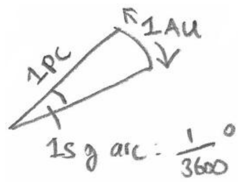
The **Cosmos** "is all that is or ever was or ever will be." (Carl Sagan, *Cosmos* pp20). *Cosmology* is the *study of the Universe at the largest of scales* i.e. *planets* to *stars* to *nebulae* to *galaxies* to *galactic clusters*, to the *observable Universe.*

Key scientific questions: *How big* is the Universe? *How many stars*, how many planets, and how many of these could sustain life? *How far away* are the stars, and how difficult might it be for inter-planetary intelligent civilizations to contact each other? *How fast* are the stars moving, towards and away from each other? *How old* is the Universe? Did it have a 'beginning,' and will it have an end? What are the average *physical properties* of the cosmos? i.e. average density, temperature, balance of radiation and matter?

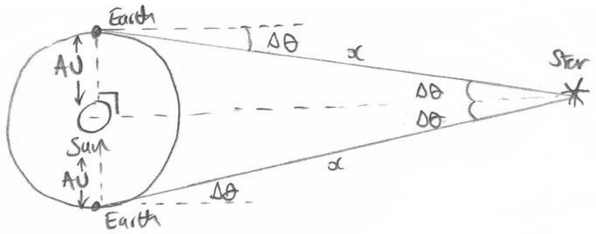
Planet and star distance scales:
Earth radius $R_{\oplus} = 6,371\text{km}$. Solar Radius $R_{\odot} = 696,340\text{km}$. Astronomical Unit (AU) is the average Earth-Sun separation: $1\text{AU} = 1.496\times 10^{11}\text{m}$.

Inter-star distance scales:



1 light-year $\text{ly} = c\times 1\text{Yr} = 2.998\times 10^{8}\text{ms}^{-1}\times 365\times 24\times 3600\text{s} = 9.461\times 10^{15}\text{m}$.
1 parsec $\text{pc} = 3.086\times 10^{16}\text{m} = 3.26\text{ly}$. $1\text{AU} = 1\text{pc}\times \frac{1}{3600}\times \frac{\pi}{180}$.
Note: 1 *arc-second* is $\frac{1}{3600}$ of a degree or $\frac{1}{3600}\times\frac{\pi}{180}$ radians.
Note most inter-galactic distances are measured in *Mega-parsecs* (Mpc). $\text{Mpc} = 10^{6}\text{pc}$.

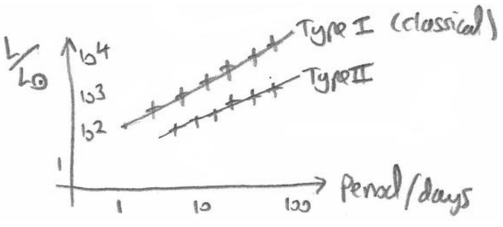
Measuring the distances of the stars using parallax



The Earth orbits the Sun every year. This means the positions of stars will vary in an angular sense throughout the year. This is called *parallax*. Since stars are many light years away, one can assume the angular deviations due to the motion of the stars themselves (relative to the Sun) are deemed negligible over a year. Hence if the Earth-Sun (1AU) distance is known, the parallax angle $\Delta\theta$ can be used to determine distance x of the star from Earth:

$$x\sin\Delta\theta = \text{AU} \therefore x \approx \frac{\text{AU}}{\Delta\theta} \text{ if } \Delta\theta \ll 1 \text{radian.}$$

Measuring the distances of the stars using radiated power
Luminosity L is the *total radiated power of a star*. In most situations one can assume this is uniform in all directions, so the power per unit area Φ received at distance x from a star is: $\Phi = L/4\pi x^2$. Φ can be measured from the brightness of a star in a suitably calibrated telescope. The *radiative flux* Φ from the Sun at Earth distance is $\Phi = 1370\text{Wm}^{-1} = L_{\odot}/4\pi\text{AU}^2 \Rightarrow L_{\odot} = 3.846\times 10^{26}\text{W}$.



Cepheid variable stars have a *periodic* luminosity relationship, that can enable their luminosity to be calculated from the period of the luminosity variation. This means a Cepheid star can be used as a 'standard candle'. If Φ and L are known for this star, the distance $x = \sqrt{L/4\pi\Phi}$. This method can be used to determine larger distances, where parallax is too small to measure precisely.

The balance of gravity and radiation pressure will result in stars being a spherical shape, or radius R. This can be calculated given both the luminosity and the average surface temperature T (in Kelvin). The latter can be determined from the wavelength associated with the peak of the spectrum of radiation from the star, using Wein's law, which can be derived from the Planck formula, which describes the variation of spectral irradiance B of a uniformly radiating body of temperature T.

