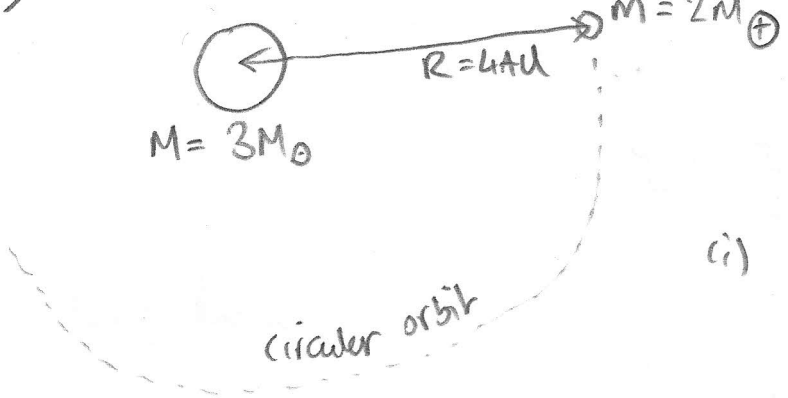
Gravitation & orbits questions

1/



Kepler's Third law

$$T^2 = \frac{4\pi^2}{G(m+M)} R^3$$

(i) For Earth - Sun

$$T_\oplus^2 = \frac{4\pi^2}{G(M_\oplus + M_\odot)} R_\oplus^3$$

$T_\oplus = 1$ year

$R_\oplus = 1$ AU

So $$\left(\frac{T}{T_\oplus}\right)^2 = \frac{M_\oplus + M_\odot}{2M_\oplus + 3M_\odot} \left(\frac{R}{R_\oplus}\right)^3$$

Now since $M_\odot \simeq 332{,}837\, M_\oplus$ ($M_\odot$ = Solar mass) $\quad M_\odot \gg M_\oplus$

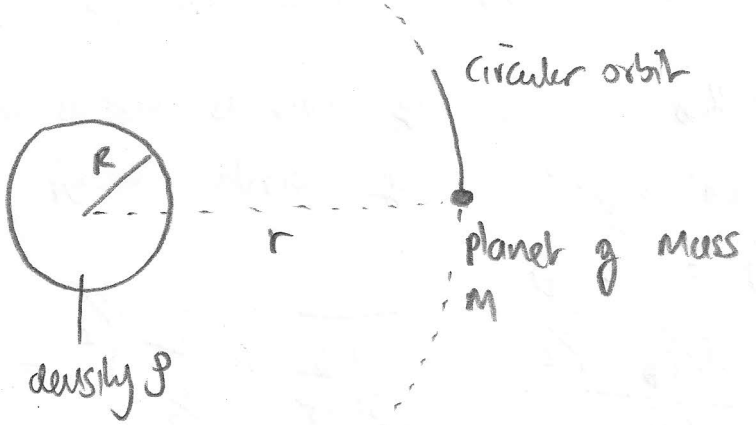
$$\therefore \left(\frac{T}{T_\oplus}\right)^2 \simeq \frac{1}{3}\left(\frac{R}{R_\oplus}\right)^3$$

i.e. mass of planet is largely irrelevant to the calculation.

$$\therefore \frac{T}{T_\oplus} = \frac{1}{\sqrt{3}} 4^{3/2} = \boxed{\frac{8}{\sqrt{3}}}$$

$$\simeq \boxed{4.62 \text{ years}}$$

2/



Newton II

$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

$$v = \frac{2\pi r}{T}$$

T ↑ period

$$\therefore \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

Now $M = \frac{4}{3}\pi R^3 \rho$

$$\therefore \frac{4\pi^2 r^3}{T^2} = \frac{4\pi G}{3} R^3 \rho$$

$$\therefore T^2 = \frac{3\pi r^3}{G\rho R^3}$$

orbital period

$$\boxed{T = \sqrt{\frac{3\pi}{G}} \frac{1}{\sqrt{\rho}} \left(\frac{r}{R}\right)^{3/2}}$$

So $T \propto \frac{1}{\sqrt{\rho}}$ and $\propto \left(\frac{r}{R}\right)^{3/2}$

Now for Sun $\rho_\odot \simeq 1406\ kgm^{-3}$ {Solar mass $M_\odot = 1.989 \times 10^{30} kg$}

$R_\odot \simeq 109.123\ R_\oplus$

$r_\oplus = 1.496 \times 10^{11} m$

$R_\oplus = 6.378 \times 10^6 m$ $\quad G = 6.67 \times 10^{-11} Nkg^{-2}m^2$

[So for Earth orbiting about the Sun

$$T_\oplus = \sqrt{\frac{3\pi}{6.67 \times 10^{-11}}} \times \frac{1}{\sqrt{1406}} \left(\frac{1.496 \times 10^{11}}{109.123 \times 6.378 \times 10^6}\right)^{3/2} \text{ (seconds)}$$

$= 1.001$ years $\quad (3.158 \times 10^7 s)$]

So if $R = \frac{1}{2} \times R_\odot$ (ie 'half as large as our Sun)

$\rho = 5488\ kgm^{-3}$ (ie Earth density)

$r = 5AU = 5 r_\oplus$

$$\therefore T/T_\oplus = \sqrt{\frac{\rho_\odot}{\rho}} \left(\frac{r/r_\oplus}{R/R_\odot}\right)^{3/2} = \sqrt{\frac{1406}{5488}} \left(\frac{5}{\frac{1}{2}}\right)^{3/2} = \boxed{16 \text{ years}}$$