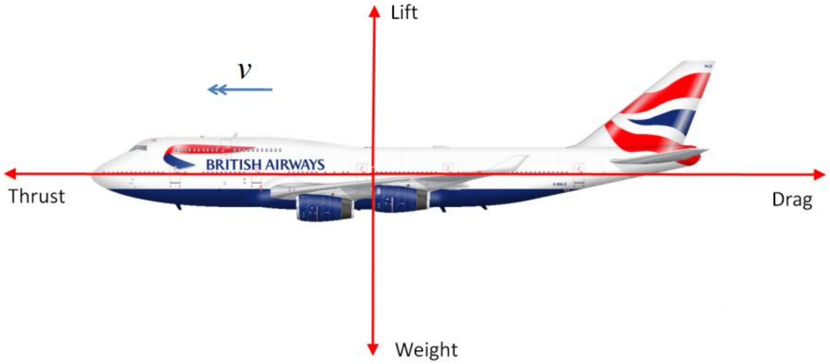
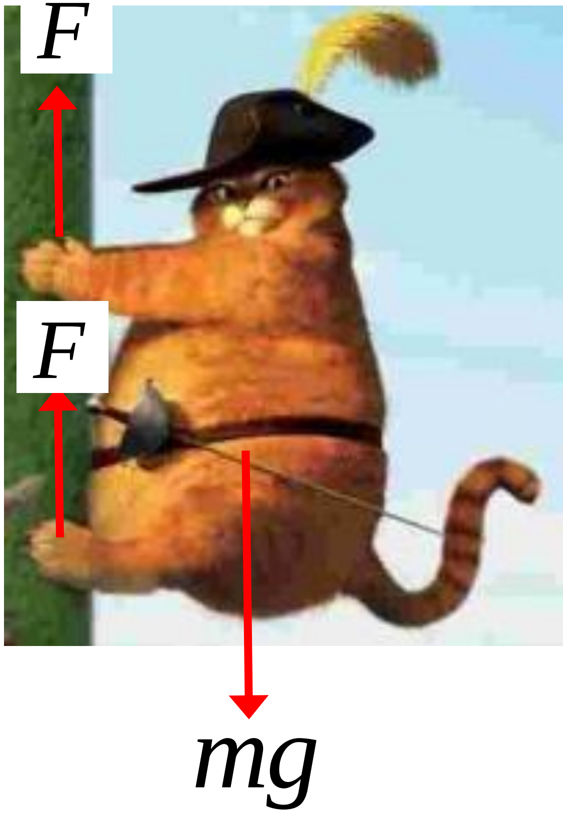


Mechanics & Machines



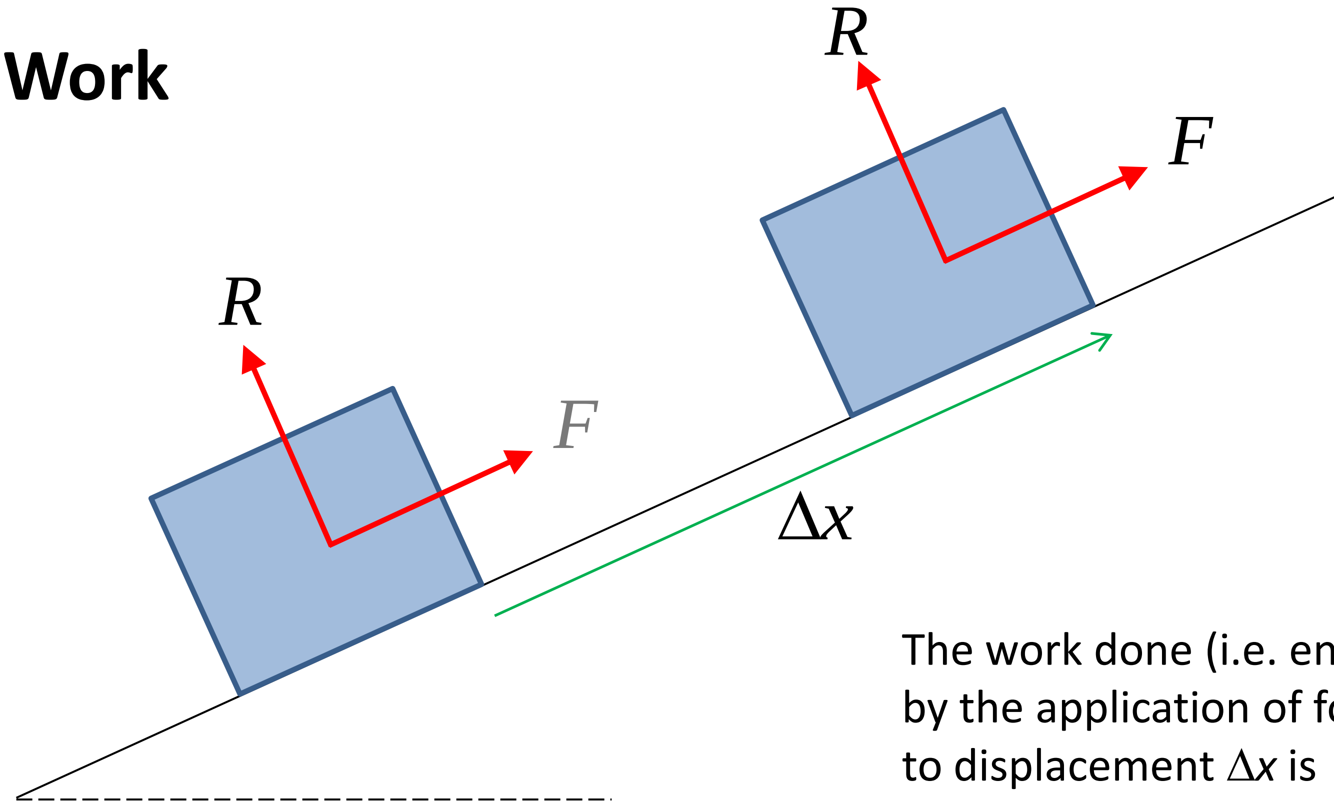
Universe by Numbers: Day 4
July 2016
Dr Andrew French

Antique mechanical washing machine mechanisms

Honda Cog advert

Crème that egg crazy machine!

Work

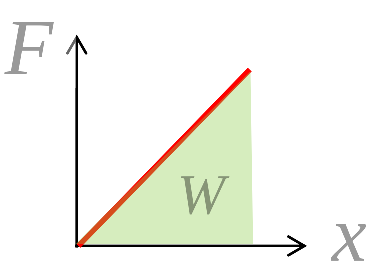


The work done (i.e. energy transferred) by the application of force *F* parallel to displacement Δx is

$$\boxed{W = F\Delta x}$$

Note there is **no work done** by any component of a force **perpendicular** to the **displacement.** i.e. force *R* does no work.

For *varying forces*, the work done is more generally the **area under the (displacement, force) graph**

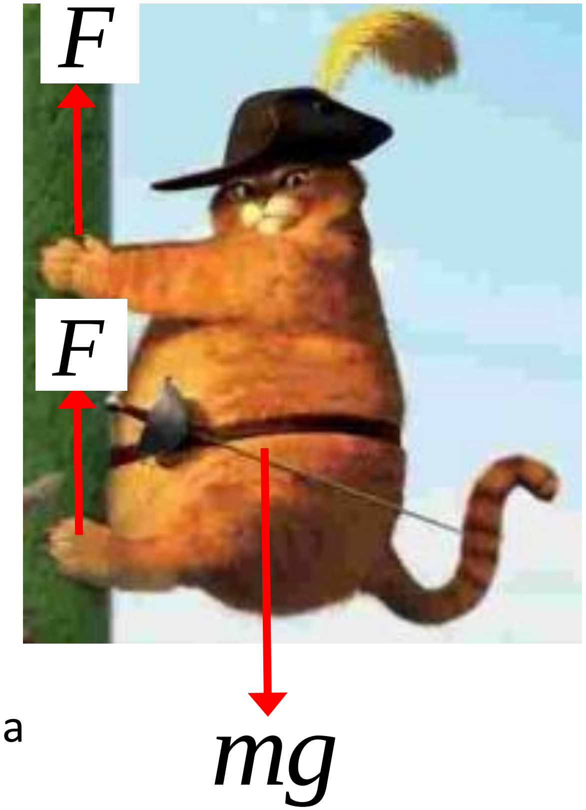


Gravity & weight

$$g = 9.81\text{ms}^{-2}$$

gravitational field strength on the surface of the Earth

A weighty puss indeed....



The force due to gravity upon a mass of m kg is ***mg*** where g is the *gravitational field strength*.

The gravitational force mg on a mass of m kg is called its ***weight***.

It is measured in Newtons.