Planck formula: $B = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}}-1}$ *Wein's Law:* $\lambda_{\max} = \frac{hc}{4.965k_B T} = \frac{b}{T} \Rightarrow (\lambda_{\max}/\text{nm}) = \frac{2.899\times 10^6}{(T/\text{K})}$

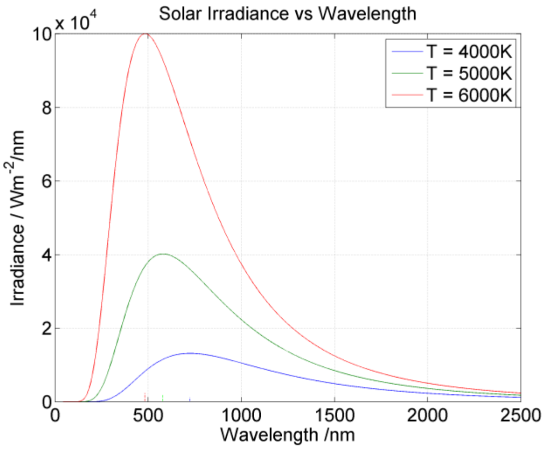
The total area under the B vs λ Planck formula curve is the radiated power per unit area. It can be shown that:

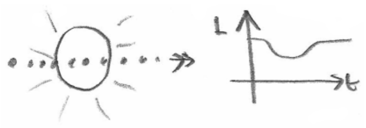
$\int_0^\infty B d\lambda = \sigma T^4$ where the Stefan-Boltzmann constant

$\sigma = \dfrac{2\pi^5 k_B^4}{15c^2h^3} = 5.67\times10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$.

This means the star luminosity $L = 4\pi R^2 \sigma T^4$ and therefore

$$R = \sqrt{\frac{L}{4\pi\sigma T^4}} = \sqrt{\frac{L}{4\pi\sigma\left(b/\lambda_{\max}\right)^4}} = \sqrt{\frac{L}{4\pi\sigma}}\left(\frac{\lambda_{\max}}{b}\right)^2.$$





Note most stars will have an associated small dip in luminosity during a *transit of a planet* i.e. a partial *eclipse*. The depth and duration of the luminosity vs time curve can enable the speed and radius of a planet to be calculated.

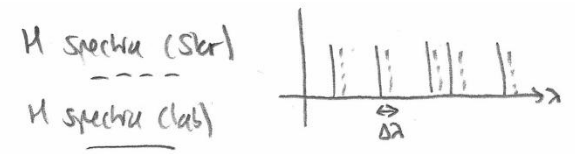
For radial motion at speeds much less than the speed of light, electromagnetic radiation from a star will be shifted in wavelength via the **Doppler formula** $\dfrac{\Delta\lambda}{\lambda} \approx \dfrac{v}{c}$ where λ is the *emitted* wavelength and $\lambda + \Delta\lambda$ is the *received* wavelength. A positive $\Delta\lambda$, or *redshift*, implies a *recessional* velocity. A *blueshift* (i.e. reduced wavelength) implies a star *approaching*.

Redshift $z = \dfrac{\lambda_{\text{observed}} - \lambda_{\text{emmitted}}}{\lambda_{\text{emmitted}}}$.

The light from a star is mostly generated by *quantum transitions in the energy levels of electrons in the hydrogen atoms*, which form the majority of stellar matter. These occur at discrete frequencies, which at a first approximation are given by the *Balmer formula*, which can be derived from the *Bohr model* of the hydrogen atom.

$\lambda_{nm} = 91.13\text{nm}\left(\dfrac{1}{m^2} - \dfrac{1}{n^2}\right)^{-1}$ where n, m are positive integers and $n > m$.

If we look at the *emission spectrum* of a star we will find peaks corresponding to λ_{nm}. The alternative is to look at an *absorption spectrum*, where a body of hydrogen gas between the star and observer, absorbs the starlight. The idea is to compare these 'spectral barcodes' to the spectra of hydrogen (or other elements such as Oxygen) observed in an Earth laboratory. The characteristic spectral pattern identifies the element, and the *doppler shift* $\Delta\lambda$ allows the recessional velocity to be calculated.



