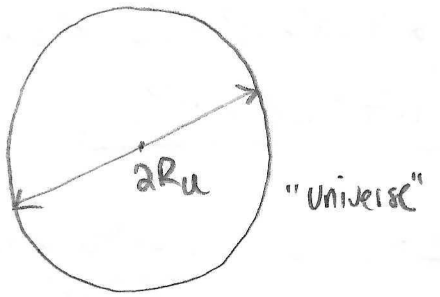
COSMOLOGY & ASTROPHYSICS

1/ (i)



$2R_u = 93 \times 10^9$ light years

$M_u = 10^{53}$ kg

$M_p = 1.673 \times 10^{-27}$ kg proton mass

a) $1\,ly = 2.998 \times 10^8\ m/s \times 365 \times 24 \times 3600\ s$

$= \boxed{9.461 \times 10^{15}\ m}$

$\therefore\ R_u = \dfrac{93 \times 10^9}{2} \times 9.461 \times 10^{15}\ m$

$= \boxed{4.40 \times 10^{26}\ m}$ Radius of the Universe in metres.

b) $V = \frac{4}{3}\pi R_u^3$

$V = \frac{4}{3}\pi \left(4.40 \times 10^{26}\right)^3 = \boxed{3.56 \times 10^{80}\ m^3}$

c) # protons that are equivalent to $M_u = 10^{53}$ kg are $N = \dfrac{10^{53}}{1.673 \times 10^{-27}} = \boxed{5.98 \times 10^{79}}$

$\therefore$ Density of the universe in protons/m^3 is

$\rho = \dfrac{5.98 \times 10^{79}\ \text{protons}}{3.56 \times 10^{80}\ m^3} = \boxed{0.17\ \text{protons}/m^3}$

Note according to the WMAP study of the cosmic microwave background radiation, the density is more like $\boxed{0.25\ \text{protons}/m^3}$

BUT, if we take into account the mass-energy (ie $E = mc^2$ equivalent) this results in about $\boxed{6\ \text{protons}/m^3}$

$\therefore$ 95% of mass-energy in the Universe appears to be 'dark matter' or 'dark energy' ... or our models of gravity at the largest scales are incorrect.

(ii) a) 1 light minute $= 2.998 \times 10^8 \, m/s \times 60 s$

$= \boxed{1.7988 \times 10^{10} m}$

$\therefore$ Earth Sun distance $1 AU = \dfrac{1.496 \times 10^{11} m}{1.7988 \times 10^{10} m/\text{light minute}}$

$= \boxed{8.32 \text{ light minutes}}$

(ie, Sunlight takes 8 mins 20s to reach Earth)

b) Distance to Sirius A is $\dfrac{8.6 \times 9.461 \times 10^{15} m}{1.496 \times 10^{11} m/AU}$

$= 543,880$ AU

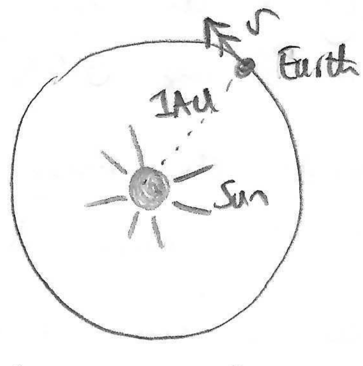
$= \boxed{5.44 \times 10^5 \text{ AU}}$

c) let Spacecraft from Earth to Sirius A travel at 15.3 km/s

This will take $\dfrac{8.6 \times 9.461 \times 10^{15} m}{15.3 \times 10^3 m/s}$

$= 5.32 \times 10^{12} s = \boxed{168,631 \text{ years}}$

ie $\boxed{1.69 \times 10^5 \text{ years}}$

d) 

(Assume circular orbit)

orbital speed of Earth is

$v = \dfrac{2\pi \times 1.496 \times 10^{11} m}{365 \times 24 \times 3600 s}$

$= \boxed{29.8 \text{ km/s}}$

ie $1.95 \times$ Voyager 2 speed.

(iii) NEW HORIZONS Spacecraft

	Speed / kms^{-1}	Distance / m	time / yr
Earth $\rightarrow$ Jupiter	16.26	$4.202 \times 1.496 \times 10^{11}$	t_1
Jupiter $\rightarrow$ Pluto	20.26	x	t_2

Assume Earth $\rightarrow$ Jupiter is closest approach

Furthest: $x = 7.5 \times 10^9 \times 10^3 - 4.202 \times 1.496 \times 10^{11} = 6.87 \times 10^{12}$ m

Closest : $x = 4.28 \times 10^9 \times 10^3 - 4.202 \times 1.496 \times 10^{11} = 3.65 \times 10^{12}$ m

$$t_1 = \frac{4.202 \times 1.496 \times 10^{11}}{16.26 \times 10^3} \quad \frac{1}{365 \times 24 \times 3600} \text{ years}$$

$= \boxed{1.23 \text{ years}}$ (!) to reach Jupiter

$$\text{min } t_2 = \frac{3.65 \times 10^{12}}{20.26 \times 10^3} \quad \frac{1}{365 \times 24 \times 3600} = \boxed{5.71 \text{ years}}$$

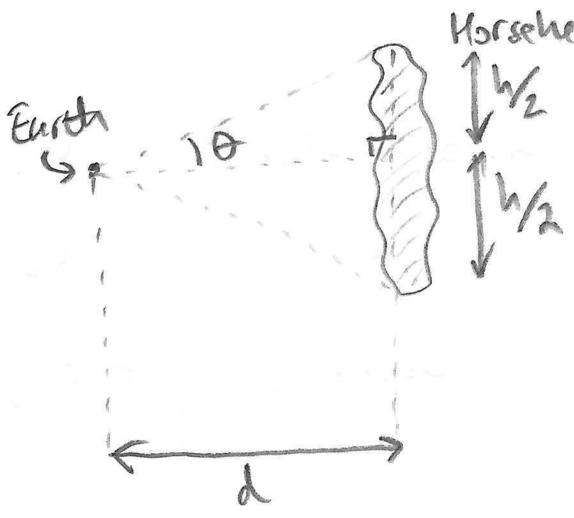
$$\text{max } t_2 = \frac{6.87 \times 10^{12}}{20.26 \times 10^3} \quad \frac{1}{365 \times 24 \times 3600} = \boxed{10.75 \text{ years}}$$

So total time to reach pluto is between 6.94 and 11.98 years.