Therefore a 70kg man weighs 686.7N *on Earth*.

g depends on the **mass** and **radius** of a planet

$$g_{mars} = 3.71\text{ms}^{-2}$$

$$g_{moon} = 1.63\text{ms}^{-2}$$

Work done rising a mass to height h against gravity = **force x distance**

$$\boxed{\text{GPE} = mgh}$$

We call this *Gravitational Potential Energy*, as this is what would be *released* if the mass fell height h

$$v = 0$$

$$\mathrm{GPE} = mgh$$
$$\mathrm{KE} = 0$$

By Conservation of Energy

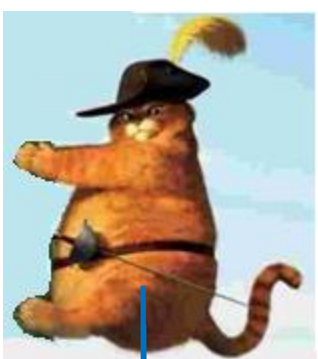
$$\boxed{\tfrac{1}{2}mv^2 = mgh}$$

$$h$$

If an object of mass *m* moves with ***speed*** *v* it has kinetic energy

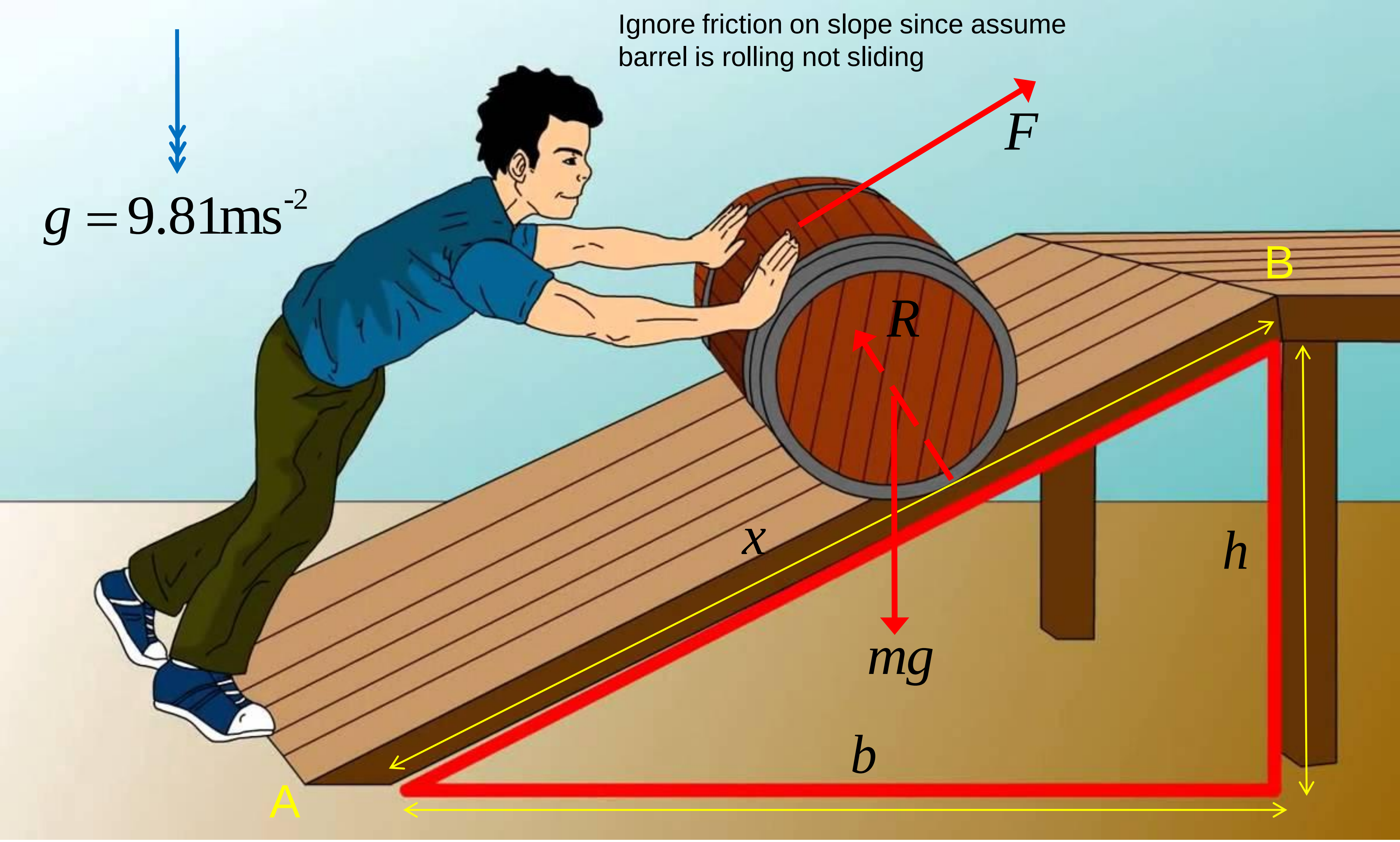
$$\boxed{\mathrm{KE} = \tfrac{1}{2}mv^2}$$

Speed is the *magnitude* of **velocity**, which is a *vector quantity* (i.e. has both magnitude and direction)