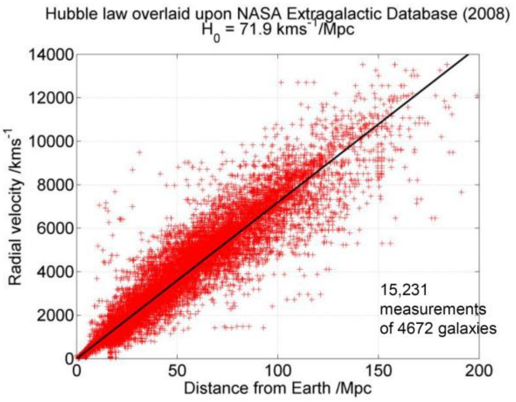
Olber's Paradox and the Big Bang theory

If all stars had an average luminosity L and there were n stars per unit volume, uniformly distributed in the universe, we can compute the expected power per unit area of starlight received on Earth by summing the contributions from shells volumes of radius r. At radius r there are $n \times 4\pi r^2 dr$ stars, so the total flux is $d\Phi = n \times 4\pi r^2 dr \times \dfrac{L}{4\pi r^2} = nLdr$.

Hence the total flux is: $\Phi = \int d\Phi = nL\int_0^\infty dr = \infty$ (!)

Why the night sky is *not* infinitely bright is known as *Olber's Paradox* (Heinrich Olbers, 1759-1840), and is resolved by the experimental fact that the Universe is *expanding*. This is known as the *Big Bang theory*, since by working backwards, the Universe must have *originated explosively from a single point.*

Hubble's constant and the expanding Universe



Edwin Hubble (1889-1953) recorded a strong positive correlation of recessional velocities v of galaxies with distance d from Earth.

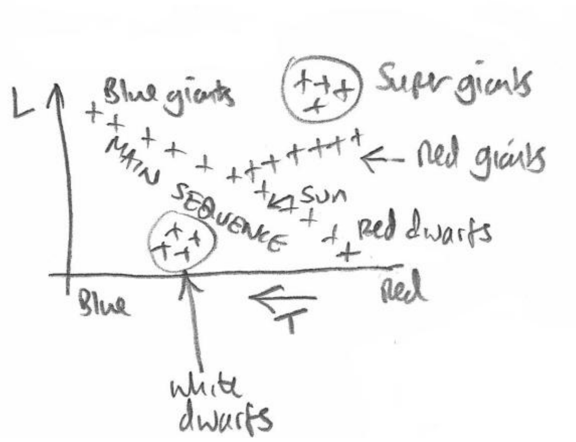
$v = H_0 d$ where the Hubble constant , the gradient of the graph of v against d is: $H_0 = 71.9\text{kms}^{-1}/\text{Mpc}$. Since v/d has units of time, this implies $1/H_0$ is a measure of the age of the Universe.

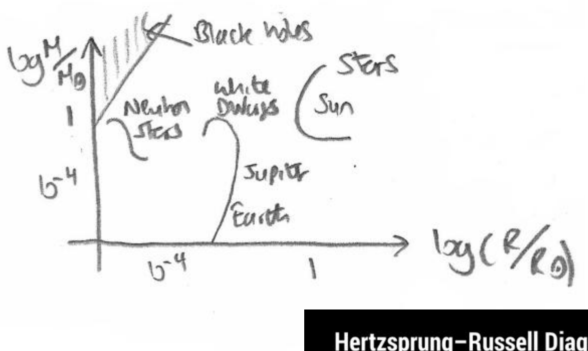
Modern analysis, using various cosmological models (which may include an exponential inflationary epoch in the early universe) plus study of the cosmic microwave background radiation (CMBR) yields 13.8 billion years.

Stars are typically spherical accretions of mostly hydrogen, bound via gravity and supported from collapse by the radiation pressure due to electromagnetic rays emitted from fusion reactions in the star interior (e.g. fusion of hydrogen isotopes to helium). Stars will expend fusion fuel, in the form of electromagnetic radiation, at different rates depending on the star mass, and their radius (and luminosity) will change over time. Once the fusion fuel is used up, stars will either shed matter and leave a compact object such as a white dwarf or neutron star, or if the star is massive enough, collapse and then explode in extreme violence in the form of a supernova. The energy of a supernova is such that heavier elements can be created via nuclear fusion. This process of *nucleosynthesis* is how all our elements of life (e.g. carbon, nitrogen) plus heavier elements from iron to uranium are formed. ***We are all stardust, the residue of supernovae.***

Stars form a pattern of *luminosity* vs *temperature* (or *wavelength*, via Wien's law) when plotted in a **Hertzsprung-Russell diagram**. Most stars exist on the '*main sequence'* band, and then branch off into the *red giant* branches, and then *white dwarf* branch, following mass shedding. The lifetime of stars in the main sequence is typically about

$t = 10^{10}$ years $\times \left(\frac{M}{M_\odot}\right)^{\frac{5}{2}}$. *Red dwarfs* have small luminosities, small masses and longest lifetimes. *Blue giants* have the highest luminosity, largest masses and shortest lifetimes (in the main sequence).





