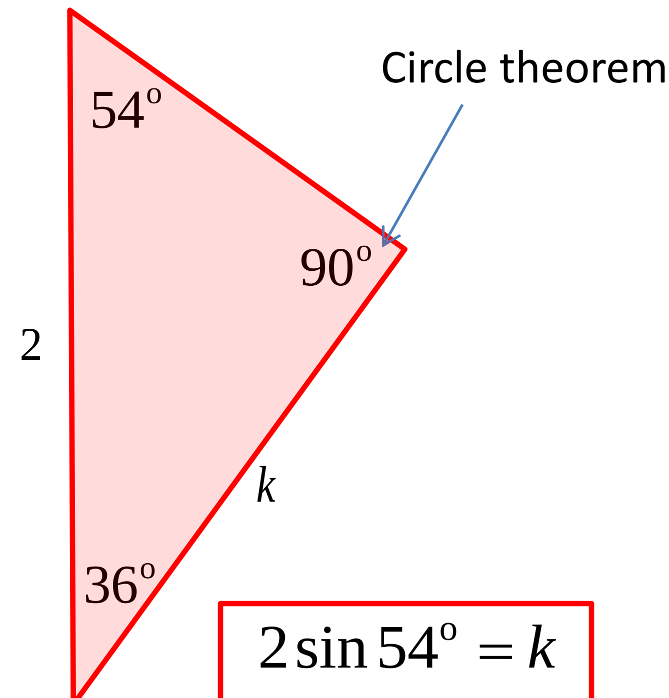
$$\Rightarrow a = \frac{5+\sqrt{5}}{10}$$

is the −ve solution

$$k = \frac{5\sqrt{5}+5}{10} \Rightarrow \boxed{k = \tfrac{1}{2}\left(1+\sqrt{5}\right)}$$



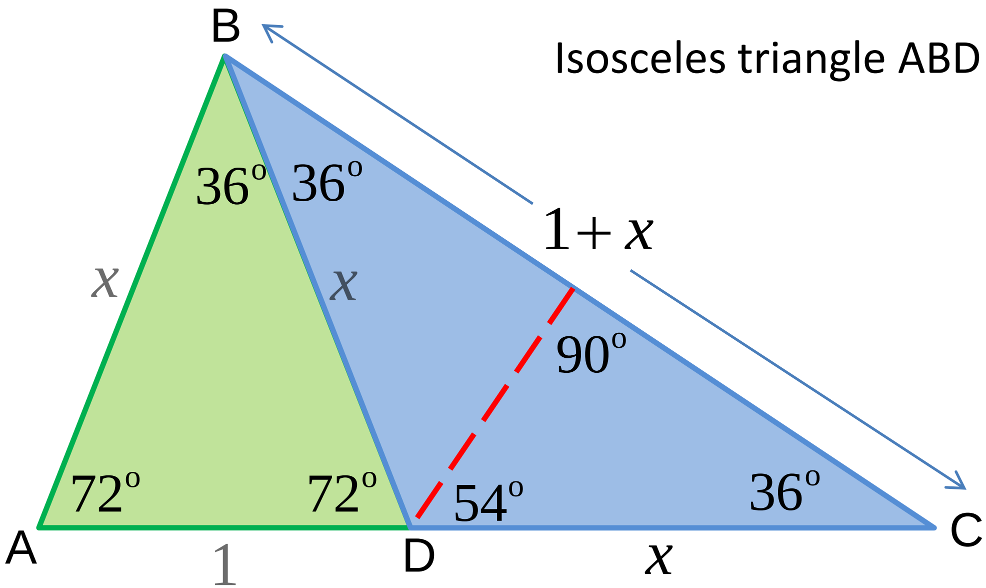
$$\phi = \frac{360^o}{5} = 72^o$$
$$180 = \phi + 2\theta$$
$$\therefore \theta = \frac{180^o - 72^o}{2} = 54^o$$



$$\boxed{\begin{aligned} 2\sin 54^o &= k \\ 2\cos 36^o &= k \end{aligned}}$$

Now consider the regular pentagon separately.
To demonstrate why the construction works we need an expression for the triangle side *k*, derived *only* from properties of the regular pentagon. i.e. *independent of the construction.* If everything agrees then the construction works!

Consider the construction below. This can be used to derive an exact expression for cos36°



Isosceles triangle ABD is *similar* to triangle ABC, hence:

$$\frac{BC}{BA}=\frac{BA}{AD}$$

$$\frac{1+x}{x}=\frac{x}{1}$$

Note

from diagram

$x>1$

$$\therefore x^2-x-1=0 \Rightarrow x=\frac{1+\sqrt{5}}{2}$$

$$\therefore x^{-1}=\frac{2}{1+\sqrt{5}}=\frac{2\left(1-\sqrt{5}\right)}{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)}$$