Flyby was on July 14th 2015 and launch was Jan 19th 2006

ie transit time was 3463 days $\approx$ $\boxed{9.49 \text{ years}}$

(iv)



$2\theta = \frac{8}{60}$ degrees.

(8 "arc minutes")

$h/2 = d \tan\theta$

$d = 1500$ light years.

$\therefore \quad h = 2 \times 1500 \tan\left(\frac{8}{120}\right)$ light years

$= \boxed{3.49 \text{ light years}}$

$h = 3.49 \times 9.461 \times 10^{15}\ \text{m} \quad = \boxed{3.30 \times 10^{16}\ \text{m}}$

1 parsec $= 3.26$ ly $\quad \therefore \quad h = \frac{3.49}{3.26}$ pc

$= \boxed{1.07\ \text{pc}}$

$h = \frac{3.30 \times 10^{16}\ \text{m}}{1.496 \times 10^{11}\ \text{m/AU}} \quad = \boxed{220{,}800\ \text{AU}}$

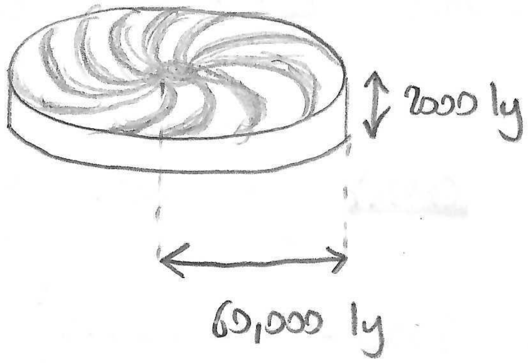
$\boxed{(2.21 \times 10^{5}\ \text{AU to } 3\ \text{s.f})}$

b) To traverse the Nebula at 1% of light speed will take:

$\Delta t = \frac{3.49 \text{ light-years}}{0.01} = \boxed{349 \text{ years}}$

$\left(\text{or} \quad \frac{3.30 \times 10^{16}\ \text{m}}{0.01 \times 2.998 \times 10^{8}\ \text{m/s}} \cdot \frac{1}{365 \times 24 \times 3600\ \text{s/yr}} = 349 \text{ years} \right).$

(v) a) Milky way model



$$V = 2000 \times \pi (60{,}000)^2 \times (9.461 \times 10^{15}\ m/ly)^3$$
$$= \boxed{1.92 \times 10^{61}\ m^3}$$

b) $1\,AU^3 = (1.496 \times 10^{11}\ m)^3$

$$\therefore V = \frac{1.92 \times 10^{61}\ m^3}{(1.496 \times 10^{11})^3\ m^3/AU^3} = \boxed{5.72 \times 10^{27}\ AU^3}$$

c) Milky way has 400×10^9 stars, each with volume $\frac{4}{3} \times \pi \times (50\ AU)^3$ which corresponds to a solar system.

$\therefore$ Fraction of Milky way volume occupied by stars and their solar systems is:

$$\frac{400 \times 10^9 \times \frac{4}{3}\pi(50)^3}{5.72 \times 10^{27}} = \boxed{3.66 \times 10^{-11}}$$

ie a very tiny proportion. Most of the volume is 'deep interseller space'.

d) orbital speed of sun about galactic centre is:

$$v_{\odot} = \frac{2\pi \times 27{,}000 \times 9.461 \times 10^{15} \times \frac{1}{1000}\ km}{240 \times 10^6 \times 365 \times 24 \times 3600\ s} = \boxed{212\ km/s}$$

i.e. about 7 × the orbital speed of the Earth about the Sun, 29.8 km/s.

(vi) let Milky Volume consist of 40 million spheres of radius R

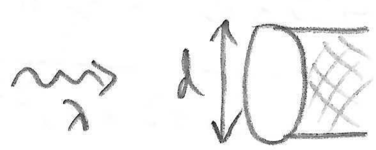
$$\therefore 40 \times 10^6 \times \frac{4}{3}\pi R^3 = 2000 \times \pi \times (60{,}000)^2 \ (ly)^3$$

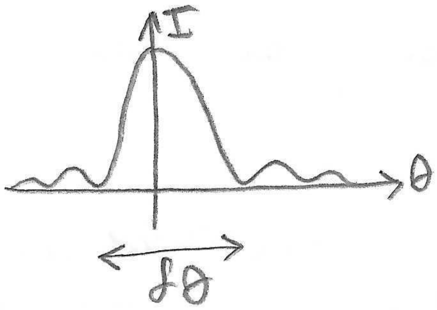
$$\therefore R = \left(\frac{2000 \times \pi \times (60{,}000)^2}{40 \times 10^6 \times \frac{4}{3}\pi} \right)^{1/3} \ ly$$

$$= \boxed{51.3 \text{ light years}}$$ is the

average separation between civilizations, if the Drake equation is correct. This is too large a distance for physical contact (based upon current propulsion technology and our understanding of relativistic effects), but radio contact may already have occurred! (The first radio broadcast to leave the Earth was in the 1936 Berlin olympics). Hence the mission of $\boxed{\text{SETI}}$ (Search for extra-terrestrial intelligence, which scans the sky for 'unnatural' radio broadcasts from the cosmos).