$$v$$

$$\mathrm{GPE} = 0$$
$$\mathrm{KE} = \tfrac{1}{2}mv^2$$

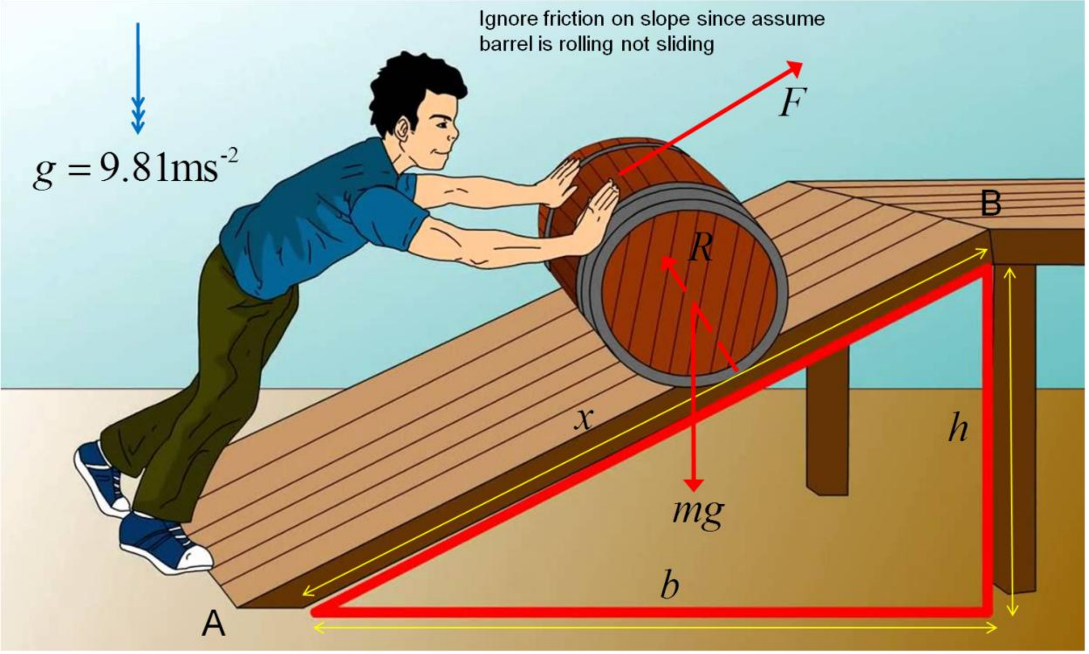


Work done A to B $= Fx$

Pythagoras: $x = \sqrt{b^2 + h^2}$

Conservation of energy $Fx = mgh$

$$\therefore F = mg\frac{h}{x} = mg\frac{h}{\sqrt{b^2 + h^2}}$$



If barrel lifted vertically

$$F = mg$$

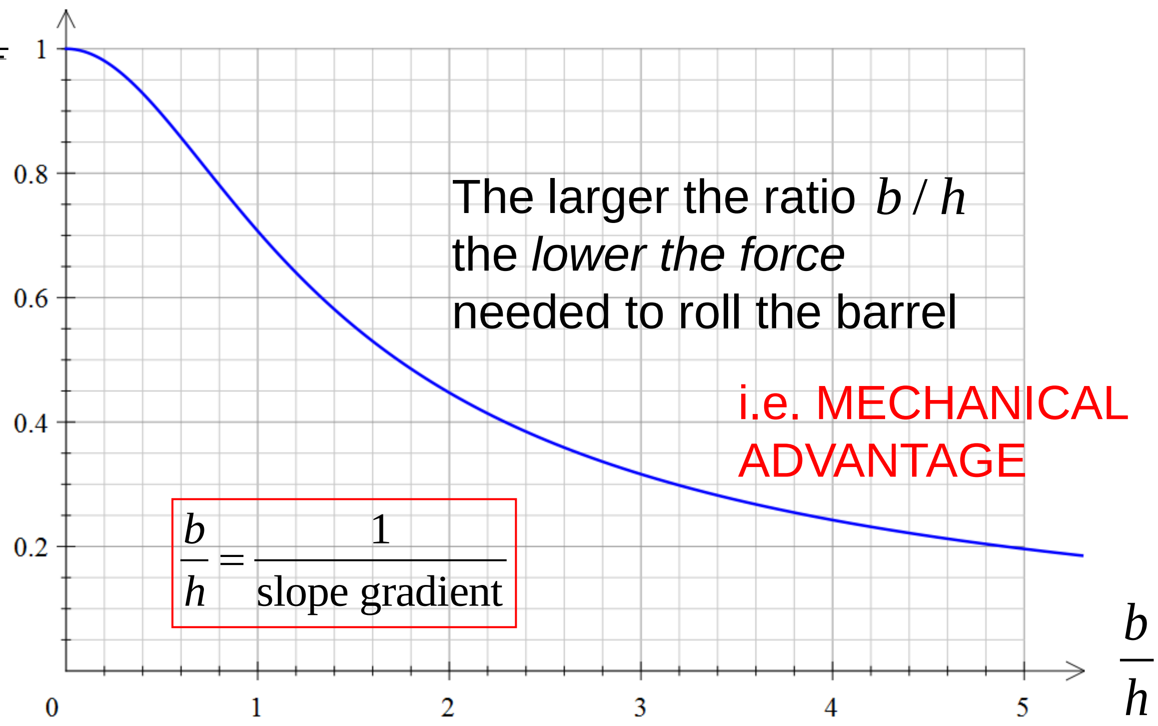
Using the *inclined plane*

$$F = mg\frac{h}{\sqrt{b^2 + h^2}}$$

$$\frac{F}{mg} = \frac{1}{\sqrt{1+\left(\frac{b}{h}\right)^2}}$$

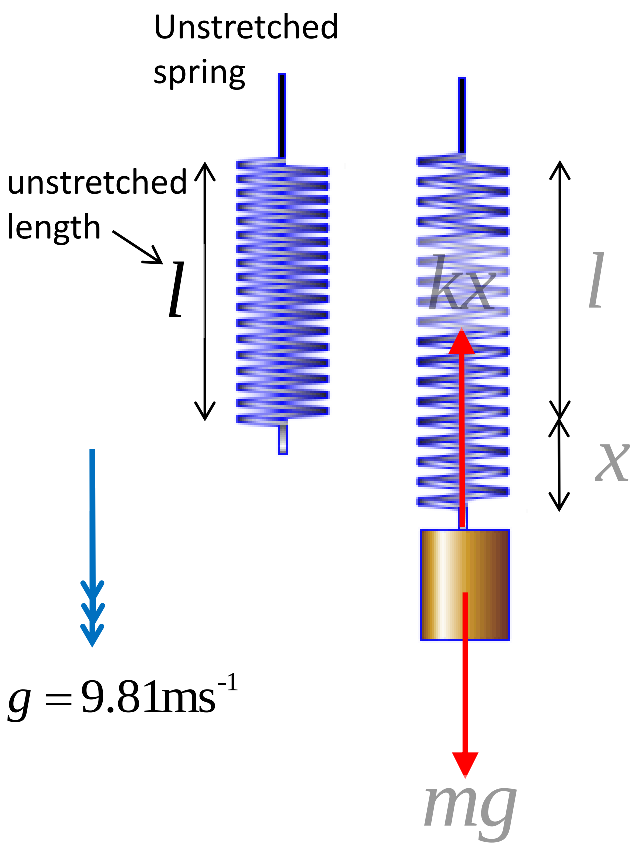
$$\frac{h}{\sqrt{b^2+h^2}} = \frac{h}{h\sqrt{\frac{b^2}{h^2}+1}}$$

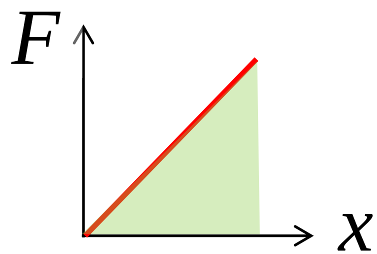
$$= \frac{1}{\sqrt{1+\left(\frac{b}{h}\right)^2}}$$



Elasticity

Elastic materials can be modelled by springs. Hooke's law means the *restoring force* due to a spring stretched by extension *x* is *proportional to the extension*





The work done by the restoring force, if 'left to its own devices' is called the **elastic potential energy**. This is the area under the (displacement, force) graph. Since triangular in shape for a 'Hookean spring' :

$$\boxed{E = \tfrac{1}{2}kx^2}$$

Hookes' Law
k is the spring constant, alternatively expressed in terms of an *elastic modulus λ*

$$\boxed{F = kx = \frac{\lambda}{l}x}$$

If in equilibrium
the forces balance

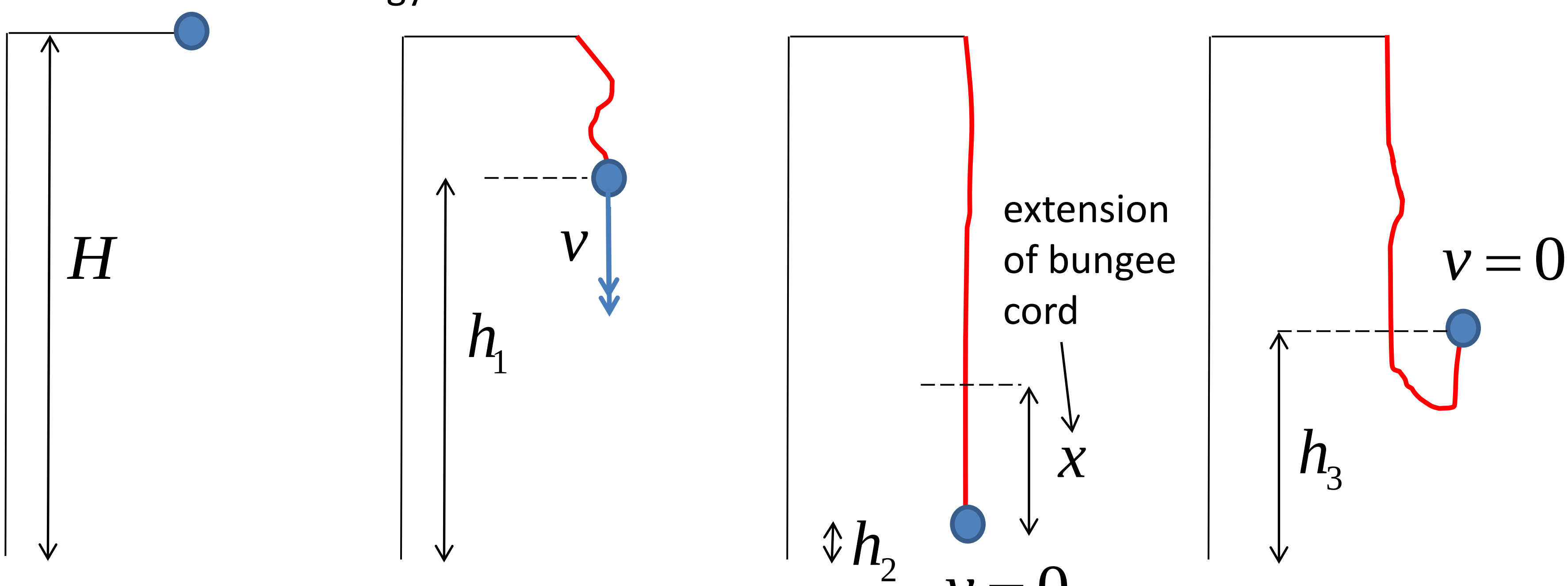
$$mg = kx \therefore k = \frac{mg}{x}$$

Robert Hooke
1635-1703

Conservation of energy

$$E = \text{constant}$$

$$E = \underbrace{\tfrac{1}{2}mv^2}_{\text{Kinetic*}} + \underbrace{\tfrac{1}{2}kx^2}_{\text{Elastic potential energy}} + \underbrace{mgh}_{\text{Gravitational potential energy}} + \overbrace{\text{losses}}^{\text{Drag, friction etc}}$$



$E = mgH$

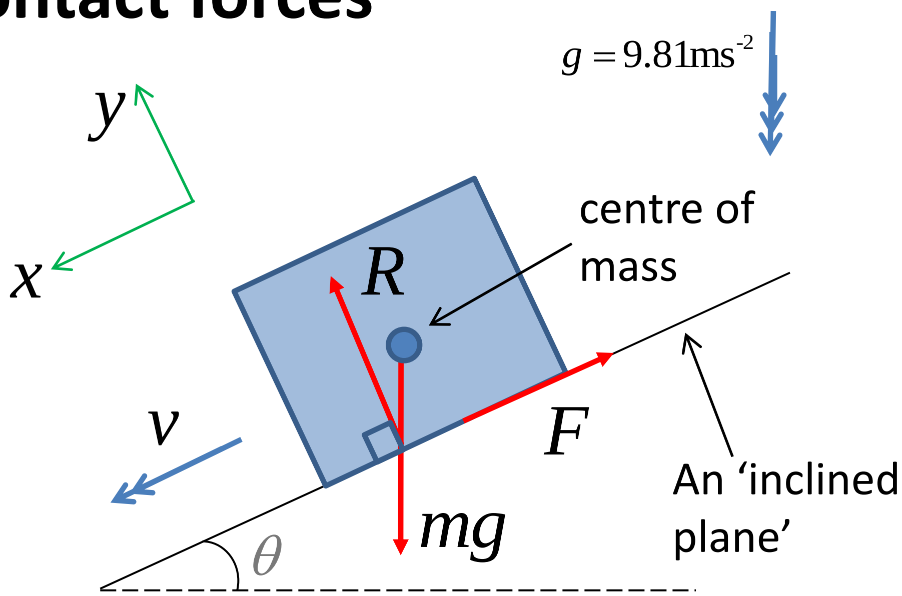
$E = mgh_1 + \tfrac{1}{2}mv^2$

$E = mgh_2 + \tfrac{1}{2}kx^2$

$E = mgh_3 + \text{losses}$

*Not just movement of the centre of mass, in general we must include vibration, rotation etc

Friction & Normal contact forces



Newton's Third Law: If you push against a surface with force *R*, the surface will push back at you with a force of the same magnitude, but in the opposite direction

Contact forces can be usefully decomposed into **normal contact** (perpendicular to a surface) and **friction** (parallel to the surface), which always *opposes* motion.