$$=\frac{2\left(1-\sqrt{5}\right)}{-4}=\frac{-1+\sqrt{5}}{2}$$

Now: $$x\cos 36^\circ=\frac{1+x}{2}$$

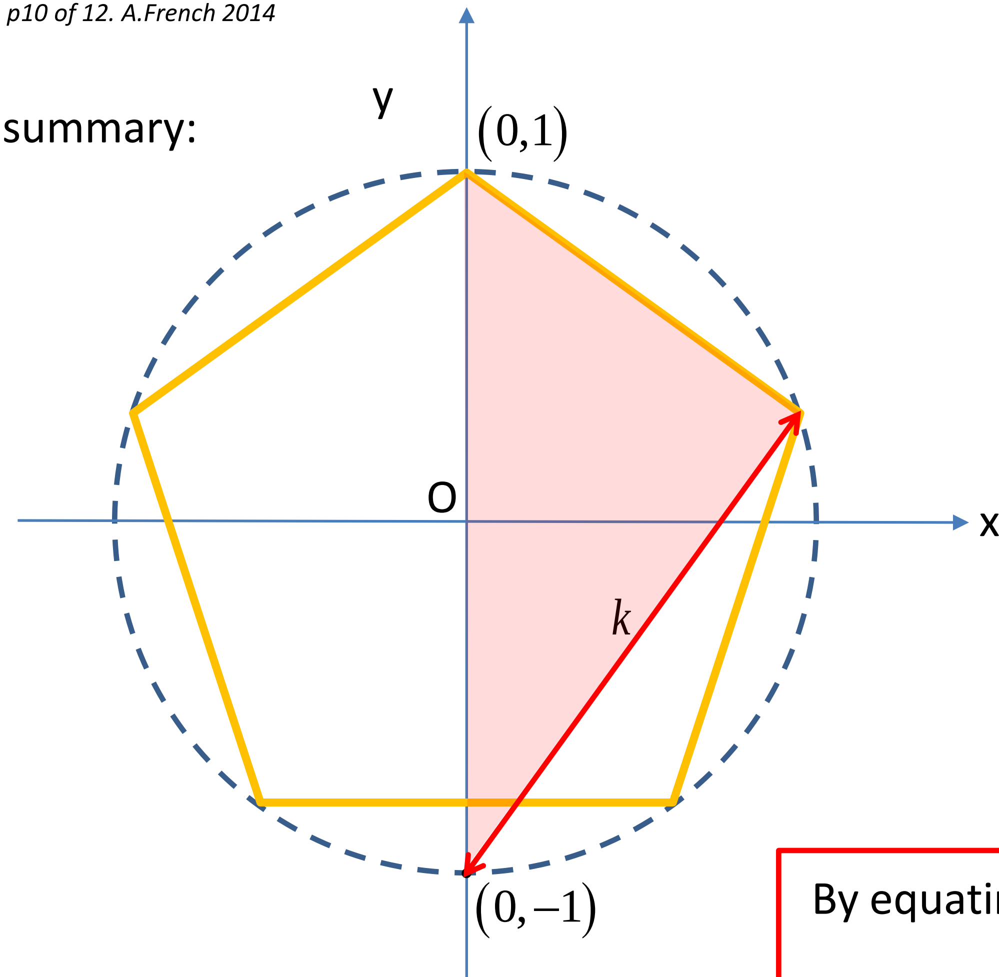
$$\therefore \cos 36^\circ=\frac{1+x^{-1}}{2}=\tfrac{1}{2}+\frac{-1+\sqrt{5}}{4}$$

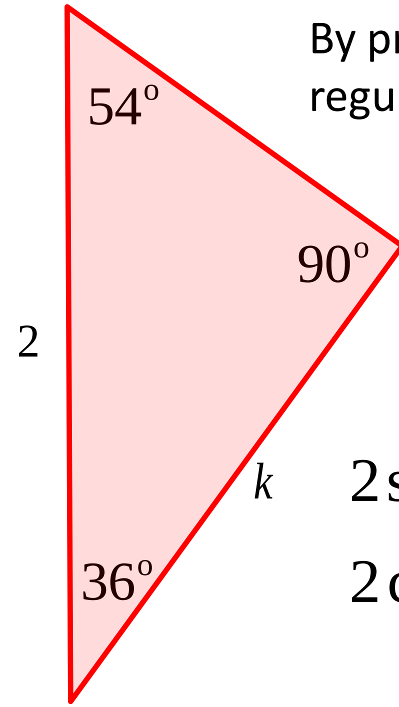
$$\therefore \cos 36^\circ=\frac{1+\sqrt{5}}{4}$$

Hence: $$\boxed{\cos 36^\circ=\sin 54^\circ=\frac{1+\sqrt{5}}{4}}$$

[Construction suggested by Hugh Hill, 5/3/14]

In summary:





By properties of a regular pentagon

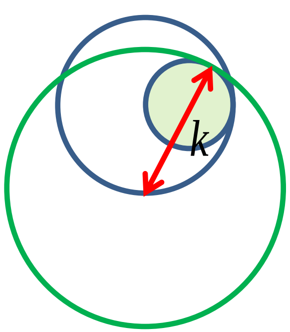
$$2\sin 54^{\mathrm{o}} = k$$

$$2\cos 36^{\mathrm{o}} = k$$

By equating *k* we find

$$\cos 36^{\mathrm{o}} = \sin 54^{\mathrm{o}} = \frac{1+\sqrt{5}}{4} \approx 0.809$$