Constants:

Speed of light $c = 2.998 \times 10^8 \text{ms}^{-1}$

Planck's constant $h = 6.63 \times 10^{-34} \text{m}^2\text{kgs}^{-1}$

Boltzmann's constant $k_B = 1.381 \times 10^{-23} \text{m}^2\text{kgs}^{-2}\text{K}^{-1}$

Gravitational constant $G = 6.67 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$

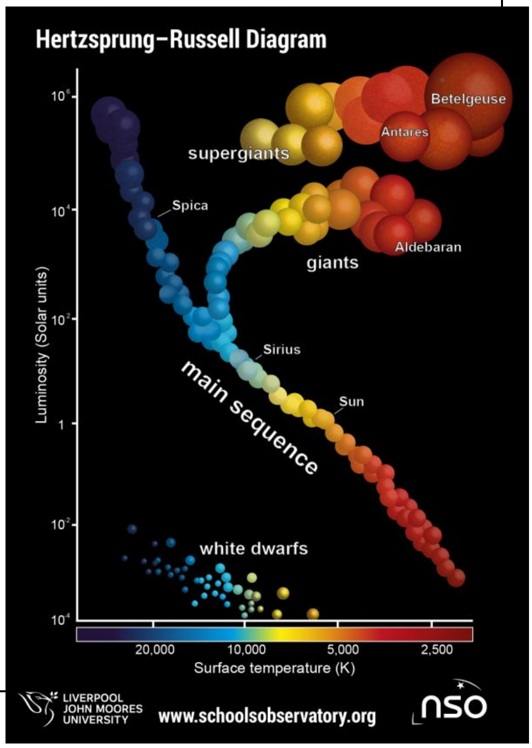
Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8} \text{Wm}^{-2}\text{K}^{-4}$

Solar mass: $M_\odot = 1.99 \times 10^{30}\text{kg}$ Earth mass: $M_\oplus = 5.97 \times 10^{24}\text{kg}$

Proton mass: $m_p = 1.673 \times 10^{-27}\text{kg}$

Neutron mass: $m_n = 1.675 \times 10^{-27}\text{kg}$

Electron mass: $m_e = 9.109 \times 10^{-31}\text{kg}$

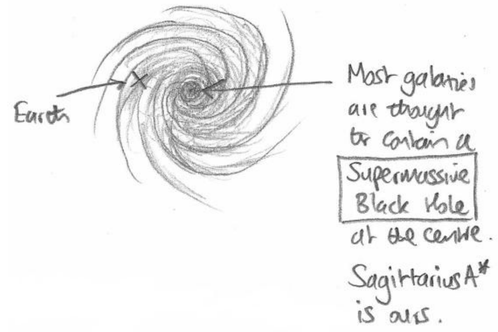


Supernovae

If a white dwarf star gains mass above the *Chandrasekhar limit* of about 1.4 solar masses, then gravity overcomes *electron degeneracy pressure* and the white dwarf implodes. Depending on whether the white dwarf is 'fed' by accreting matter from a companion star, or forms from the core of a collapsing giant star, the ensuing extremely violent explosion, called a *supernova* ,will takes slightly different forms (I or II). Both events will temporarily rival the *total luminosity of a galaxy.* The characteristic luminosities of supernovae (particularly Type Ia) enable them to be used as a 'standard candle.' i.e. if you can measure the radiative flux, you can therefore determine the distance to the supernova.

Supernova type	**Star mass**	**Mechanism**	**Where observed?**	**Spectral and light curve characteristics**
Ia	White dwarf above 1.4 solar masses in a binary system.	Carbon-oxygen white dwarf star (end state for main sequence star between 0.4 and 4 solar masses) accretes matter from a *companion.* Once it reaches the Chandrasekhar limit of about 1.4 solar masses, then gravity overcomes degeneracy pressure and the white dwarf implodes. As this occurs, the temperature rises until *spontaneous carbon fusion* occurs. This results in a chain reaction which causes the white dwarf to explode violently. Unlike a Type II supernova, nothing remains of the white dwarf core following the supernova. Gaseous Type I supernova remnants (nebulae) will be iron rich.	In all types of galaxies.	Peak luminosity of about $10^{10}L_{\odot}$, then decays gradually with time, consistent with one to two $\times 10^{44}$ J of energy released. No hydrogen spectral lines.
II	Giant star between 8 and 50 solar masses	Gravitational collapse, which raises core temperature sufficient for nuclear fusion of successively heavier elements until Iron and Nickel are formed. Energy from these reactions, plus *degeneracy pressure* resulting from electrons not being allowed to occupy the same quantum state, arrests the core collapse. Once Fe and Ni are formed, no further fusion can occur. If compacted core then exceeds the Chandrasekhar limit of about 1.4 solar masses, then gravity overcomes degeneracy pressure and the inner core collapses. The outer core then implodes, and rebounds, creating a highly energetic shockwave. This ejects the outer matter of the star in a supernova explosion. The remaining core becomes a neutron star of typical mass about two solar masses, or a black hole if the core mass exceeds about 3 solar masses, which *means neutron degeneracy pressure* can also be overcome by gravity. Gaseous Type II supernova remnants will contain elements heavier than iron.	Spiral arms of galaxies. Older, lower mass stars, or young, massive stars. Sites of recent star formation.	Plateau of luminosity of about $10^{9}L_{\odot}$, then decays rapidly with time. Hydrogen spectral lines.