The normal contact force 'acts' at the point of intersection of a vertical '*plumb line*' from the *centre of mass* of the object.

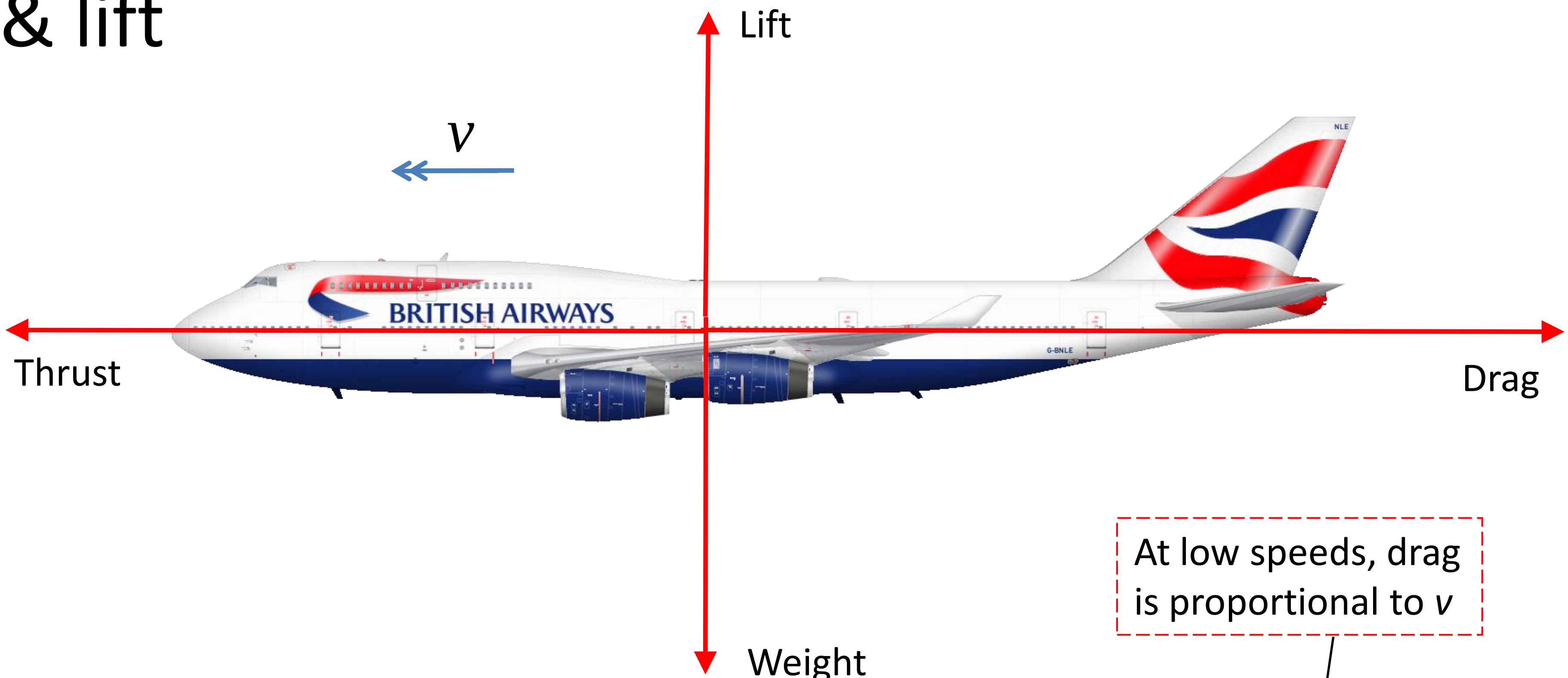
Models of friction & sliding

$F < \mu_{static} R$	No sliding, and object is in **static equilibrium**
$F = \mu_{static} R$	Object is on the point of sliding – friction is 'limiting'
$F = \mu_{slide} R$	$v > 0$ i.e. object is sliding

μ **Coefficients of friction**. Typically <<1. We often assume $\mu_{static} \approx \mu_{slide}$

Air resistance & lift

If an aircraft has a constant airspeed then it is not accelerating. Therefore the vector sum of all forces must be zero



At low speeds, drag is proportional to *v*

At 'modest speeds' (i.e. several ms^{-1}), both lift an drag forces are typically $\propto v^2$

$$F_{drag} = \tfrac{1}{2} c_D \rho A v^2$$

Drag coefficient
Typically << 1

Density of air

Cross sectional area of aircraft perpendicular to velocity

Aerodynamics of a sportscar (and driver!) being analysed using a wind tunnel

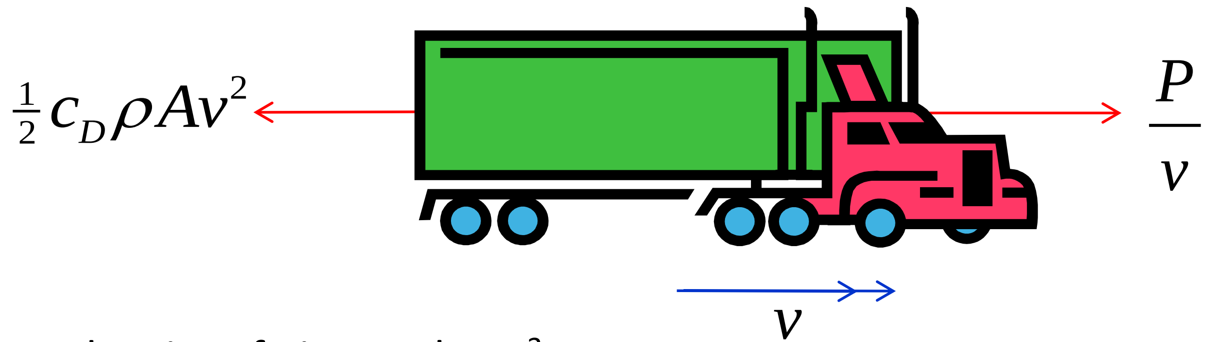
The *rate* of work done is **power** $P = F\dfrac{\Delta x}{\Delta t}$ $\therefore \boxed{P = Fv}$

A lorry is travelling a constant speed of 60 mph. If friction between the tyres and the road can be ignored at this speed, and internal losses such has heating etc can be ignored, the *driving force* of the engine is balanced by *air resistance*. If the cab has a cross section of 8 m^2, estimate the engine power *P*.

Since lorry is in equilibrium, **driving force = air resistance**

$$\tfrac{1}{2}c_D\rho A v^2 = \frac{P}{v}$$

$$\therefore \boxed{P = \tfrac{1}{2}c_D\rho A v^3}$$



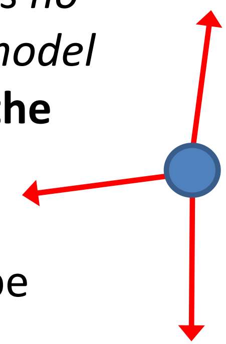
Assume drag coefficient c_D = 1, density of air ρ = 1kgm^{-3}
v = 60/2.34 = 25.64ms^{-1}

$$P = \tfrac{1}{2}\times 1\times 1\times 8\times 25.64^3$$

$$\boxed{P \approx 67.4\text{kW}}$$

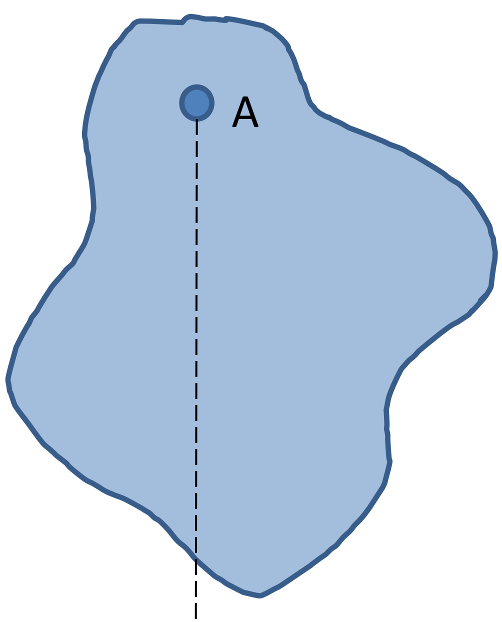
Particles & centre of mass

A *particle* is an object which has mass (and forces can act upon it) but it has *no extension.* i.e. it is located at a point in space. If objects are **rigid**, we can '*model them as particles'* since one can decompose motion into **displacement of the centre of mass + rotation of an object about the centre of mass.**

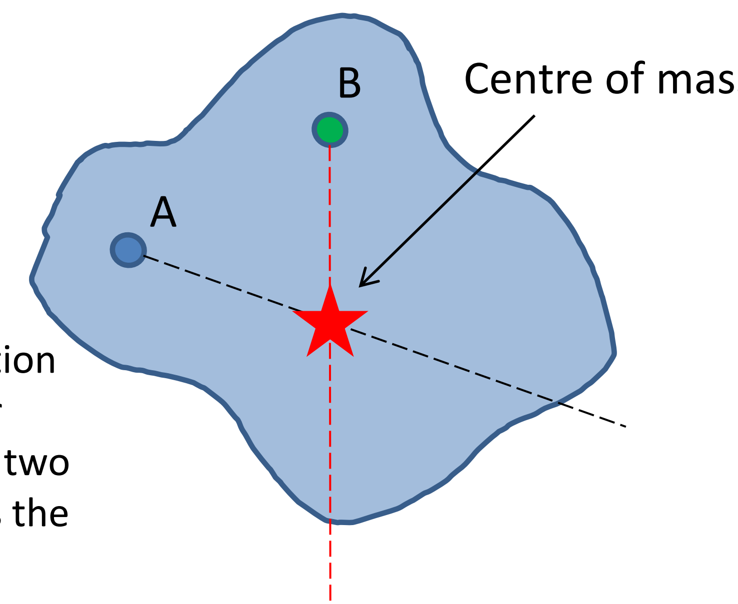


The centre of mass is the point where the entire weight of the object can be balanced without causing a turning moment about this point.