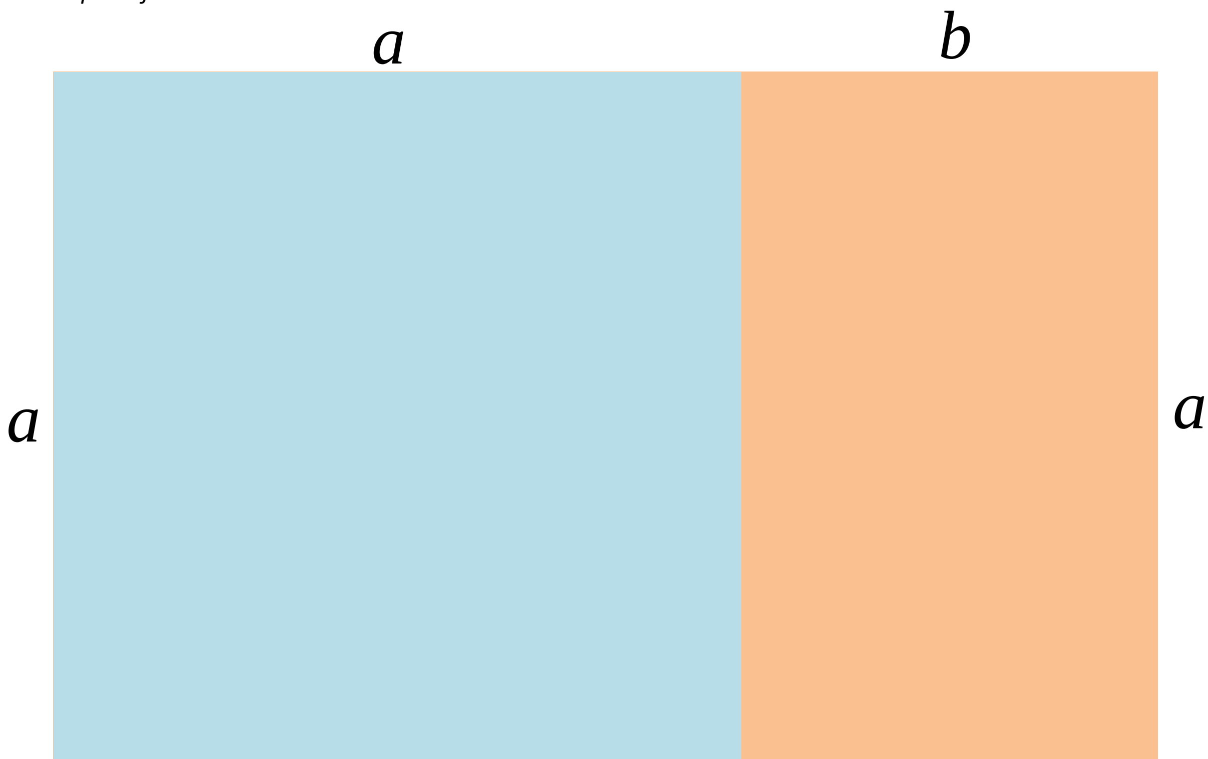
which indeed is the case from the previous isosceles triangle construction!



$$k = \tfrac{1}{2}\left(1+\sqrt{5}\right)$$

From the construction

Note $k = \dfrac{1+\sqrt{5}}{2}$ is the GOLDEN RATIO



The Golden Ratio

Define
'Golden rectangle'
by ratio of sides
a/b

$$\phi = \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{1}{\phi}$$

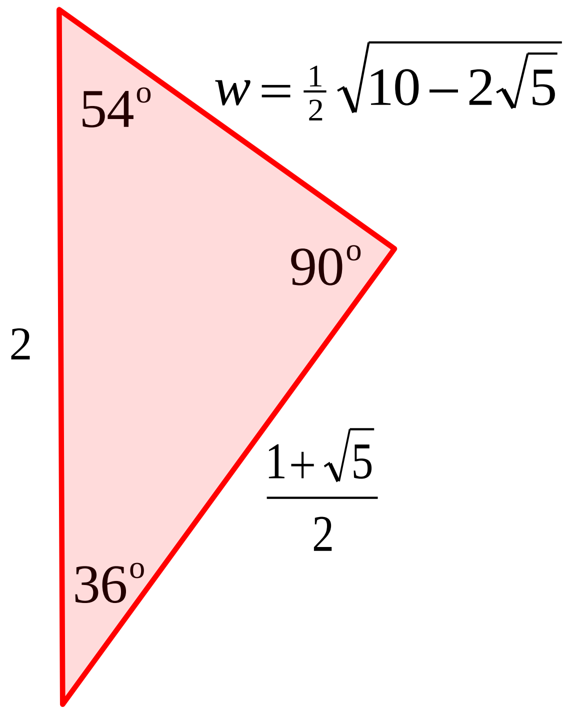
$$\Rightarrow \phi^2 - \phi - 1 = 0$$

$$\Rightarrow \phi = \frac{1 \pm \sqrt{5}}{2}$$

Typically take *a* > *b* so
GOLDEN RATIO

$$\boxed{\phi = \frac{1+\sqrt{5}}{2} \approx 1.618}$$

The construction of the regular pentagon therefore gives us another set of *exact* right-angled triangle relationships:



$$\cos 36^{\mathrm{o}} = \frac{1+\sqrt{5}}{4}$$

$$\sin 36^{\mathrm{o}} = \tfrac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$\cos 54^{\mathrm{o}} = \tfrac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$\sin 54^{\mathrm{o}} = \frac{1+\sqrt{5}}{2}$$

$$\tan 36^{\mathrm{o}} = \frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$$