(vii) A space telescope has an angular resolution of $\boxed{\delta\theta \simeq \lambda/d}$





This is ≃ the angular width of the diffraction pattern cause by the finite aperture d.

So $\boxed{d \simeq \frac{\lambda}{\delta\theta}}$ $\delta\theta = 10^{-3} \times \frac{\pi}{180} \times \frac{1}{3600}$ radians

a) $d_{vis} \approx \dfrac{10^{-7} m}{10^{-3} \times \frac{\pi}{180} \times \frac{1}{3600}} = \boxed{20.63\ m}$

$\longrightarrow 4.85 \times 10^{-9}$ radians

b) $d_X \approx \boxed{0.021\ m}$

c) $d_{IR} \approx \boxed{2063\ m}$ (!)

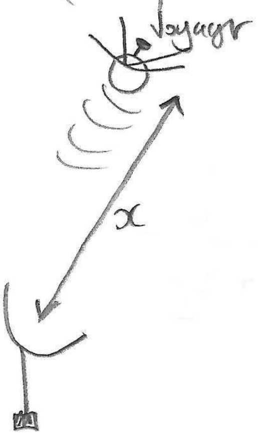
This is probably impractical, so an array of telescopes would be required.

Note the Hubble optical telescope has a resolution of about 0.05 arc-seconds $\therefore \boxed{d \approx 0.413 m}$

This means the optics is a little more complicated than this requirement indicates, since the primary mirror of Hubble is 2.4m, with a 0.3m secondary mirror.

So perhaps are d values should be 10 × larger!

(viii) Φ from Voyager is: $\boxed{\dfrac{PG}{4\pi x^2}}$ (ie flux in Wm^{-2})



where G is the gain, P is the transmit power

let collecting area of antenna be $\boxed{A = \pi D^2/4}$ where D is the diameter

$\therefore$ power received $P_R = \dfrac{PG}{4\pi x^2} \pi D^2/4 = \boxed{\dfrac{PGD^2}{16x^2}}$

$$= \frac{22.4 \times 10^{48/10} \times 70^2}{16 \times (30.2 \times 1.496 \times 10^{11})^2} = \boxed{2.12 \times 10^{-17}\ W}$$

Not a lot!

(ix)

(a)

$$\frac{\Delta\lambda}{\lambda} \approx \frac{v}{c} \qquad \therefore \quad v = \frac{c\,\Delta\lambda}{\lambda}$$

$$\frac{\Delta\lambda}{\lambda} = -z = 3.668 \times 10^{-4} \quad \text{in our case}$$

(i.e. a negative redshift, or a 'blueshift')

$$\therefore \quad v = \frac{2.998 \times 10^{8} \times 3.668 \times 10^{-4}}{1000}\ \mathrm{km/s}$$

$$= \boxed{110\ \mathrm{km/s}}$$

[Note: $\Delta f = +\frac{v}{c} f_{em}$, if v is approaching, is the Doppler formula.

$$f_{obs} = f_{em} + \Delta f = f_{em}(1 + v/c)$$

$$c = f\lambda \quad \therefore \lambda = c/f \quad \therefore \lambda_{obs} = \frac{c}{f_{em}(1 + v/c)}$$

$$\frac{c}{f_{em}} = \lambda_{em} \Rightarrow \quad \lambda_{obs} = \frac{\lambda_{em}}{1 + v/c}$$

$$\therefore \quad z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{1}{1 + v/c} - 1$$

$$= \frac{1 - 1 - v/c}{1 + v/c} = \frac{-v/c}{1 + v/c}$$

$$\therefore \quad \boxed{z = \frac{-v}{c + v}} \qquad \therefore \quad v(z+1) = -zc$$

$$\therefore \quad \boxed{v = \frac{-zc}{1+z}} \quad]$$

(b) However in our case $|z| \ll 1$ so this correction makes little difference

[Correction! $\boxed{z = \frac{v}{c}}$ for source moving along line of sight is correct, not an approximation. (See Electromagnetic Doppler shift notes in waves section). $\Delta f \approx -\frac{v}{c} f_e$ is the approximation.

$$\boxed{1 + z = \lambda_o / \lambda_e} \Rightarrow z = \lambda_o / \lambda_e - 1$$

$$\boxed{z = \frac{\lambda_o - \lambda_e}{\lambda_e}}$$]

b) $2.5 \times 10^6 \times 9.461 \times 10^{15}$ m is the distance of Andromeda galaxy from Earth.

So if Andromeda is approaching at 110 km/s this means the Milky Way and Andromeda will be merged after

$$\frac{2.5 \times 10^6 \times 9.461 \times 10^{15}}{110 \times 10^3 \times 365 \times 24 \times 3600} \text{ years}$$

$$= \boxed{6.8 \text{ billion years}}$$

In about 5.4 billion years our Sun will become a red giant and engulf the earth. This will eventually shed about half a solar mass into a planetary nebula, leaving a white dwarf star.

The earth won't survive this! If humans do, then hopefully they will occupy other planets by this point!

(x) a) Distance to Betelgeuse is $d = \frac{AU}{\Delta\theta}$

$$= \frac{1.496 \times 10^{11}\,m}{5.95 \times \frac{\pi}{180} \frac{1}{3600} \times \frac{1}{1000}}$$

$$= \boxed{5.186 \times 10^{18}\,m}$$

$$= \boxed{548\,ly} \qquad (\because 9.461 \times 10^{15}\,m)$$

[Though on the internet (!) it says 642.5 ly from Earth...?]