A **black hole** results when gravitational collapse of a star cannot be arrested by outward pressure resulting from radiation from fusion reactions, or at 'last resort' the *degeneracy pressure* resulting from the application of the *Pauli exclusion principle* applied to firstly electrons and, finally, neutrons. The radius of a black hole is known as the '*event horizon'* or '*Schwarzschild radius*' $R_S = 2GM/c^2$. At this proximity, the gravitational fields are so strong that *not even light can escape*, and the difference in field strength over a short distance may be sufficient to 'spagettify' matter.[1] The theory of *General Relativity* predicts a severe distortion of space-time, perhaps towards an infinity (a 'singularity') at the centre of a black hole. It is thought that supermassive black holes (i.e. millions or billions of solar masses) form the centers of most large galaxies. A supermassive black hole surrounded by an *accretion disc* of in-falling matter can produce high energy jets of electromagnetic radiation. These are known as *quasars*, and can be thousands of times more luminous than entire galaxies!



[1] Interestingly, this 'tidal' effect of a severe gravitational field gradient, is less severe for more massive black holes.

Question 1

(i) The known Universe has a *diameter* of 93 billion light years, and has a total mass of about 10^{53} kg. By contrast, a *proton* has a mass of 1.673×10^{-27} kg. Calculate the following, expressing your answers in standard form to 3.s.f:

(a) The radius of the Universe in meters

(b) The volume of the Universe in meters, assuming it is spherical

(c) The average density of the Universe, expressed as the *number of protons per cubic metre*[2]

(ii) (a) Calculate the distance in meters that light travels in a minute (a 'light-minute'), and hence the average Earth-Sun distance (1AU) in light-minutes.

(b) Calculate the distance (in AU) to the star Sirius A, which is 8.6 light-years away.

(c) Voyager 2 left the *heliosphere* in November 2018, a distance of 122AU. It was moving at 15.3km/s. Calculate the time (in years) for a spacecraft travelling at this speed to travel from Earth to Sirius A.

(d) Compare the speed of Voyager 2 to the orbital speed of the Earth about the Sun.

(iii) The NASA space probe *New Horizons* was launched on January 19 th 2006 with the mission of making a close encounter with the dwarf planet Pluto and its moons. It escaped Earth with a velocity of 16.26 kms^{-1} and gained a further 4kms^{-1} following a gravitational assist from Jupiter. Pluto varies in distance between 4.28 billion km and 7.5 billion km from Earth. Jupiter is at minimum 4.202 AU from Earth. Use this information to estimate the time in years it will take New Horizons to reach Pluto. Compare this to the actual flyby date of July 14th 2015.

(iv) The *Horsehead nebula* in the constellation of Orion is about 1,500 light years from Earth. In angular terms, it is 8 arc minutes high.

(a) Use some basic trigonometry to work out the height of the nebula in light years. Note 1 arc minute is 1/60 of a degree. Convert this distance to AU and also meters and parsecs.

(b) If a futuristic alien spacecraft can travel at 1% of the speed of light, how long would it take to cover the full length of the nebula? (Ignore any relativistic effects).

(v) The Milky Way Galaxy has up to 400 billion stars. It can be thought of as a disk of thickness 2000 light years and diameter 120,000 light years.

(a) Calculate the volume of the Milky Way in cubic meters.

(b) Calculate the volume in AU^3 .

(c) If each star has a solar system contained within a sphere of radius 50 AU, estimate the fraction of the milky way's volume which is occupied by stars and their solar system.

(d) It takes the Sun 240 million years to orbit the centre of the Milky Way. It is approximately 27,000 light years from the galactic centre. Calculate its average speed in kms^{-1} as it rotates, assuming a circular orbit.