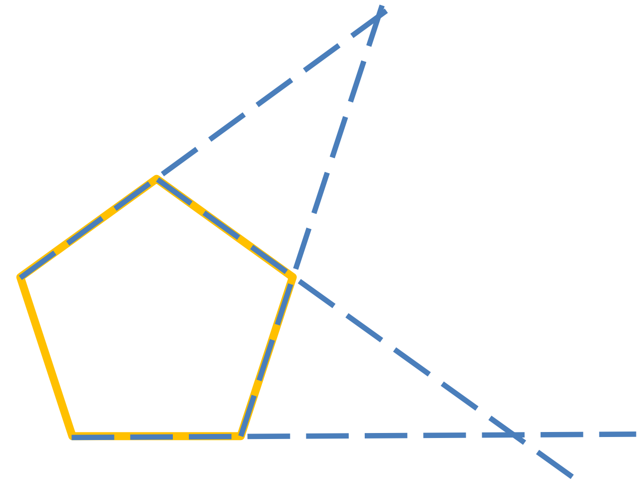
$$\tan 54^{\mathrm{o}} = \frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$$

From Pythagoras' theorem:

$$4 = \frac{\left(1+\sqrt{5}\right)^2}{4} + w^2$$

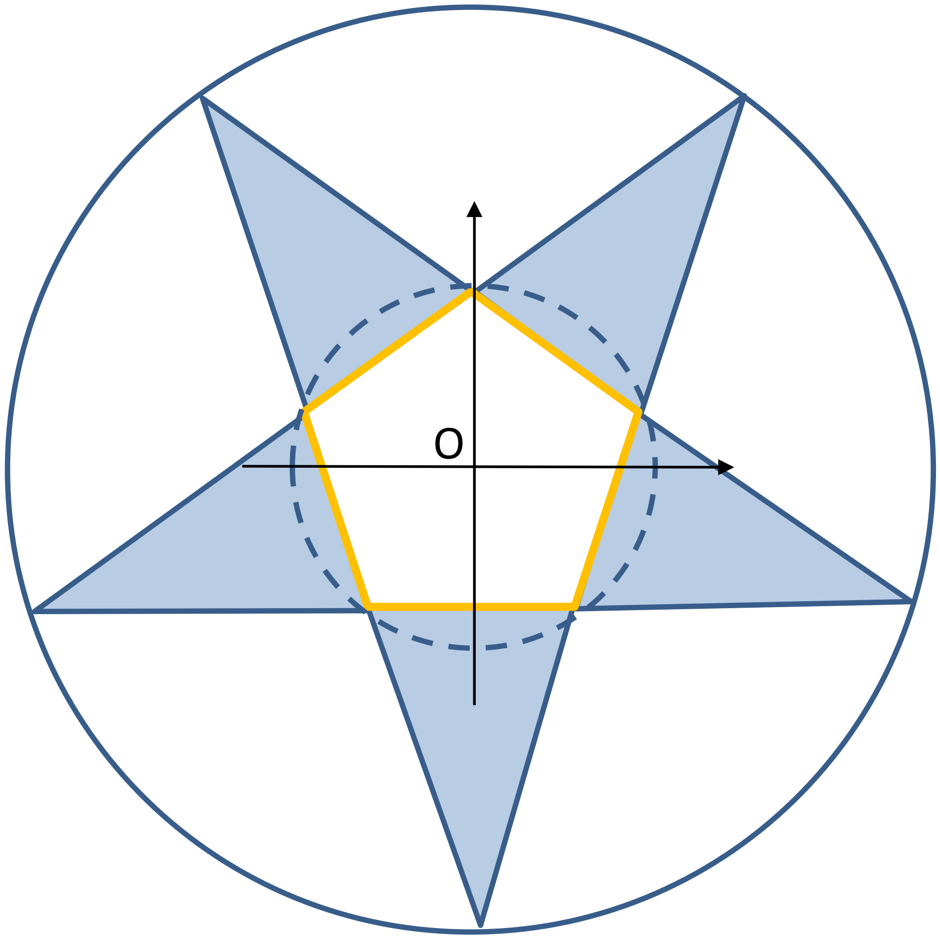
$$\Rightarrow w = \sqrt{4 - \frac{\left(1+\sqrt{5}\right)^2}{4}} = \sqrt{4 - \frac{1+2\sqrt{5}+5}{4}} = \tfrac{1}{2}\sqrt{16-1-2\sqrt{5}-5}$$