↑ google search

b) Wien's law : $\boxed{\lambda_{max}/nm = \frac{2.899 \times 10^6}{T/k}}$

$$\therefore \quad T = \frac{2.899 \times 10^6}{805} \quad (k)$$

$$= \boxed{3600\,k} \qquad \text{(Surface temperature)}$$

c) $L = 4\pi R^2 \sigma T^4 \quad \therefore \quad \frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4$

$$\therefore \quad R/R_\odot = \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}} \left(\frac{T_\odot}{T}\right)^2$$

Now $\frac{\lambda_{max}}{502\,nm} = \frac{T_\odot}{T}$ from Wein's law

$$\Rightarrow \quad R/R_\odot = (126000)^{\frac{1}{2}} \left(\frac{805}{502}\right)^2$$

$$= \boxed{913}$$

But L is estimated too!

{ which is larger than 764^{+116}_{-62} quoted in Wikipedia }

d) $\frac{\Delta\lambda}{\lambda} = v/c$ where v is the recessional velocity

$\Delta\lambda = 5.88 \times 10^{-2}$ nm

$\lambda = 805$ nm

$\therefore v = 2.998 \times 10^{8} \times \frac{5.88 \times 10^{-2}}{805} \frac{1}{1000}$ km/s

$= \boxed{21.9 \text{ km/s}}$

e) Average density $\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{\frac{1}{2}(16.5+19) \times 1.99 \times 10^{30}}{\frac{4}{3}\pi\left(913 \times 696340 \times 10^{3}\right)^3}$

from c)

$= \boxed{3.28 \times 10^{-5} \text{ kg/m}^3}$

(According to Google it is 1.2×10^{-5} kg/m^3, not surprising given uncertainties in both mass and radius).

Note $\rho_\odot = \frac{1.99 \times 10^{30}}{\frac{4}{3}\pi(696340 \times 10^3)^3} = \boxed{1407 \text{ kg/m}^3}$

which gives one some idea how much less dense a red giant star is than a yellow main sequence star like the sun.

(xi) Age of Universe is $t \propto \frac{1}{H_0}$ ← Hubble constant.

$H_0 = 71.9$ km/s / Mpc

$= \frac{71.9 \times 10^{3} \text{ m/s}}{10^6 \times 3.086 \times 10^{16} \text{m}} = 2.33 \times 10^{-18} \text{ s}^{-1}$

$\therefore t/\text{years} = \frac{1}{2.33 \times 10^{-18} \times 365 \times 24 \times 3600}$

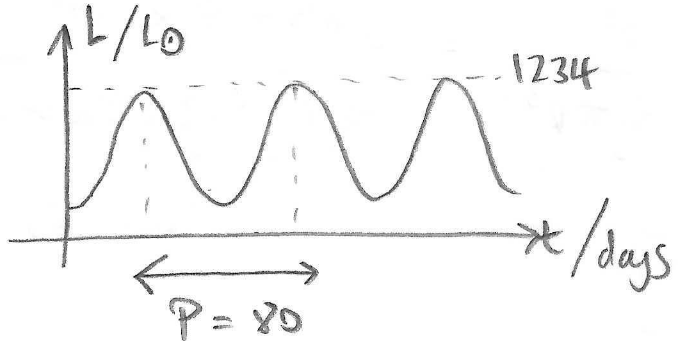
$= \boxed{13.6 \text{ billion years}}$

ie 0.2 billion years (1.5%) short of 13.8 billion years

which is the modern multi-method calculation.
(13.813 ± 0.038) billion years according to Planck Survey.

↳ Combining with other methods: $\boxed{(13.799 \pm 0.021) \text{ billion years}}$
as of 2015. So 13.8 billion to 3 s.f.

(xii)



Variation of luminosity of a Cepheid variable star.

So expected flux Φ received by a near-earth space telescope is:

$$\Phi = \frac{L}{4\pi x^2}$$

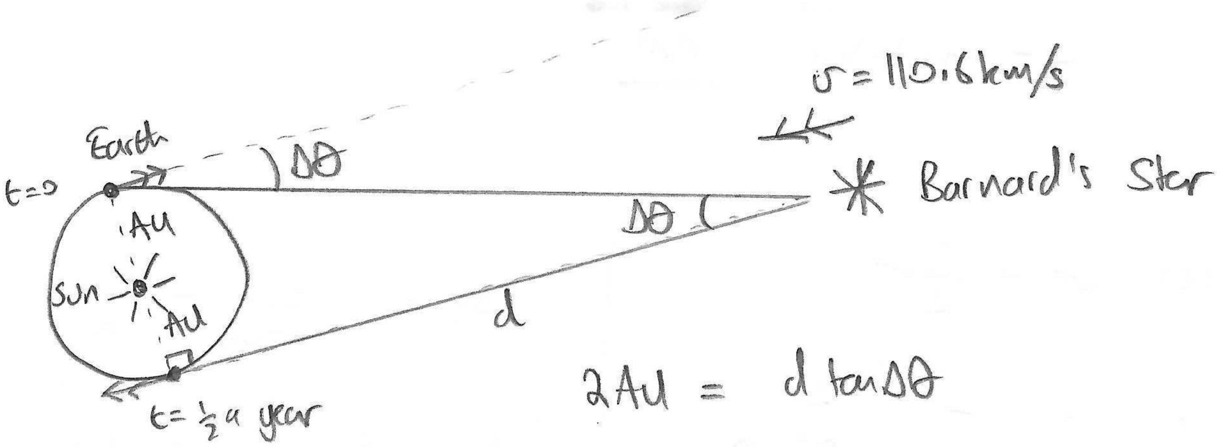
$$x = 4321\ \text{pc} = 4321 \times 3.086 \times 10^{16}\ \text{m}$$

$$\therefore \quad \Phi = \frac{1234 \times 3.846 \times 10^{26}\ \text{W}}{4\pi \times (4321 \times 3.086 \times 10^{16}\ \text{m})^2}$$

$$= \boxed{2.12 \times 10^{-12}\ \text{W/m}^2}$$

2/ See spreadsheet.

