VIBk Cosmology problems

AF. June 2015.

Kepler's laws and orbits - using notes

1. If two stars attract each other via Newton's Law of Universal Gravitation, prove the centre of mass of this system moves at a constant velocity i.e. does not accelerate.

2. Use your notes to prove that the angular momentum of a binary star system is

$$\mathbf{J} = \frac{m_1 m_2}{m_1 + m_2}\mathbf{r}\times\dot{\mathbf{r}}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector displacement between masses m_1 and m_2

3. The total energy of a binary star system is

$$E = \tfrac{1}{2}m_1\left|\dot{\mathbf{r}}_1\right|^2 + \tfrac{1}{2}m_2\left|\dot{\mathbf{r}}_2\right|^2 - \frac{Gm_1m_2}{r}$$

If $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$

Use $\mathbf{J} = \frac{m_1 m_2}{m_1 + m_2}\mathbf{r}\times\dot{\mathbf{r}}$ to show that

$$\dot{\theta} = \frac{\left(m_1 + m_2\right)J}{m_1 m_2 r^2}$$

$$E = \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left(\dot{r}^2 + \frac{\left(m_1 + m_2\right)^2 J}{\left(m_1 m_2\right)^2 r^2}\right) - \frac{Gm_1m_2}{r}$$

4. Using the variable change $u = \frac{1}{r}$ and the chain rule result $\frac{du}{d\theta} = \frac{du}{dt}\times\frac{dt}{d\theta} = \frac{\dot{u}}{\dot{\theta}}$

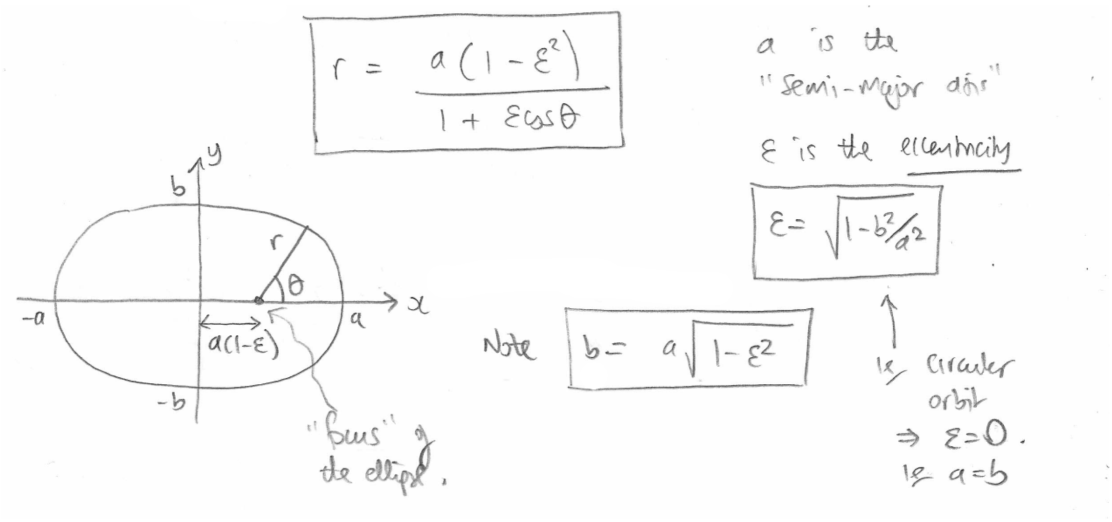
Use the formulae derived in Question 3 to derive the orbital equation

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm_1^2m_2^2}{\left(m_1 + m_2\right)J^2}$$

Show that ellipses in polar coordinates of the form $r(\theta) = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$ are a solution, and hence show that the magnitude of the angular momentum is

$$J^2 = \frac{Gm_1^2m_2^2\left(1-\varepsilon^2\right)a}{m_1 + m_2}$$

Note a , b , ε are defined by



5. Use the results in Question 4 to show that the total energy of the system is

$$E = -\frac{Gm_1 m_2}{2a}$$

6. Use the results in Question 1-4 to show that the velocities of the stars are

$$\dot{\mathbf{r}}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{v}$$

$$\dot{\mathbf{r}}_2 = \frac{m_1}{m_1 + m_2}\mathbf{v}$$

$$\mathbf{v} = \sqrt{\frac{G(m_1 + m_2)}{a\left(1-\varepsilon^2\right)}}\left(1+\varepsilon\cos\theta\right)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\right)$$

7. Use the results in Question 4 to prove *all three of Kepler's Laws*

Gravitation & orbits questions

1. A planet of twice the mass of Earth orbits, in a circular fashion, a star three times the mass of the Sun at an orbital radius of four astronomical units.

   (i) Calculate the orbital period of the planet in years

   (ii) Which piece of information is largely irrelevant to the calculation, and why?

2. A planet orbits a star of mean density ρ and radius R . The orbit is assumed to be circular with radius r .

   (i) Determine how the period of the orbit varies with these parameters.

   (ii) The density of our Sun is about 1406 kgm^{-3}, whereas Earth has an average density of 5488 $kgm^{-}3$. If a star was half as large as our sun and had the density of the Earth, calculate the period of a circular orbit of radius 5AU about this star.

3. Calculate the area 'swept' by the orbit of the Earth about the Sun each day. You will need to look up the various parameters and revisit your notes on Kepler's Second Law. Then work out the ratio of area swept per day between Pluto and Earth.

4. A cloud of anti-matter passes through the solar system and causes one third of the mass of the Sun to be annihilated. Assuming the resulting burst of gamma rays does not destroy the Earth, calculate the eccentricity of the resulting orbit of the Earth. You may assume the Earth was orbiting at 1AU in a circular orbit prior to the event.

   Show that the orbital period is $2\sqrt{3}$ years. What will the new seasons be like?

5. In the film *Interstellar*, a spacecraft flies directly into the super-massive Black Hole *Gargantua*, which has a mass of 100 million solar masses.

   (i) By considering the escape velocity from a spherical mass of mass M and radius R to be the speed of light c , determine the maximum radius of a Black Hole. (This is called the 'Event Horizon'. It is not clear what happens to the Laws of Physics beyond this point!)

   (ii) In reality, gravity has a profound effect upon both space and time as one approaches a Black Hole. However, for the moment let us ignore these effects and consider just Newton's Law of Universal Gravitation. If a spacecraft is stationary at radius r from a Black Hole, and uses gravity alone to pull it in, determine formulae for:

      (a) The velocity it reaches the Event Horizon

      (b) How long this takes

   (iii) In *Interstellar*, *Miller's Planet* orbits *Gargantua* at about 2 x the radius of the Black Hole. If a spacecraft falls into *Gargantua* from rest from this radius (i.e. in deep space and nowhere near Miller's Planet so unaffected by its gravity), calculate how long it will take to reach the event horizon, assuming no relativistic effects.

* this will require a bit of an integration challenge!