(vi) The *Drake equation* is used to estimate how many advanced civilizations might evolve in a Galaxy of similar size as the Milky Way. The answers vary, but suggest about **40 million civilizations** at any one time is possible. Assuming the planets upon which each civilization lives are evenly spread throughout the galactic disc, work out the time it would take to send a radio signal from one civilization to another. Based upon your answer, do you think it is likely that the Earth will make contact with aliens anytime soon? *The Milky Way is a disk of thickness 2000 light years and diameter 120,000 light years. HINT: divide up the volume of the galaxy into 40 million spheres.*

(vii) A futuristic space telescope has an angular *resolution* of $\delta\theta \approx \lambda/d$ radians, which in this case is 10^{-3} arc seconds. Approximately, what would the telescope aperture diameter d have to be if the radiation was:

(a) Visible ($\lambda = 10^{-7}\,\text{m}$) (b) X-Ray ($\lambda = 10^{-10}\,\text{m}$) (c) Infra-Red ($\lambda = 10^{-5}\,\text{m}$)

(viii) *Voyager 2* passed Neptune on the 15th August 1989. Neptune is approximately 30.2 AU from Earth. Assuming an inverse square law, calculate the power (in Watts) received by a 70m diameter antenna which is part of the NASA Deep Space Network. Voyager transmits with a power of 22.4W, and the antenna gain at X-band (8-12GHz) is 48 dB i.e. a gain of $10^{48/10}$ in the direction of Earth.

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[2] Ordinary matter is thought to only constitute about 4.9% of the total mass-energy of the Universe. The remainder is 'dark matter' or 'dark energy'. The nature of these, currently unobservable quantities, is, at the time of writing, mysterious!

(ix) The *Andromeda* galaxy emits radiation at λ = 21.11cm corresponding to an energy change in Hydrogen atoms. This radiation is received on earth with a redshift of $z = -3.668 \times 10^{-4}$ (i.e. a *blueshift*).

(a) Use this information to calculate how fast (in km/s) the Andromeda galaxy is approaching the Milky Way.

(b) The Andromeda Galaxy is about 2.5 million light years away from Earth. After how many years will it take for the Milky Way and the Andromeda galaxy to merge from the point of view of Earth? Do you think it likely there will be an Earth-bound observatory to record this event?

(x) The Red Supergiant star *Betelgeuse*, of mass between 16.5 and 19 solar masses, is usually the tenth brightest in the night sky. The parallax angle measured from Earth is about 5.95 milli-arc-seconds, the luminosity of Betelgeuse is about 126,000 $L_{\odot}$, and the peak spectral wavelength is 805nm, following a Doppler shift adjustment of $+5.88 \times 10^{-2}\,\mathrm{nm}$. Use this information to calculate (a) the distance to Betelgeuse /ly, (b) the surface temperature (in K), (c) the radius (in solar radii), (d) the recessional velocity (in km/s), (e) the average density (in kg/m^3). $\lambda_{\max}$ for the Sun is 502nm.

(xi) Calculate the age of the Universe (in years), assuming Hubble's law holds true since the Big Bang, and . . $H_0 = 71.9\mathrm{kms}^{-1}/\mathrm{Mpc}$. Compare to the multi-method modern calculation of 13.8 billion years.

(xii) A *Cepheid* variable star has a peak luminosity of 1,234 $L_{\odot}$, which varies over an eighty day period. If the star is 4,321 parsecs from Earth, calculate the expected radiative flux received in W/m^2 by a space telescope.

Question 2 Construct a spreadsheet and fill in the missing columns in the table below.

Star	**Luminosity /** $L_{\odot}$	**Mass /solar masses**	$\lambda_{\max}$ **/nm**	**Parallax (milli-arc seconds)**	**Redshift / 10^{-5}**	Distance /ly	Radius /solar radii	Recessional velocity /kms^{-1}	Surface temperature /K	Average density /kgm^{-3}	Surface gravity /Nkg^{-1}
Sun	1	1	502	-	-						
Sirius A	25.4	2.06	292	379.21	-1.83						
Canopus	10,700	8.0	394	10.55	6.77						
Arcturus	170	1.08	676	88.83	-1.73						
Vega	40.12	2.14	302	130.23	-4.64						
Capella Aa	78.7	2.57	583	75.02	9.99						

$R_{\odot} = 696{,}340\mathrm{km}$, $M_{\odot} = 1.99 \times 10^{30}\,\mathrm{kg}$, $L_{\odot} = 3.846 \times 10^{26}\,\mathrm{W}$.

Question 3 *Barnard's Star* is the fourth closest star from the Sun. When observed from Earth over the course of half a year, it is observed to change angular positions by 1.0908 arc seconds. *Barnard's Star* is currently *approaching* our solar system with a radial velocity v of 110.6 kms^{-1}.