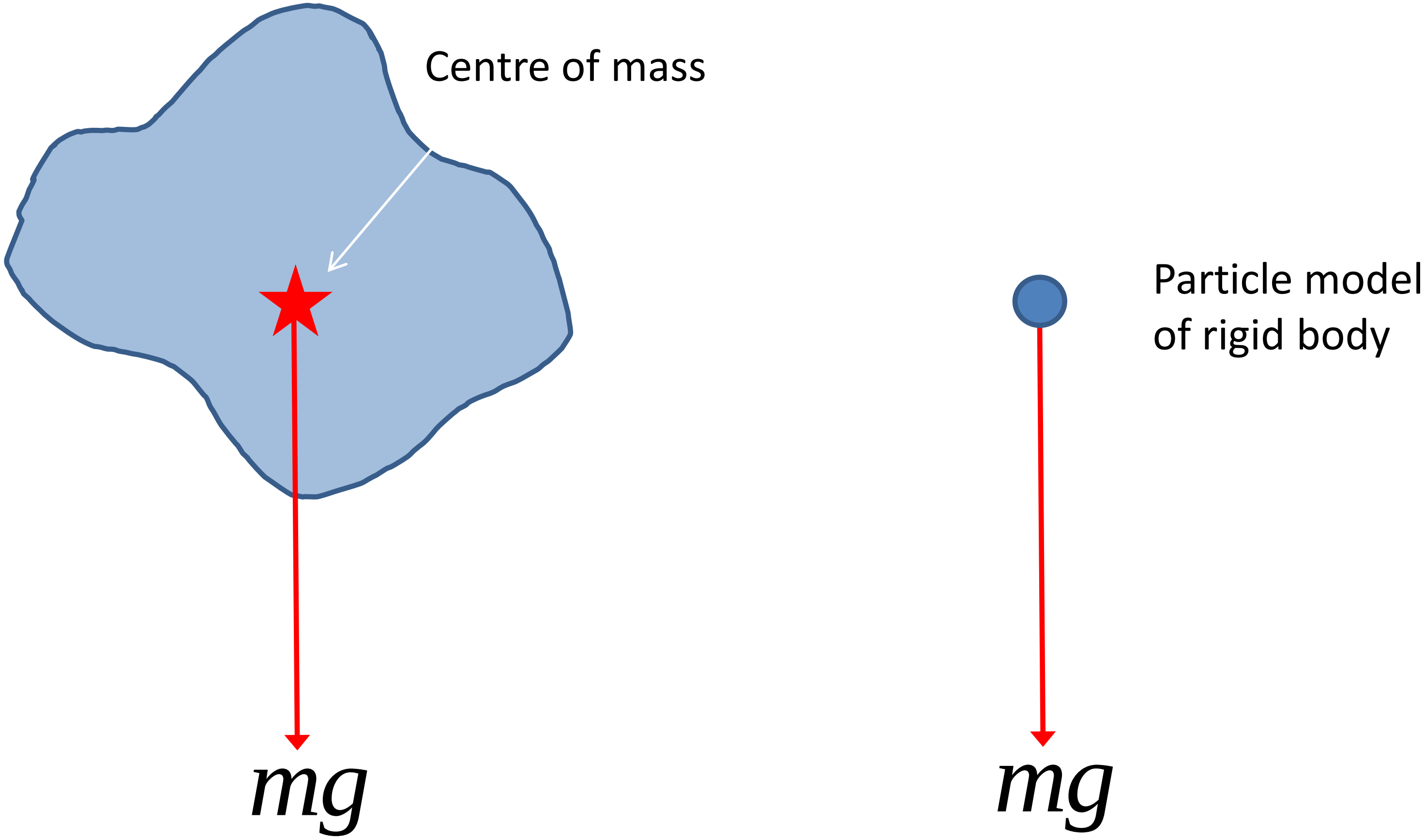
It can be found practically by hanging a 2D object from various positions and working out where the plumb lines intersect.



Hang object from position A and draw on plumb line



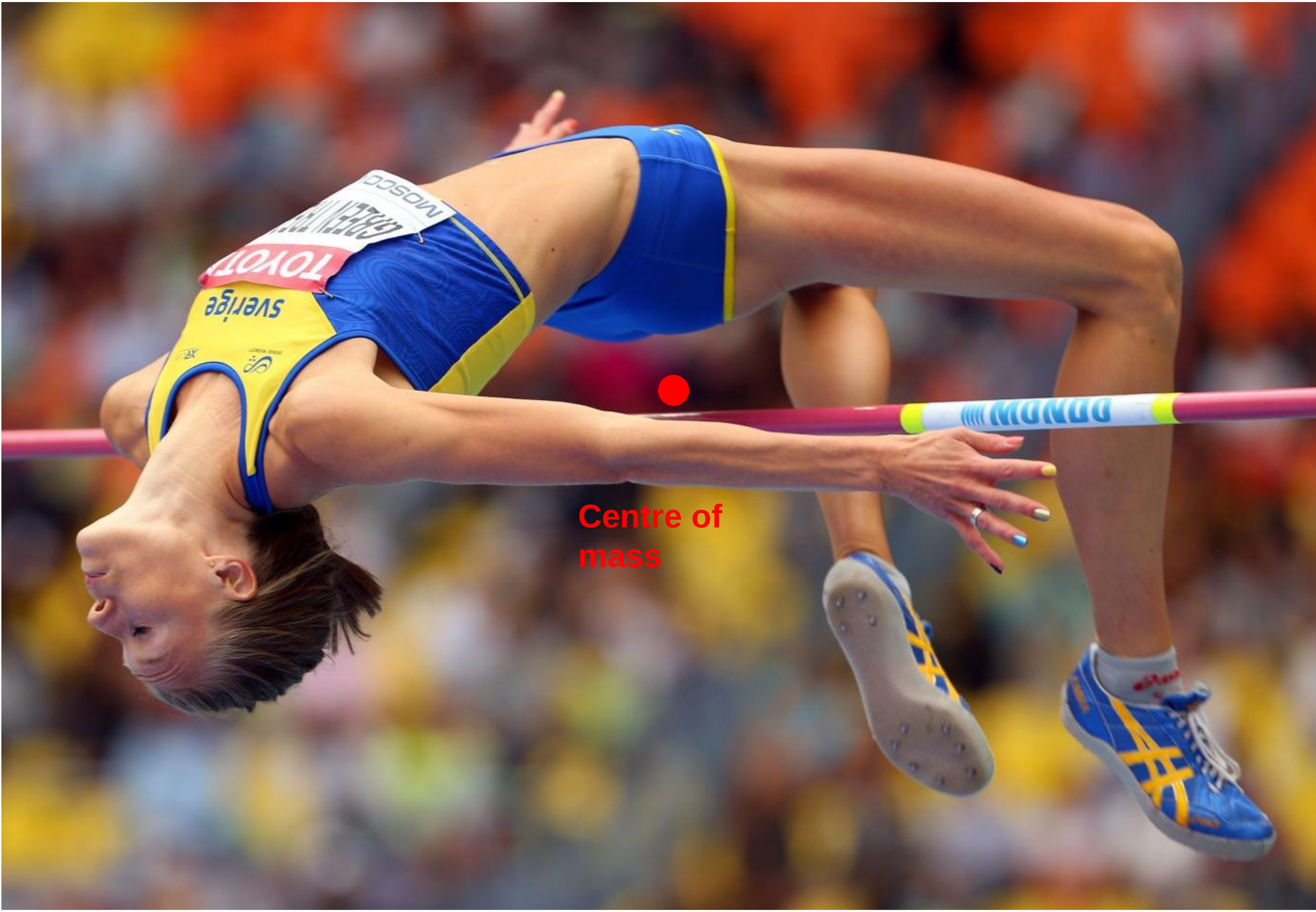
Hang object from position B and draw on another plumb line. Where the two plumb lines intersect is the centre of mass.