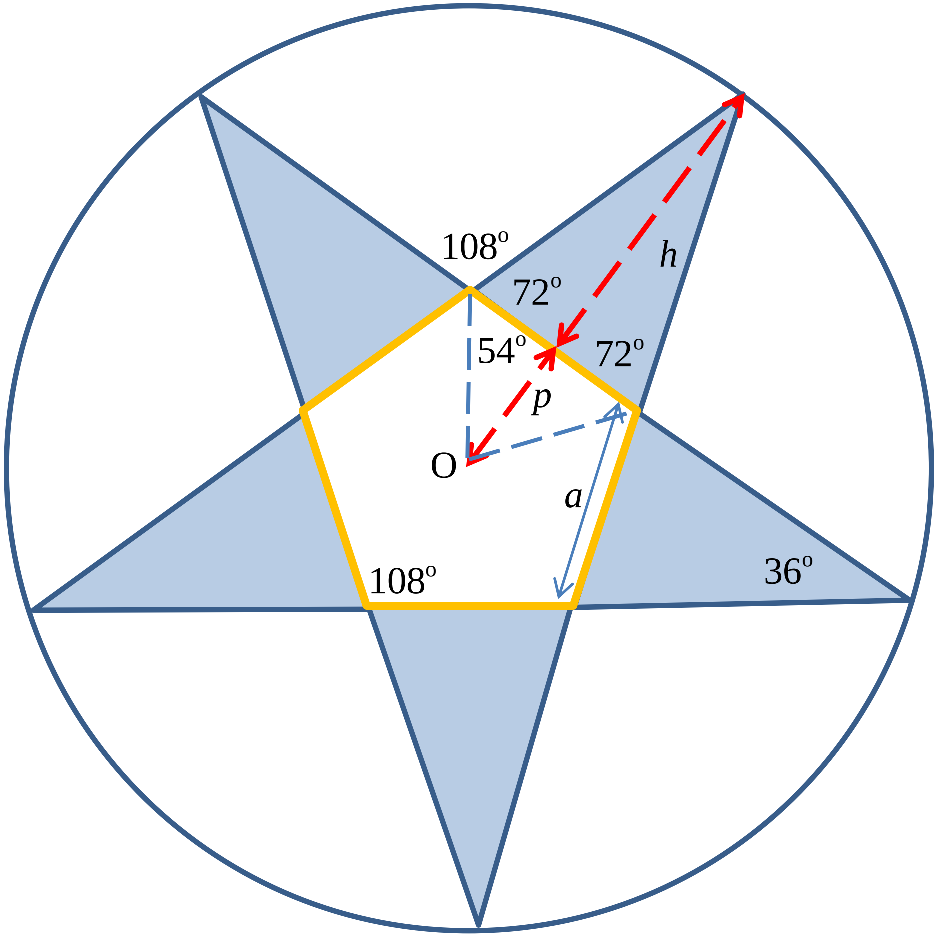
$$\Rightarrow \boxed{w = \tfrac{1}{2}\sqrt{10-2\sqrt{5}}}$$



Place a ruler on each side of the regular pentagon and extend lines. Construct isosceles triangles in this way and hence make a **pentagram.**

Circumscribe it using the original circle used to construct the regular pentagon.





Some basic trigonometry can allow us to work out formulae for the area of the regular pentagon and the pentagram

$$p = \tfrac{1}{2} a \tan 54^\circ$$

$$h = \tfrac{1}{2} a \tan 72^\circ$$

Pentagon area is (via adding triangles of base *a* and perpendicular height *p*)

$$A = 5 \times \tfrac{1}{2} ap$$

$$A = 5 \times \tfrac{1}{2} a \tfrac{1}{2} a \tan 54^\circ$$

$$\boxed{A = \tfrac{5}{4} a^2 \tan 54^\circ}$$

Pentagram area is (via adding triangles of base *a* and perpendicular height *h* to *A*)

$$P = A + 5 \times \tfrac{1}{2} ah$$

$$P = A + \tfrac{5}{4} a^2 \tan 72^\circ$$

$$\boxed{P = \tfrac{5}{4} a^2 \left(\tan 54^\circ + \tan 72^\circ \right)}$$

$$\tan 72^{\circ} = \tan\left(36^{\circ} + 36^{\circ}\right)$$

$$= \frac{2\tan 36^{\circ}}{1-\tan^{2} 36^{\circ}}$$

$$\tan^{2} 36^{\circ} = \frac{10-2\sqrt{5}}{\left(1+\sqrt{5}\right)^{2}}$$

$$\tan^{2} 36^{\circ} = \frac{10-2\sqrt{5}}{1+2\sqrt{5}+5} = \frac{10-2\sqrt{5}}{6+2\sqrt{5}}$$

$$= \frac{\left(10-2\sqrt{5}\right)\left(6-2\sqrt{5}\right)}{\left(6+2\sqrt{5}\right)\left(6-2\sqrt{5}\right)} = \frac{60-32\sqrt{5}+20}{36-20}$$

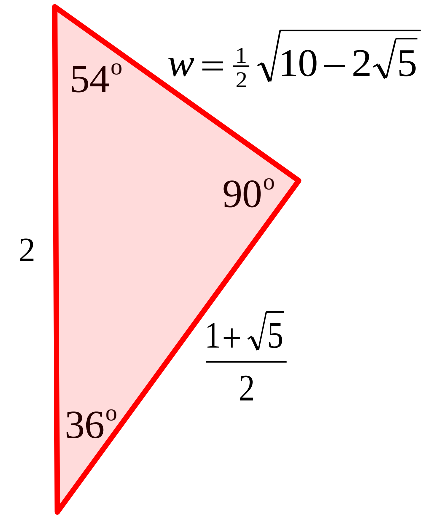
$$= \frac{80-32\sqrt{5}}{16} = 5-2\sqrt{5}$$

$$\tan 36^{\circ} = \sqrt{5-2\sqrt{5}}$$

$$\tan 72^{\circ} = \frac{2\sqrt{5-2\sqrt{5}}}{1-5+2\sqrt{5}}$$

$$\tan 72^{\circ} = \frac{2\sqrt{5-2\sqrt{5}}}{2\sqrt{5}-4} = \frac{2\sqrt{5-2\sqrt{5}}\times\left(2\sqrt{5}+4\right)}{\left(2\sqrt{5}-4\right)\left(2\sqrt{5}+4\right)}$$

$$\tan 72^{\circ} = \frac{\left(4\sqrt{5}+8\right)\sqrt{5-2\sqrt{5}}}{20-16} = \left(\sqrt{5}+2\right)\sqrt{5-2\sqrt{5}}$$



$$\cos 36^{\circ} = \frac{1+\sqrt{5}}{4}$$

$$\sin 36^{\circ} = \tfrac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$\cos 54^{\circ} = \tfrac{1}{4}\sqrt{10-2\sqrt{5}}$$