(a) Show that Barnard's Star is approximately 5.98 light years away. How many AU is this?

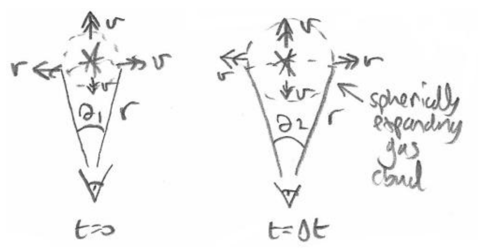
(b) If a spacecraft travels at 57,888 kmh^{-1}, (i.e. Voyager probe speeds), how many years would it take to reach Barnard's Star, starting from Earth?

Changes in the energy state of Hydrogen atoms in Barnard's star cause the emission of ultra-violet photons with wavelength 121.6 nm.

(c) Calculate the Doppler frequency shift Δf of this radiation using the formulae $c = f\lambda$ and $\Delta f = \frac{v}{c} f$.

(d) Hence calculate the *redshift* z of this radiation $z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emmitted}}}{\lambda_{\text{emmitted}}}$. Comment on the sign.

Question 4 The *Crab Nebula* is a remnant from a *Supernova*. It is essentially a gas cloud which is expanding radially at about 1,500 kms^{-1}. In twenty years the angle between the centre of the nebula and the outer rim has changed by 3.16 arc seconds. Use this information to calculate the distance to the Crab Nebula. Express your answer in light years.



Question 5 In a Type 1a supernova, a white dwarf star accretes mass from a companion in a binary system until it reaches the *Chandrasekhar limit* of about 1.4 solar masses. $M_{\odot} = 1.99 \times 10^{30}\,\mathrm{kg}$

(a) Explain briefly the processes which result in the violent explosion of a white dwarf once its mass exceeds $1.4M_{\odot}$.

(b) Using Einstein's mass-energy equivalence $E = mc^2$, calculate the energy released by a Type 1a supernova. Initially assume all of the mass of the white dwarf is released as electromagnetic waves in the explosion.

(c) A supernova has an average luminosity of $10^9 L_{\odot}$. Using (b), estimate how long the supernova will remain bright in the night sky. $L_{\odot} = 3.846 \times 10^{26}\,\mathrm{W}$. Supernovae explosions *actually* last a matter of days or months. Speculate in what form the majority of the mass-energy of the white dwarf ends up, and determine the actual fraction of E released as EM waves if the supernova lasted 100 days.

(d) Sketch $L/L_{\odot}$ vs time (in days) for the supernova, and overlay what you might expect for a Type II supernova. Explain in bullet points the processes that results in a Type II explosion.

Question 6 If the neutron-matter remnant of a red giant star following a Type II supernova exceeds about three solar masses, then neutron degeneracy pressure is insufficient to arrest further gravitational collapse, and a black hole will form. The *Schwarzschild radius* R_S of a black hole of mass M is $R_S = \frac{2GM}{c^2}$, so criteria for a spherical mass M of radius R to form a black hole is $R < R_S$.

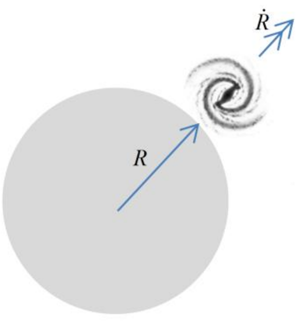
(a) Calculate the *Schwarzschild radius* for our Sun, in km. $M_{\odot} = 1.99 \times 10^{30}\,\mathrm{kg}$.

(b) TON 618 is a 'hyperluminous' quasar, with a mass of 66 *billion* solar masses. Calculate its Schwarzschild radius, in AU. Note: $1\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$, and for comparison, the orbital radius of Saturn is about 9.6 AU.

(c) If a black hole has uniform density ρ, show that $\rho > \frac{3c^6}{32\pi G^3 M^2}$. Sketch the variation of this lower limit of ρ with M. What is the minimum black hole mass (in solar masses) such that $\rho < 1{,}000\,\mathrm{kgm^{-3}}$, i.e. the density of water?

(d) If a neutron of mass $m_n = 1.675 \times 10^{-27}\,\mathrm{kg}$ can be modeled as having a radius of about $r_n = 0.8 \times 10^{-15}\,\mathrm{m}$, calculate its density ρ_n. If the maximum density of a black hole is that of pure neutron matter, i.e. neutrons pushed together without any gaps or further squeezing, calculate a lower limit for the mass of a black hole (in solar masses). Compare this to the opening statement of "exceeds about three solar masses."