3/ BARNARD'S STAR
(4th closest star from Sun)



$$2\text{AU} = d \tan\Delta\theta$$

a)

$$\Delta\theta = \frac{1.0908}{3600} \text{ degrees.}$$

$$\therefore \; d/\text{AU} = \frac{2}{\tan\Delta\theta} = \boxed{378{,}000}$$

$$\therefore \; d = \frac{2}{\tan\Delta\theta} \times 1.496\times10^{11} \text{ m}$$

$$= 5.66\times10^{16} \text{ m}$$

$\div\; 9.461\times10^{15}$ m/ly

$$= \boxed{5.98 \text{ ly}} \checkmark$$

b) If a spacecraft travels at 57888 km/h (Voyager probe speed)

it will take
$$\frac{5.66\times10^{16}\text{ m}}{\frac{57888\times10^{3}\text{ m}}{3600\text{ s}}} \times \frac{1}{365\times24\times3600} \text{ yr/s}$$

$$= \boxed{112{,}000 \text{ years}}$$

c) $\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$ and $v = -110.6$ km/s

$\lambda = 121.6$ nm (UV photons from Hydrogen energy state change).

(13)

$$\therefore \quad \Delta\lambda = \frac{-110.6\times10^{3}}{2.998\times10^{8}} \times 121.6 \text{ nm}$$

c) (cont...)

$$= -\boxed{4.49\times10^{-2} \text{ nm}}$$

let $\Delta f = \frac{v}{c} f$

$$f + \Delta f = \frac{c}{\lambda + \Delta\lambda}$$

$$f = \frac{c}{\lambda}$$

$$\therefore \Delta f = \frac{c}{\lambda + \Delta\lambda} - \frac{c}{\lambda}$$

$$\therefore \Delta f = \frac{c}{\lambda}\left(\frac{1}{1+\Delta\lambda/\lambda} - 1\right)$$

$$\therefore \Delta f = \frac{2.998\times10^{8}}{121.6\times10^{-9}}\left(\frac{1}{1-4.49\times10^{-2}} - 1\right) \quad (\text{Hz})$$

$$= 2.465\times10^{15} \text{ Hz} \times 3.6905\times10^{-4}$$

$$= \boxed{9.099\times10^{11} \text{ Hz}}$$

$$\left(\Delta f/f = 0.037\%\right)$$

d) $\boxed{z = \frac{\lambda_o - \lambda_e}{\lambda_o}}$ $= \frac{\Delta\lambda}{\lambda_o}$

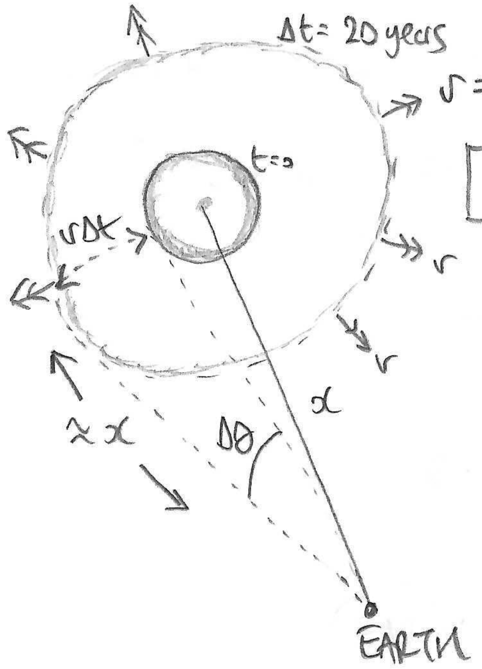
Redshift

$$= \frac{-4.49\times10^{-2}}{121.6}$$

$$= \boxed{-3.7\times10^{-4}}$$

A negative redshift is a blueshift, which means the source is approaching, not receeding.

4/



$v = 1500$ km/s ← get this from doppler shift $\frac{\Delta\lambda}{\lambda} = -v/c$ (near nebula approaches)

$\boxed{\text{Expanding Crab nebula}}$ (radial expansion)

$$\Delta\theta = 3.16 \times \frac{\pi}{180} \times \frac{1}{3600} \text{ radians}$$

$$v\Delta t \approx x\Delta\theta \qquad (x \gg v\Delta t)$$

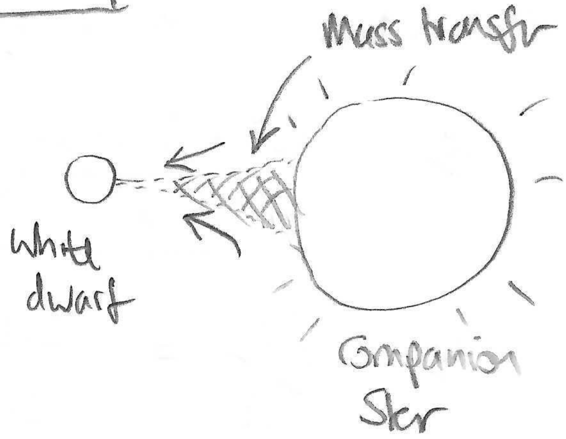
so distance to Crab nebula is

$$\boxed{x \approx \frac{v\Delta t}{\Delta\theta}}$$

$$\therefore x = \frac{1500 \times 10^3 \times 20 \times 365 \times 24 \times 3600}{3.16 \times \frac{\pi}{180} \times \frac{1}{3600} \times 9.461 \times 10^{15}} \text{ ly}$$

$$x = \boxed{6530 \text{ light years}}$$

5/ $\boxed{\text{Type 1a Supernova}}$



a) * Carbon-Oxygen white dwarf Star (end state of a Main Sequence Star between 0.4 and 4 $\times M_\odot$) accretes matter from a companion