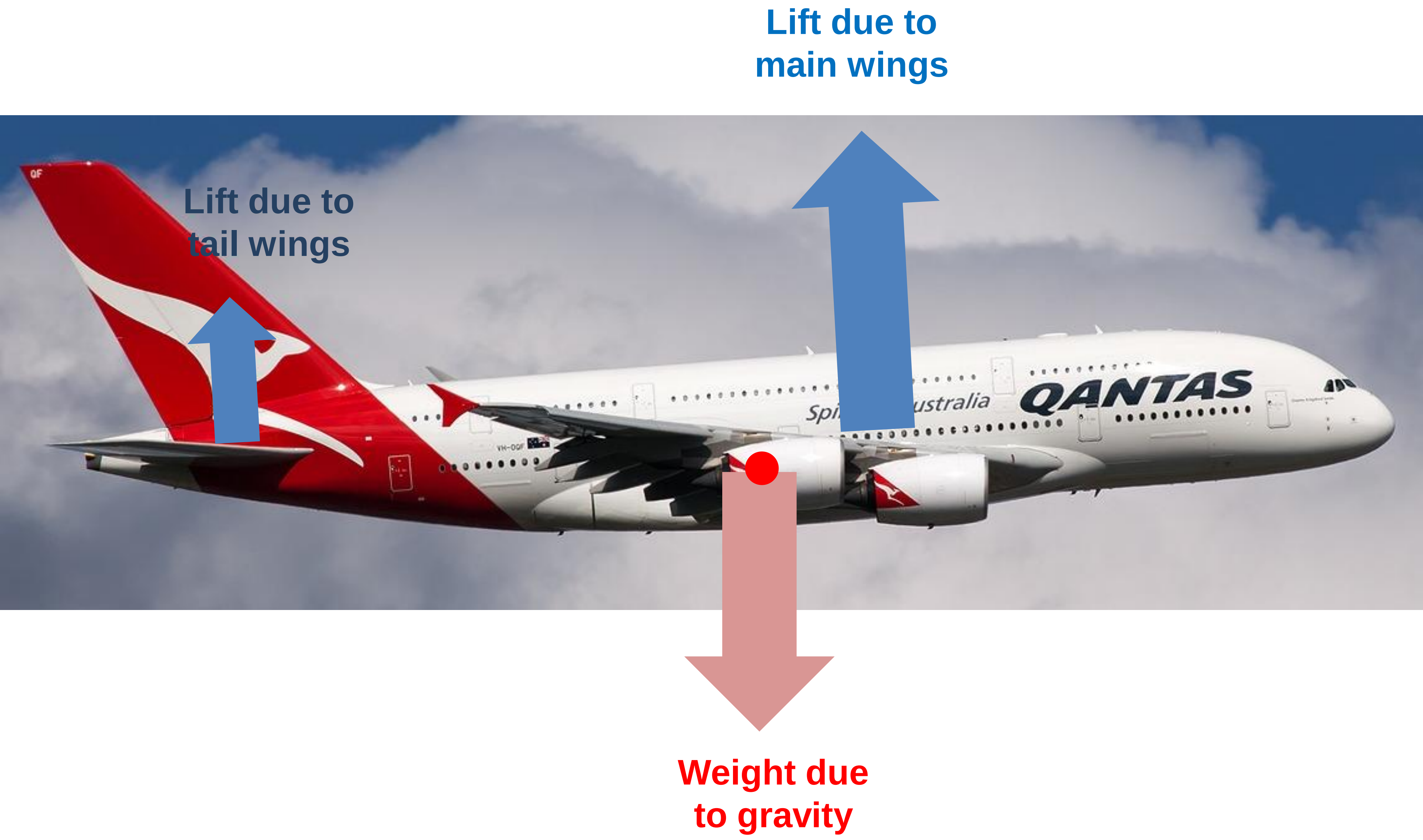


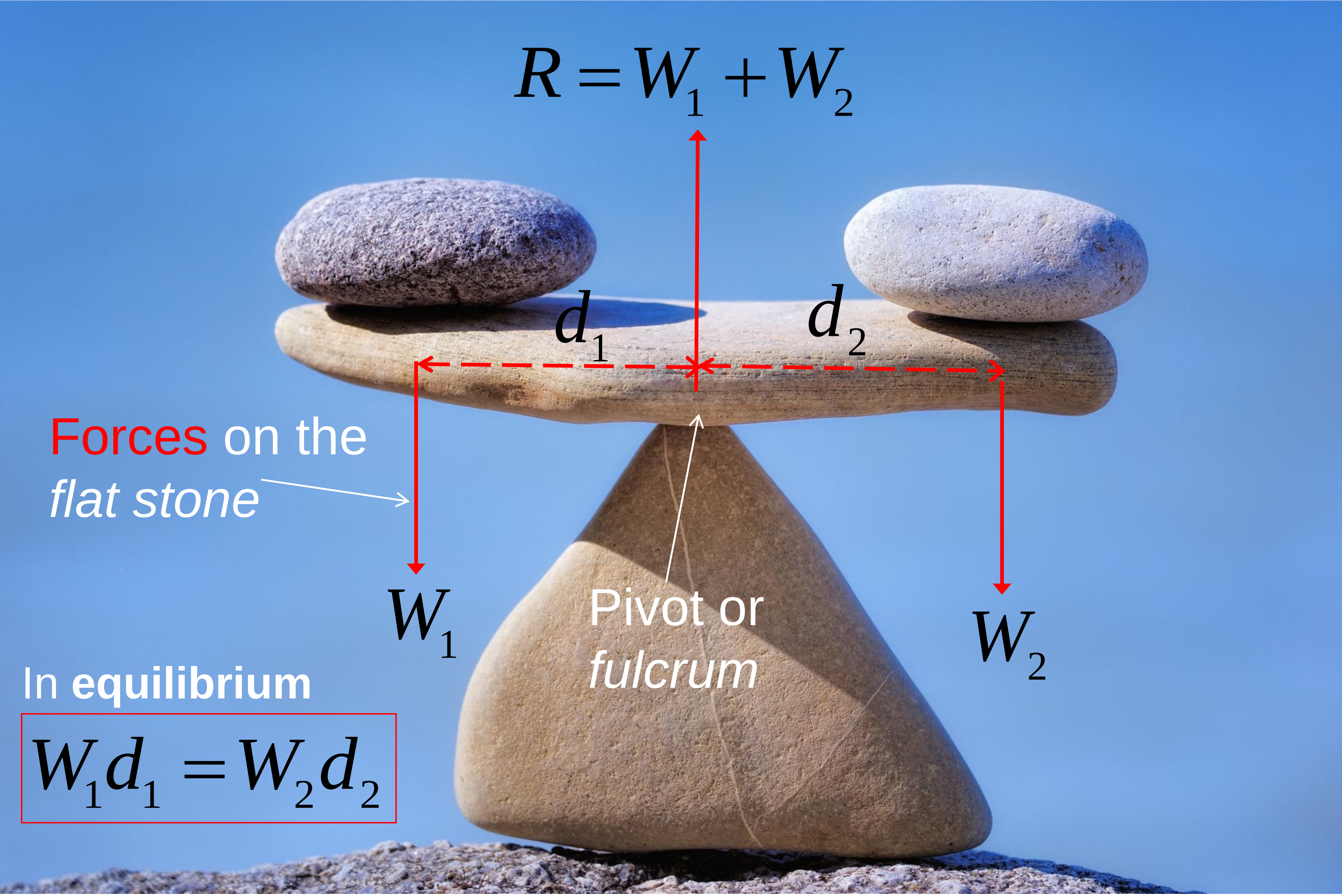
The entire weight of a rigid object effectively acts upon its **centre of mass**.

If rotation is ignored, we can model a rigid object as a **particle** i.e. just consider the motion of the centre of mass


Centre of
mass


Centre of
mass


Lift due to
main wings
Lift due to
tail wings
Spirit of Australia
QANTAS
Weight due
to gravity



Moment = Force x Distance from the Pivot

A rigid body is in **mechanical equilibrium** when:

- the sum of all forces on all particles of the system is zero
- the sum of all torques on all particles of the system is zero

Essentially another word for **Moment**

Forces from different directions balance

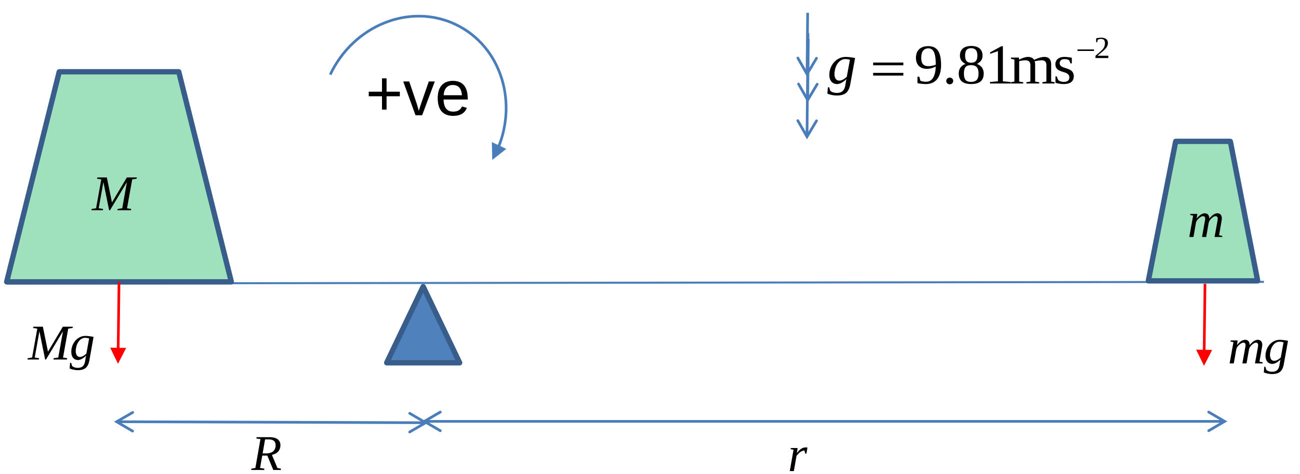
Clockwise moments = Anti-clockwise moments

Moment = Force x Distance from the Pivot

Moment = Force x Distance from the Pivot

If a body is not rotating, or rotating with a constant angular velocity, then the *sum of moments must equate to zero*. This is very useful in calculating forces in equilibrium problems involving rigid bodies, since it doesn't matter in this case which point 'we take moments about.' (They must always sum to zero).