Question 7 The diagram below illustrates the large-scale dynamics of a galaxy of mass m on the periphery of a 'Universe' of radius R and density ρ. The total energy (KE + GPE) of the galaxy is $E = \frac{1}{2}m\dot{R}^2 - \frac{G\frac{4}{3}\pi\rho R^3}{R}$.



(a) If $E = 0$ (what would this mean?) show that $\rho = \frac{3H_0^2}{8\pi G}$ where Hubble's constant is taken to be $H_0 = \frac{\dot{R}}{R} = 71.9\,\mathrm{kms^{-1}/Mpc}$. Evaluate this density in terms of *protons per cubic meter*, and compare to the answer to Q1 (i) (c), which is about 0.4 protons per cubic meter. Use this to speculate about the fraction of matter in the observable Universe.

(b) The *Cosmic Microwave Background Radiation* (CMBR), thought to be the radiation signature of the *Big Bang*, wavelength-stretched by the expansion and cooling of the Universe over 13.8 billion years, was analyzed by the WMAP study, which operated from space from 2001 to 2010. WMAP measured tiny (micro-kelvin) deviations of temperature from an average of 2.725K. Use Wein's Law to calculate the frequency of electromagnetic waves that correspond to this temperature. What part of the electromagnetic spectrum does this correspond to? Speculate on the characteristic dimensions of antennae used to measure properties of the CMBR.

Question 8 Consider the following selection of stars[3] that occupy the *Main Sequence* of the *Hertzsprung-Russell* diagram of luminosity vs star surface temperature.

Radius $R/R_{\odot}$	**Mass** $M/M_{\odot}$	**Luminosity** $L/L_{\odot}$	**Surface temperature /K**	**Star name**
0.13	0.1	8.00E-04	2,660	Van Biesbroeck's star
0.32	0.21	7.90E-03	3,120	EZ Aquarii A
0.51	0.6	7.20E-02	3,800	Lacaille 8760
0.74	0.69	1.60E-01	4,410	61 Cygni A
0.85	0.78	4.00E-01	5,240	70 Ophiuchi A
0.93	0.93	7.90E-01	5,610	Alpha Mensae
1	1	1	5,780	Sun
1.05	1.1	1.26E+00	5,920	Beta Comae Berenices
1.2	1.3	2.50E+00	6,540	Eta Arietis
1.3	1.7	6.00E+00	7,240	Gamma Virginis
1.7	2.1	2.00E+01	8,620	Beta Pictoris
2.5	3.2	8.00E+01	10,800	Alpha Coronae Borealis A
3.8	6.5	8.00E+02	16,400	Pi Andromedae A
7.4	18	2.00E+04	30,000	Phi1 Orionis
18	40	5.00E+05	38,000	Theta1 Orionis C

(a) Type the data above into a spreadsheet and add extra columns of: $\log_{10}(L/L_{\odot})$, $\log_{10}(M/M_{\odot})$, $\log_{10}(T/T_{\odot})$ and $2\log_{10}(R/R_{\odot})+4\log_{10}(T/T_{\odot})$.

(b) On separate graphs, plot:

$\log_{10}(L/L_{\odot})$ vs $\log_{10}(T/T_{\odot})$

$\log_{10}(L/L_{\odot})$ vs $\log_{10}(M/M_{\odot})$

$\log_{10}(M/M_{\odot})$ vs $\log_{10}(T/T_{\odot})$

and determine a line of best fit for each. Hence determine powers α, β, γ such that:

$$\left(\frac{L}{L_{\odot}}\right)=\left(\frac{T}{T_{\odot}}\right)^{\alpha}; \quad \left(\frac{L}{L_{\odot}}\right)=\left(\frac{M}{M_{\odot}}\right)^{\beta}; \quad \left(\frac{M}{M_{\odot}}\right)=\left(\frac{T}{T_{\odot}}\right)^{\gamma}$$

(c) By plotting $\log_{10}(L/L_{\odot})$ vs $2\log_{10}(R/R_{\odot})+4\log_{10}(T/T_{\odot})$, confirm the validity of Stefan's law of luminosity as a function of star radius and surface temperature: $L=4\pi R^2\sigma T^4$.

(d) A main-sequence star has a spectral peak at $\lambda_{\max}=456\text{nm}$. Use the correlations above to determine: its surface temperature in K, its luminosity $L/L_{\odot}$, its radius $R/R_{\odot}$, its mass $M/M_{\odot}$. Also estimate the lifetime of the star using $t=10^{10}$ years $\times\left(\frac{M}{M_{\odot}}\right)^{\frac{5}{2}}$. $\lambda_{\max}$ for the Sun is 502nm.

[3] *Main sequence* star selection listed at: https://en.wikipedia.org/wiki/Main_sequence