* once Mass $\approx 1.4 M_\odot$, gravity overcomes the degeneracy pressure of the electrons in the white dwarf atoms. ⇒ Star implodes. (Pauli exclusion principle ⇒ spin $\frac{1}{2}$ particles like electrons and neutrons can't share the same quantum state).

* As white dwarf implodes, temperature rises. Eventually it is hot enough for spontaneous carbon fusion. This thermonuclear reaction is energetic enough to cause a chain reaction of carbon fusion, which causes the white dwarf to explode violently. Nothing is left!

b) $\Delta E = 1.4 M_{\odot} c^2$ for a type Ia Supernova.

$$\Rightarrow \quad \Delta E = 1.4 \times 1.99 \times 10^{30} \times (2.998 \times 10^8)^2 \quad (J)$$

$$= \boxed{2.50 \times 10^{47} \ J}$$

c) If average luminosity is $10^9 L_{\odot}$

then $10^9 L_{\odot} \Delta t = \Delta E$

$$\therefore \quad \Delta t = \frac{2.50 \times 10^{47} \ J}{10^9 \times 3.846 \times 10^{26} \ J/s}$$

$$= 6.51 \times 10^{11} s \quad = \boxed{20{,}650 \text{ years}}$$

This is <u>much</u> longer than most observed Type Ia Supernovae. In reality they are bright for $\approx$ 100 days.

ie $10^9 \times 3.846 \times 10^{26} \times 100 \times 24 \times 3600$ J

$= \boxed{3.33 \times 10^{42} \ J}$ released $\quad (\delta E)$

ie $\dfrac{\delta E}{\Delta E} = \dfrac{3.33 \times 10^{42}}{2.50 \times 10^{47}} \approx \boxed{1.33 \times 10^{-5}}$

$= 0.00133\,\%$

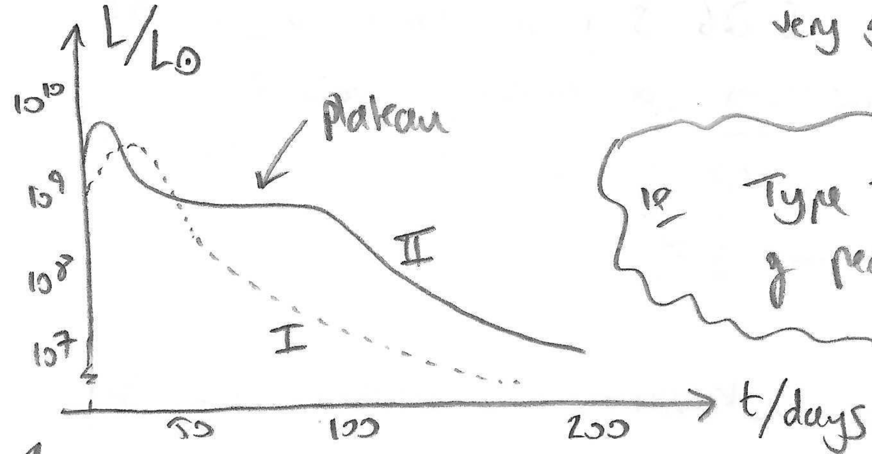
one assumes δE is the EM radiation released, the rest are $\boxed{\text{matter particles}}$ such as neutrons and neutrinos from electron and proton fusion. Also heavier elements too (eg from Carbon - Carbon fusion).

one assumes the violent shockwave from the rapid core collapse and runaway C-C fusion will blast the white dwarf matter into space.

(16)

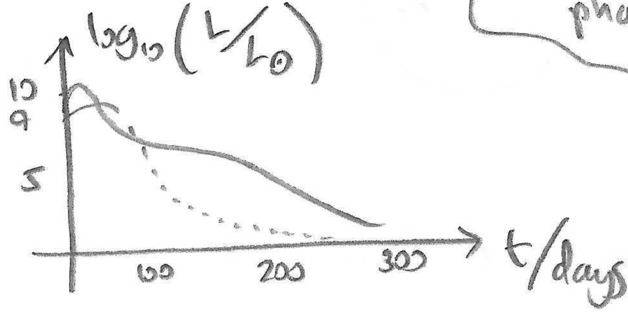
d) This is the curve in $\boxed{\text{Hyperphysics}}$ (Type I and II Supernovae)

↑ website. Very good! By R. Nave.



ie Type II plateaus $\simeq \frac{1}{10}$ of peak of Type I, and decays more rapidly, but lasts longer due to plateau phase

↑ ie a logarithmic curve (perhaps better to plot a $\log_{10} \frac{L}{L_\odot}$ y axis).