Only when there is net torque is it a good idea to choose the centre of mass of an object.



Using the convention that *clockwise moments are positive*

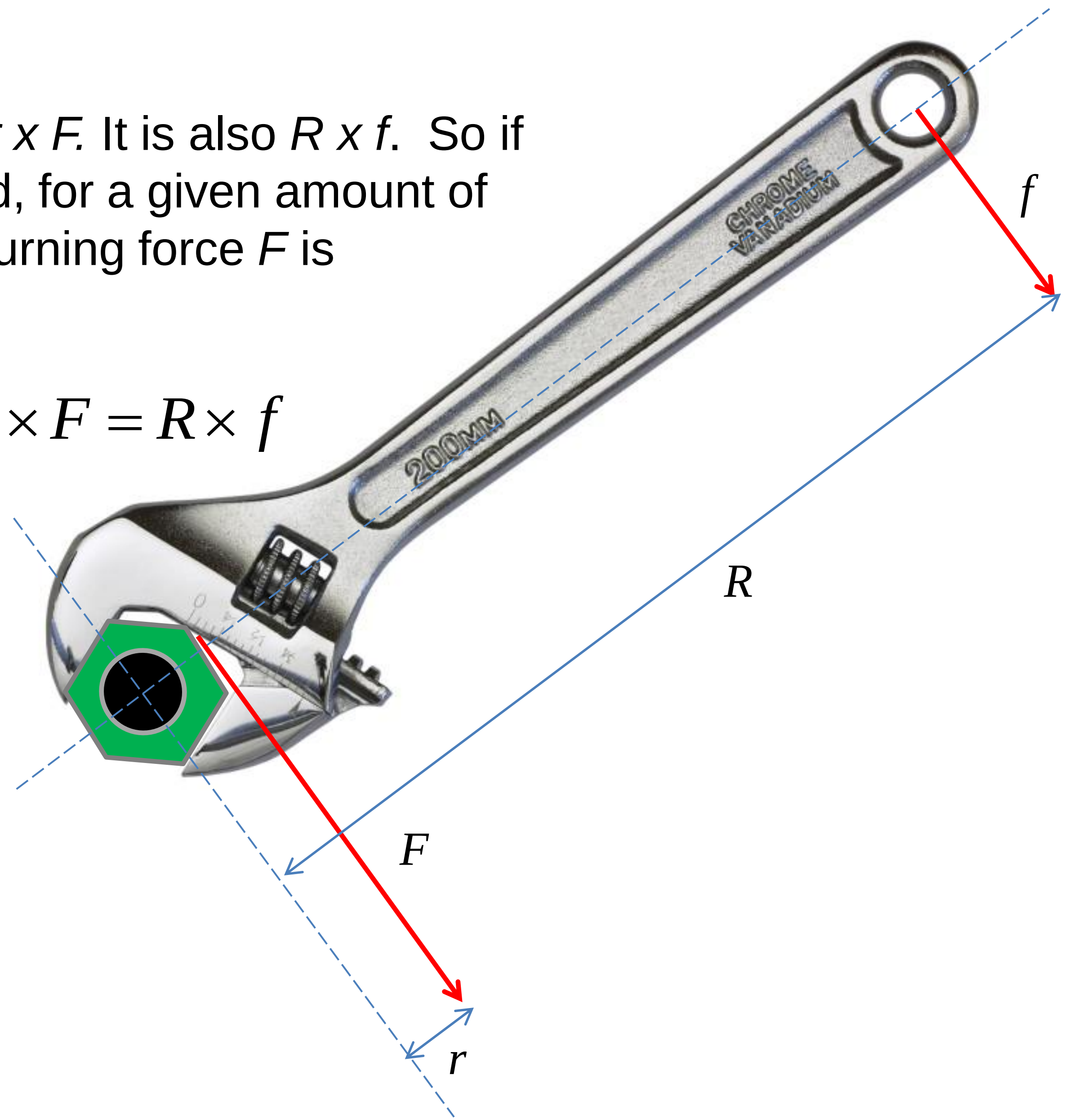
$$-MgR + mgr = 0 \quad \therefore \quad \boxed{mr = MR}$$

A **tool** like a screwdriver or a wrench can deliver the required turning moment to tighten a screw or nut *with less force*.

The torque on the nut is *r x F.* It is also *R x f*. So if the ratio *R / r* is increased, for a given amount of force f, the effective nut-turning force *F* is magnified.

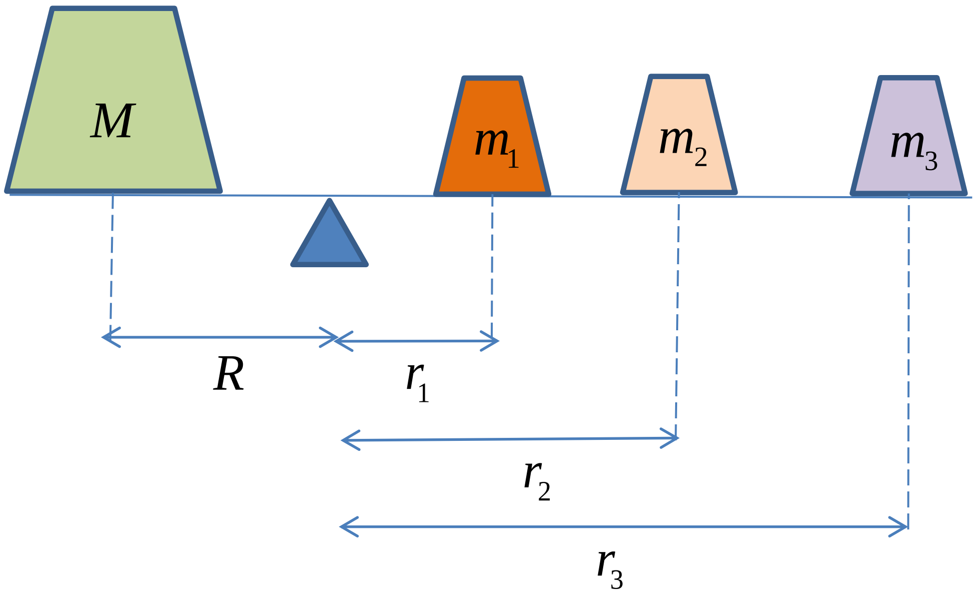
$$r \times F = R \times f$$

$$\boxed{F = \frac{R}{r} \times f}$$



This is called a **mechanical advantage**

A simple 1D example of centre of mass

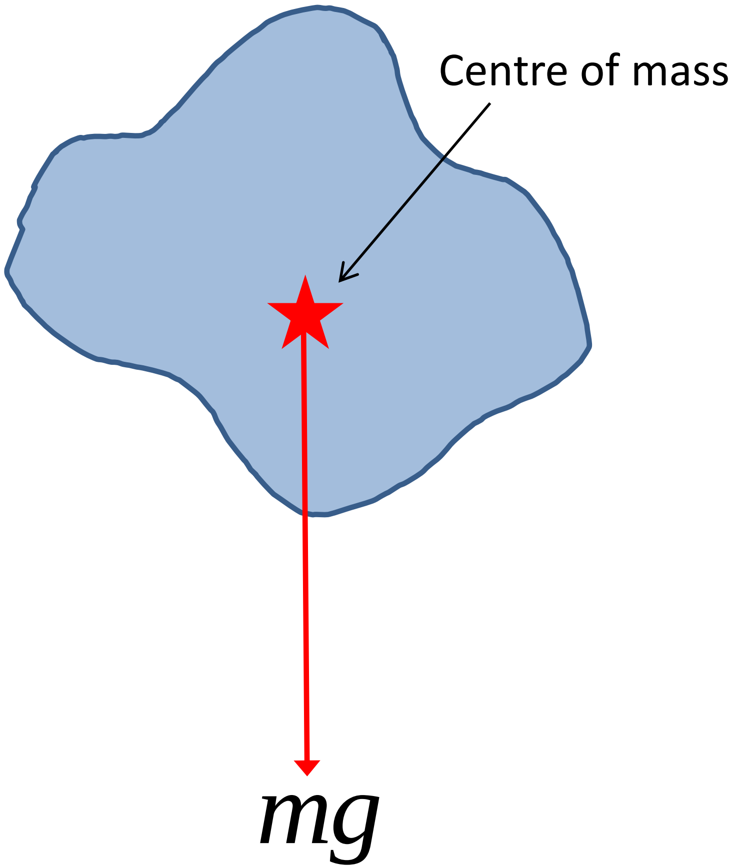


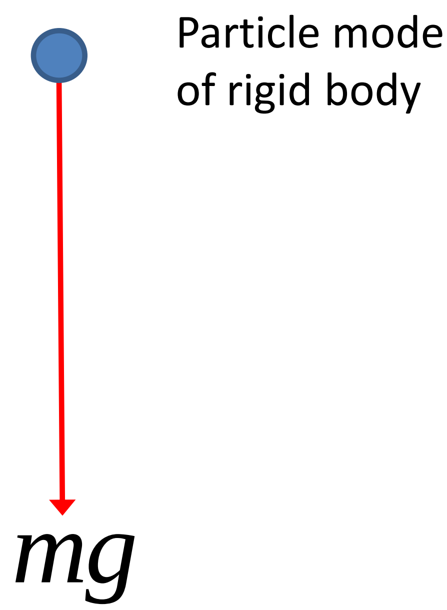
In order for the above system to be in equilibrium, the **total moment about the pivot must be zero**