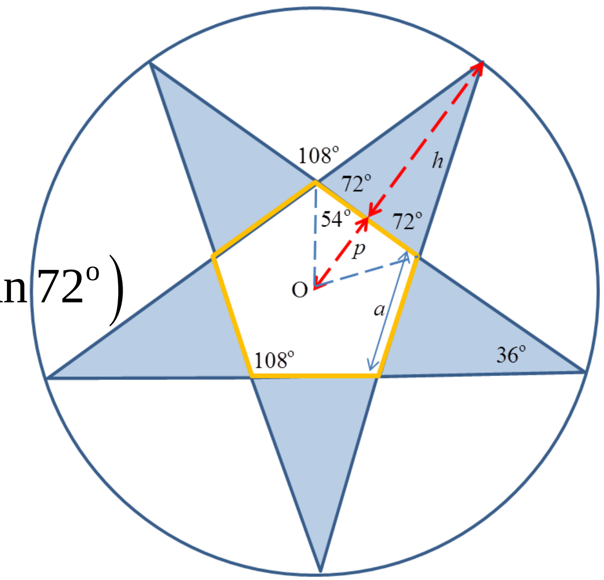
$$\sin 54^{\circ} = \frac{1+\sqrt{5}}{2}$$

$$\tan 36^{\circ} = \frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$$

$$\tan 54^{\circ} = \frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$$

$$A = \tfrac{5}{4}a^{2}\tan 54^{\circ}$$

$$P = \tfrac{5}{4}a^{2}\left(\tan 54^{\circ} + \tan 72^{\circ}\right)$$

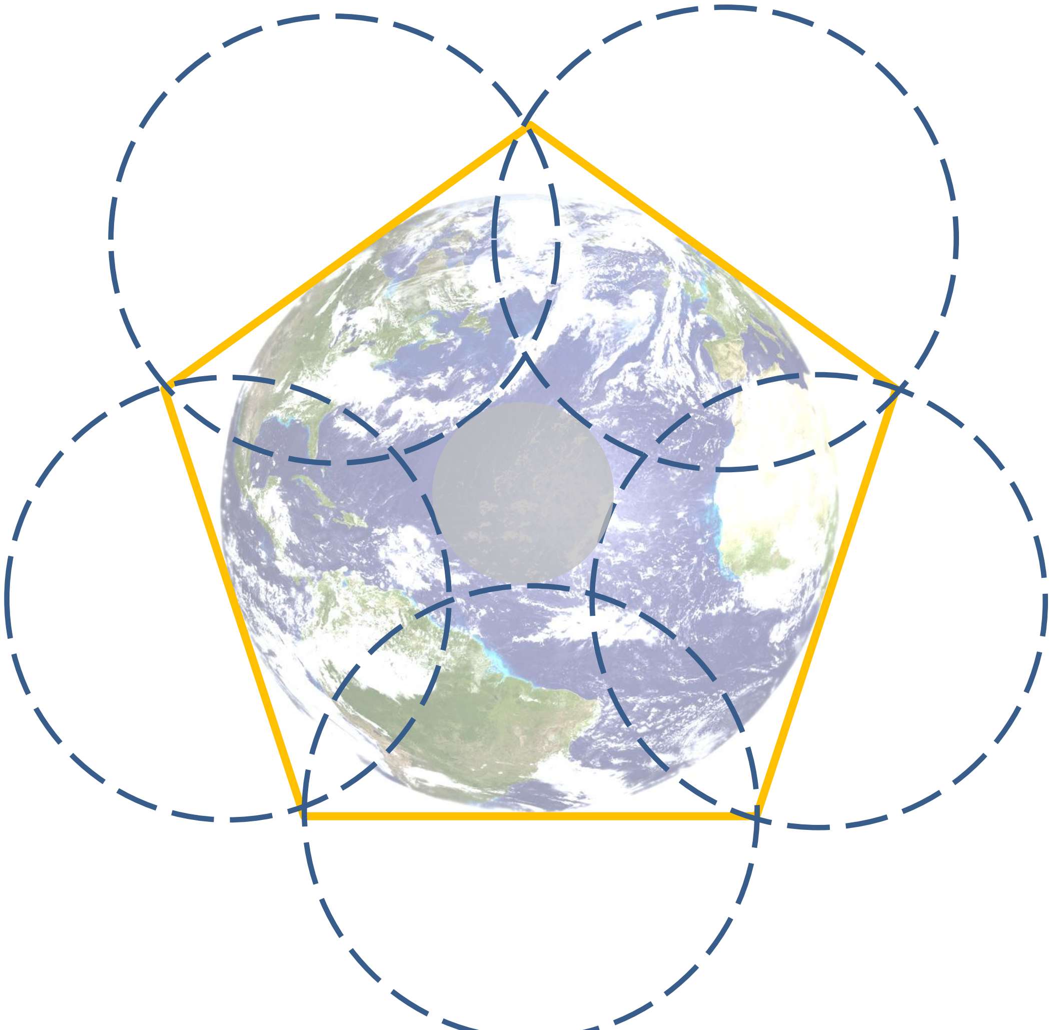


$$A = \tfrac{5}{4}a^{2}\,\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$$

$$P = \tfrac{5}{4}a^{2}\left(\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} + \left(\sqrt{5}+2\right)\sqrt{5-2\sqrt{5}}\right)$$