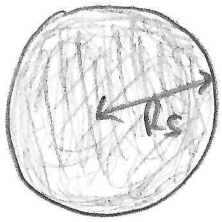
$\boxed{\text{Type II}}$:

* Giant star between 8 — 50 solar masses runs out of 'normal' fusion fuel (eg H isotopes). ∴ radiation pressure is overcome by gravity and the star implodes
* Implosion causes the core temperature sufficient to initiate fusion of heavier elements. This occurs until Iron / Nickel are formed. At this point, not energetically favourable to cause further fusion (Fe/Ni have a BINDING ENERGY maximum).
* ∴ once Fe/Ni are formed, core collapses further to form a white dwarf (but this is just the <u>core</u>, the star still will have an outer envelope).
* If core mass > 1.4 $M_\odot$ then gravity > degeneracy pressure and core will implode. It will rebound (against the iron/Ni ?) and the resulting violent shockwave ejects the outer envelope of the star.

or perhaps 'neutron matter' →

* The inner core becomes a neutron star. This will be stable unless mass > 3 or so solar masses. [Q6 suggest about 5]. At this point neutron degeneracy pressure is insufficient to arrest gravitational collapse, and a black hole will form.

6/ a)



Schwarzschild radius R_S of a black hole

$$\boxed{R_S = 2GM/c^2}$$

a) For our Sun to form a black hole

$$R_\odot < R_S$$

$$R_\odot < 2GM_\odot/c^2$$

$$R_\odot < \frac{2 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{(2.998 \times 10^8)^2}$$

$$\boxed{R_\odot < 2954\ m}$$ ie about 3 km (!)

[$R_\odot$ = 696,340 km at present]

b) For TON 618 ("hyperluminous quasar) $M = 66 \times 10^9 M_\odot$

$$R_S = \frac{2 \times 6.67 \times 10^{-11} \times 66 \times 10^9 \times 1.99 \times 10^{30}}{(2.998 \times 10^8)^2}$$

$$= 2954\ m \times 66 \times 10^9$$

$$= 1.949 \times 10^{14}\ m$$

$\div 1.496 \times 10^{11}$

$$= 1303\ AU$$

ie 136 times the orbital radius of Saturn.
It would easily engulf the whole Solar System!

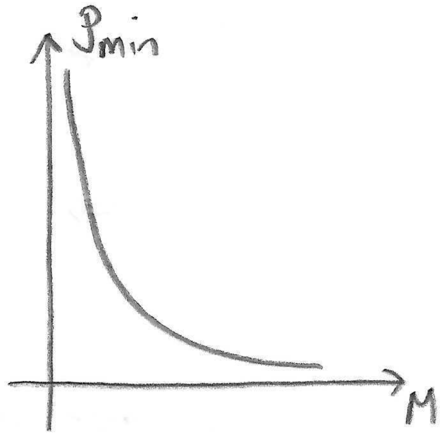
c) Density of black hole $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$

If $R < R_S$

$$\Rightarrow \quad \rho > \frac{M}{\frac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3}$$

$$\rho > \frac{3Mc^6}{4\pi \times 8G^3M^3}$$

$$\boxed{\rho > \frac{3c^6}{32\pi G^3 M^2}}$$



$$\rho_{min} = \frac{3c^6}{32\pi G^3}\,\frac{1}{M^2}$$

$$\therefore \quad M > \sqrt{\frac{3c^6}{32\pi G^3 \rho}}$$

So if $\rho = 1000\ kg/m^3$ (water density)

$$\therefore \quad \frac{M}{M_\odot} > \sqrt{\frac{3\times(2.998\times10^8)^6}{32\pi(6.67\times10^{-11})^3\times1000}} \Big/ 1.99\times10^{30}$$

$$\frac{M}{M_\odot} > 1.36\times10^8$$

ie $\boxed{\frac{M}{M_\odot} > 0.136 \text{ billion}}$ (or 136 million)

d) Neutron Matter:

$$\rho_n = \frac{M_n}{\frac{4}{3}\pi r_n^3} = \frac{1.675\times10^{-27}}{\frac{4}{3}\pi(0.8\times10^{-15})^3} \text{ kg/m}^3$$

$$= \boxed{7.810\times10^{17} \text{ kg/m}^3}$$

Assume black hole density $\rho < \rho_n$

$$\therefore \text{ if } \rho > \frac{3c^6}{32\pi G^3 M^2}$$

$$\Rightarrow \quad M > \sqrt{\frac{3c^6}{32\pi G^3 \rho}}$$

If max density is ρ_n, since $R_{HS} \propto \frac{1}{\sqrt{\rho}}$ this will result in a lower bound for a black hole mass.

$$\text{ie } M > \sqrt{\frac{3c^6}{32G^3\rho_n\pi}}$$

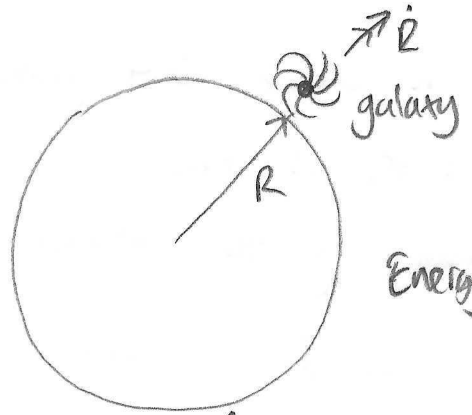
$$\Rightarrow \frac{M}{M_\odot} > \sqrt{\frac{3\times(2.998\times10^8)^6}{32\pi\times(6.67\times10^{-11})^3\times7.810\times10^{17}}}$$

$$\boxed{\frac{M}{M_\odot} > 4.86}$$

So this is not far off "exceeds about three solar masses". Perhaps the degeneracy pressure argument effectively allows for a slightly higher density of neutron matter, ie to reduce $\frac{M}{M_\odot}$ to something closer to 3.