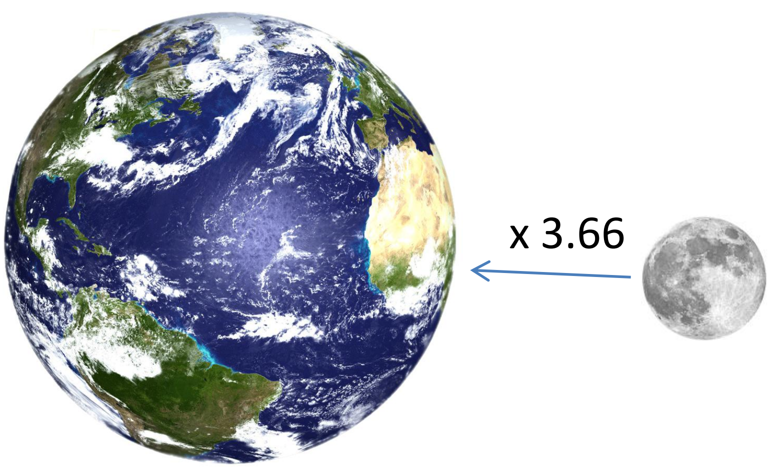
The ratio of Moon and Earth radii is approximately $\frac{R_\oplus}{R_M} \approx \frac{6353\text{km}}{1737\text{km}} = 3.66$

Find the midpoint of each edge of a regular pentagon and hence construct circles whose diameters are these edges.

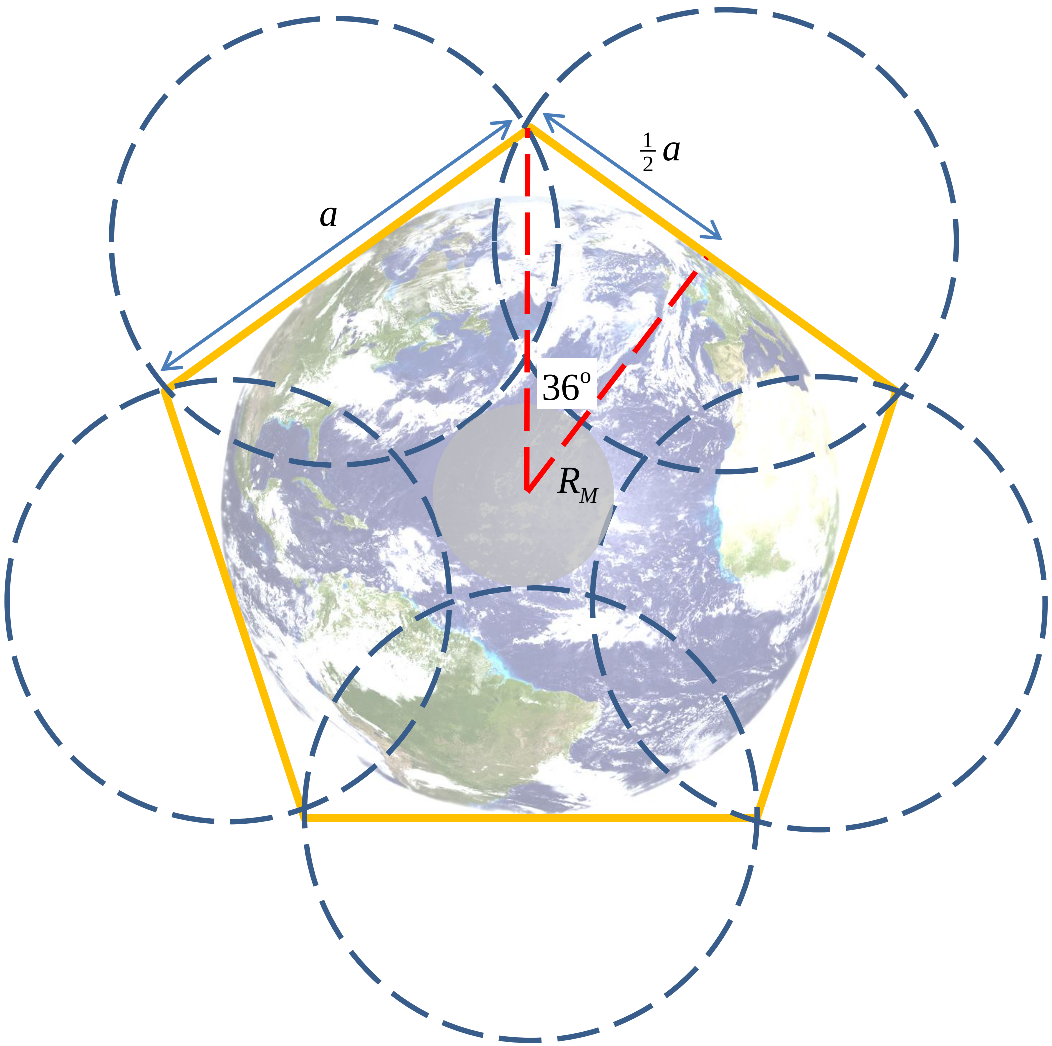


Then construct circles that
(i) inscribe the pentagon
and (ii) inscribe the 'concave pentagon' formed by the boundary of the intersecting outer circles.