7/

a)



$$E = \underbrace{\tfrac{1}{2} m \dot{R}^2}_{KE} - \underbrace{\frac{Gm\left(\frac{4}{3}\pi \rho R^3\right)}{R}}_{GPE}$$

↑ Energy of galaxy of mass m

↑ Universe of density ρ (expanding at velocity $\dot{R}$)

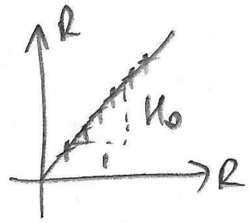
a) if $E = 0$ this means no net energy to galaxy at edge of universe is not bound gravitationally, and ∴ can expand and not contract (escape velocity idea).

So if $E = 0$:

$$\tfrac{1}{2}\dot{R}^2 = G \tfrac{4}{3}\pi \rho R^2$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho}{3}$$

Now let $\boxed{\dot{R} = H_0 R}$ Hubble law



$$\therefore \boxed{\rho = \frac{3H_0^2}{8\pi G}}$$

$2.3299 \times 10^{-18}\ s^{-1}$

let $H_0 = 71.9\ km/s/Mpc = \dfrac{71.9 \times 10^3}{10^6 \times 3.086 \times 10^{16}}\ s^{-1}$

$$\therefore \rho = \frac{3 \times \left(\dfrac{71.9 \times 10^3}{10^6 \times 3.086 \times 10^{16}}\right)^2}{8\pi \times 6.67 \times 10^{-11}} = \boxed{9.715 \times 10^{-27}\ kg/m^3}$$

$M_p = 1.673 \times 10^{-27}$ kg so $\rho = \dfrac{9.715}{1.673} = \boxed{5.8 \text{ protons}/m^3}$

Now the answer to Q1(i)(c) is about 0.4 protons/m^3

So if H_0 is correct, then matter is about

$\frac{0.4}{5.8} = 7\%$ of the mass-energy of the universe

(5% is a figure often quoted, so this is not far off).

$\boxed{\lambda\text{-CDM Model:}}$ of Cosmology

5% ordinary matter & energy
27% dark matter } currently
68% dark energy } mysterious!

b) Cosmic Microwave Background Radiation (CMBR)

WMAP probe : $T = 2.725$ K
(average temperature of CMBR).

Planck probe is more modern. Higher resolution?

Wein's law: $\lambda_{max} = \frac{2.899 \times 10^6}{2.725}$ (nm)

$\therefore f = \frac{c}{\lambda_{max}} = \frac{2.998 \times 10^8 \times 2.725}{2.899 \times 10^6 \times 10^{-9}}$ Hz

$= \boxed{282 \text{ GHz}}$

This is in the <u>microwave</u> part of the EM spectrum.

The antenna should scale with λ, ie

$\lambda = \left(\frac{2.725}{2.899 \times 10^6 \times 10^{-9}}\right)^{-1}$ m $= \boxed{1.06 \text{ mm}}$

For gain in a given direction, want aperture $\gg \lambda$ so an antenna perhaps $\simeq$ metres across should be sufficient.

The $\boxed{\text{WMAP}}$ radio antenna was based upon 1.4m × 1.6m primary reflectors.

[23 GHz → 94 GHz range

↳ It looked at the following bands: (GHz)

K	23 ± 5.5 ← "Bandwidth"
Ka	33 ± 7.0
Q	41 ± 8.3
V	61 ± 14
W	94 ± 20.5

Now this is rather strange, as Wien's law implies, for 42GHz say,

$$T = \frac{2.899 \times 10^{6}}{\dfrac{2.998 \times 10^{8}}{42 \times 10^{9}} \times 10^{9}} \quad (K)$$

$$= \boxed{0.41\ K}$$

So one assumes some form of 'mix down' process in WMAP to reduce 282 GHz → the K to W frequency bands. A good question to ask!

8/ See spreadsheet.

$$\left(\frac{L}{L_{\odot}}\right) \simeq \left(\frac{T}{T_{\odot}}\right)^{\overset{\alpha}{6.81}} \qquad \left(\frac{M}{M_{\odot}}\right) \simeq \left(\frac{T}{T_{\odot}}\right)^{\overset{\gamma}{1.95}}$$

and $\left(\frac{L}{L_\odot}\right) \simeq \left(\frac{M}{M_\odot}\right)^{3.46}$ ↖ β

d) let $\lambda_{max} = 456\,nm$

Wein's law: $\frac{\lambda_{max}}{nm} = \frac{2.899\times10^6}{T/K}$

So $\left(\frac{\lambda_{max}/nm}{502}\right) = \frac{T_\odot}{T}$

$\therefore$ $\boxed{T/T_\odot = \frac{502}{\lambda_{max}/nm}}$

So $\boxed{T/T_\odot = \frac{502}{456}} = 1.10.$

$\therefore$ $T = \frac{502}{456} \times 5780$ (K)

$= \boxed{6363\,K}$

$\therefore$ if Main Sequence correlations hold:

$L/L_\odot = \left(\frac{502}{456}\right)^{6.81} = \boxed{1.92}$

$R/R_\odot = \left(\frac{502}{456}\right)^{\frac{6.81}{2}-2} = \boxed{1.14}$

(From Stefan's law $\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4$

$\Rightarrow R/R_\odot = \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}} \left(\frac{T}{T_\odot}\right)^{-2}$)

$$M/M_\odot = \left(\frac{502}{456}\right)^{1.95} = \boxed{1.21}$$

Star lifetime $t = 10^{10}$ years $\times \left(\frac{M}{M_\odot}\right)^{5/2}$

$$\therefore \quad t = 10 \times \left(\frac{502}{456}\right)^{1.95 \times \frac{5}{2}} \quad \text{billion years}$$

$$= \boxed{16 \text{ billion years}}$$

AF 3/1/21.