You will discover the ratio of the circle radii is also 3.66 .....



The ratio of Moon and Earth radii is approximately $\dfrac{R_{\oplus}}{R_M} \approx \dfrac{6353\text{km}}{1737\text{km}} = 3.66$



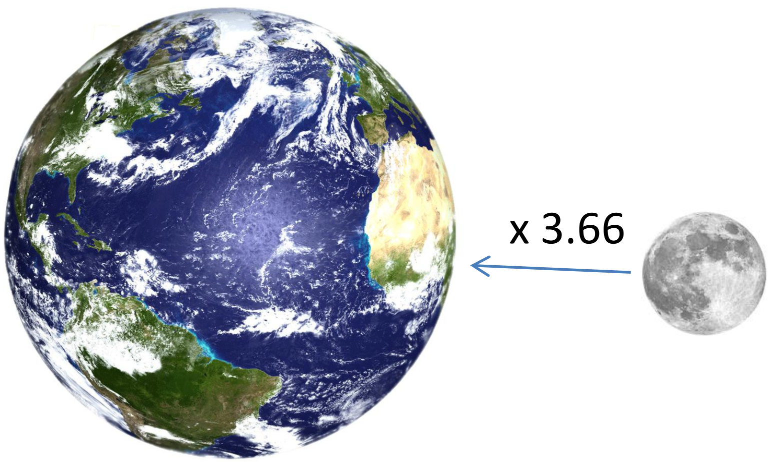
By the construction on the left:

$$R_M = R_{\oplus} - \tfrac{1}{2}a$$

$$\tfrac{1}{2}a = R_{\oplus}\tan 36^{\circ}$$

$$\therefore R_M = R_{\oplus} - R_{\oplus}\tan 36^{\circ}$$

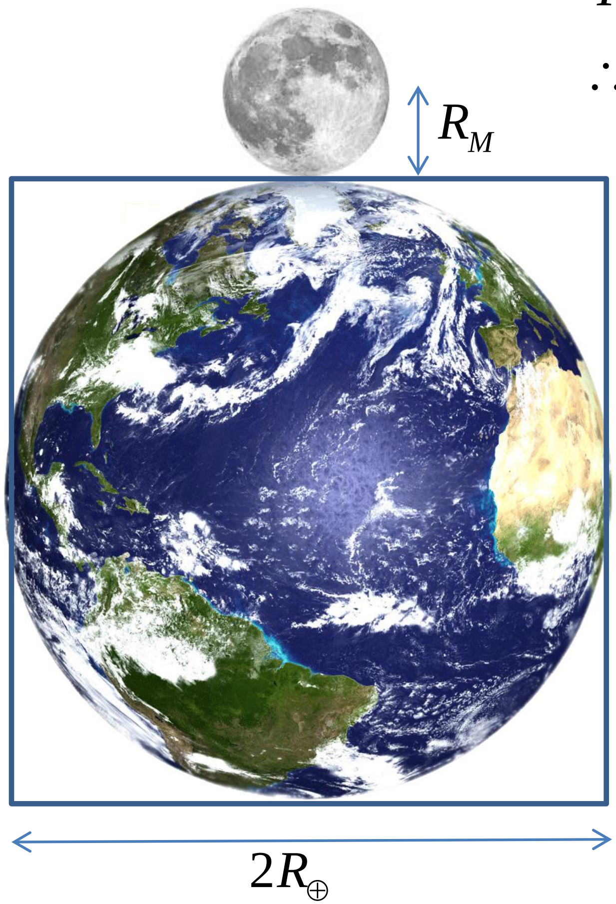
$$\frac{R_{\oplus}}{R_M} = \frac{1}{1-\tan 36^{\circ}} \approx 3.66$$



$$\frac{R_\oplus}{R_M} \approx 3\tfrac{2}{3} = \tfrac{11}{3}$$

$$\therefore R_\oplus = \tfrac{11}{3} R_M$$

This amazing coincidence also shows that 'the Moon and the Earth square the circle'



The circumference of the Earth plus the circumference of the Moon is given by:

$$C = 2\pi R_\oplus + 2\pi R_M$$

$$C = 2\pi R_M \left(\tfrac{11}{3} + 1\right)$$

$$C = 2\pi R_M \left(\tfrac{11}{3} + \tfrac{3}{3}\right) = 2\pi R_M \times \tfrac{14}{3}$$

$$C = \tfrac{4\times 7}{3}\pi R_M$$

The perimeter of a square bounding the Earth is

$$P = 4 \times 2R_\oplus = 4 \times \tfrac{22}{3} R_M$$

$$\therefore \quad \frac{P}{C} = \frac{4\times\tfrac{22}{3}R_M}{\tfrac{4\times 7}{3}\pi R_M} = \tfrac{22}{7}\times\tfrac{1}{\pi} = 1.000402\ldots$$

i.e. $P = C$ to a very good approximation!

Note this is perhaps where the popular approximation $\pi \approx \tfrac{22}{7}$ *might* have originated....