$$MR = m_1 r_1 + m_2 r_2 + m_3 r_3$$

$$\therefore \boxed{R = \frac{\sum_i m_i r_i}{M}}$$

If *M* is the sum of all the masses, then
we can think of the three-mass system *being equivalent* to a single mass *M* a distance *R* from the fulcrum.



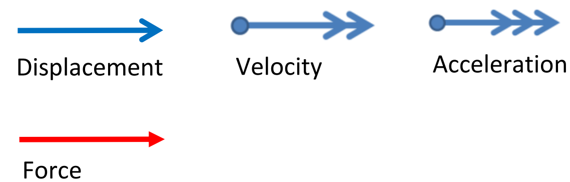


The entire weight of a rigid object effectively acts upon its **centre of mass**.

If rotation is ignored, we can model a rigid object as a **particle** i.e. just consider the motion of the centre of mass

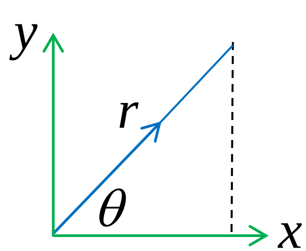
Vector and scalar quantities

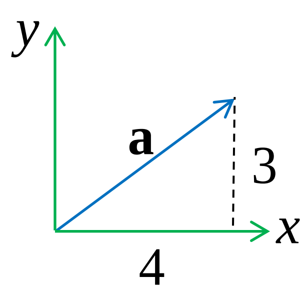
Vector quantities		Units	Scalar quantities	Units
Displacement	$\mathbf{x}$	mm,m,km	Mass m	kg
Velocity	$\mathbf{v}$	ms^{-1}, kmh^{-1}	Time t	s, mins, h
Acceleration	$\mathbf{a}$	ms^{-2}	Speed s	ms^{-1}, kmh^{-1}
Momentum	$\mathbf{p} = m\mathbf{v}$	kgms^{-1}	Length l	mm,m,km
Force	$\mathbf{f}$	N (Newtons)		



A vector has both *magnitude* and *direction*.

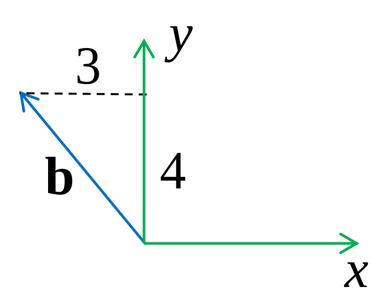
$\mathbf{r}$ $\quad\quad$ $|\mathbf{r}| = r$ $\quad\quad$ θ





$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$|\mathbf{a}| = 5$$



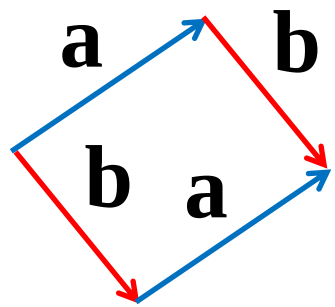
$$\mathbf{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$|\mathbf{b}| = 5$$

Addition and scalar multiplication using vectors

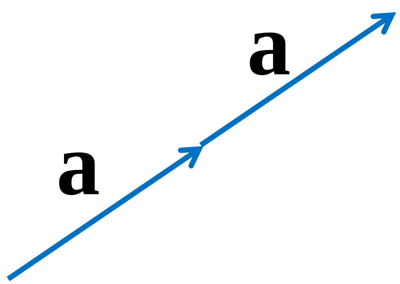
The algebra of vectors is very similar to scalars. Except *vector multiplication is very different*. This will not be discussed in this course!

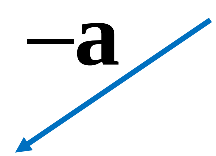
$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$



Vectors add 'tip to tail'

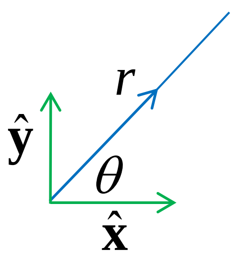
$$2\mathbf{a}=\mathbf{a}+\mathbf{a}$$





Components of a vector are with respect to a **coordinate system**

These are **unit vectors** in the x and y directions



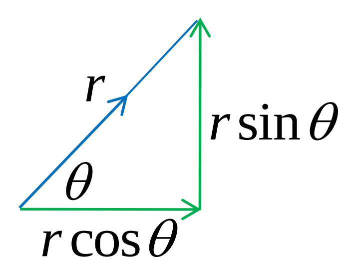
$$|\hat{\mathbf{y}}|=|\hat{\mathbf{x}}|=1$$

$$\mathbf{r}=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

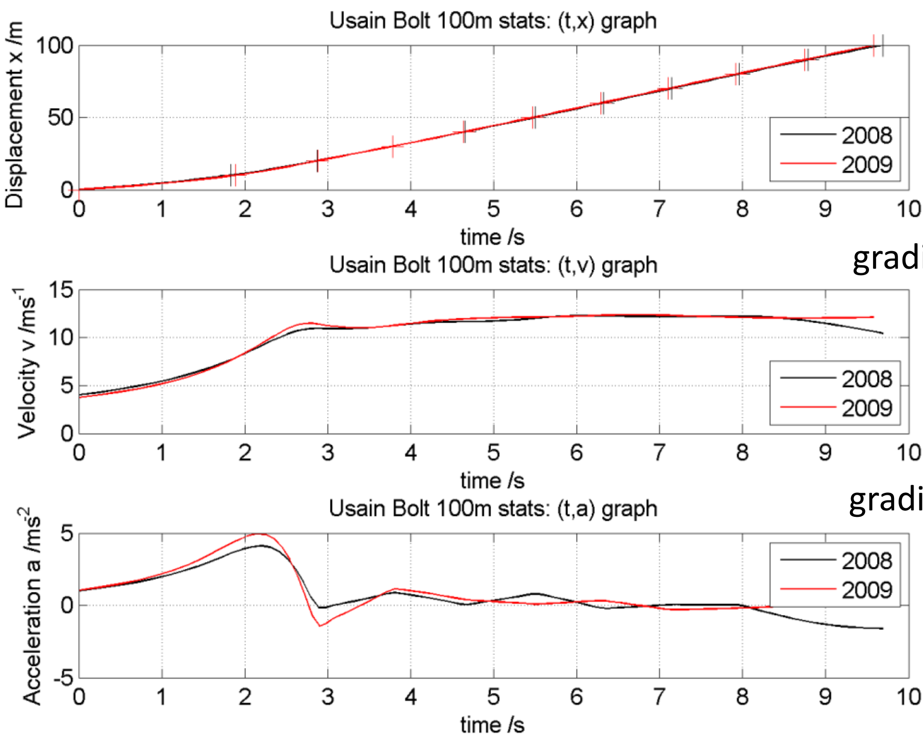
$\leftarrow$ x component

$\leftarrow$ y component

We often speak of 'resolving' a vector into components $\mathbf{r}=r\cos\theta\hat{\mathbf{x}}+r\sin\theta\hat{\mathbf{y}}$



Relationship between displacement, velocity and acceleration



gradient $\dfrac{dx}{dt}$

gradient $\dfrac{dv}{dt}$

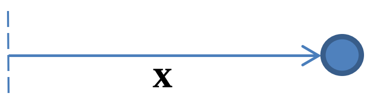
Useful speed conversions:

1 ms^{-1} = 2.24 miles per hour

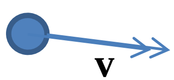
1 ms^{-1} = 3.6 km per hour

$$t\,/\,\text{min} = 60 \times \frac{x\,/\,\text{miles}}{v\,/\,\text{mph}}$$

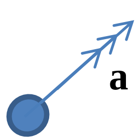
Speed in mph	Time in minutes per 10 miles
10	60
20	30
30	20
40	15
50	12
60	10
70	8.57



Displacement is the vector between a fixed origin and the point of interest. If an object is moving, the displacement will vary with time *t*



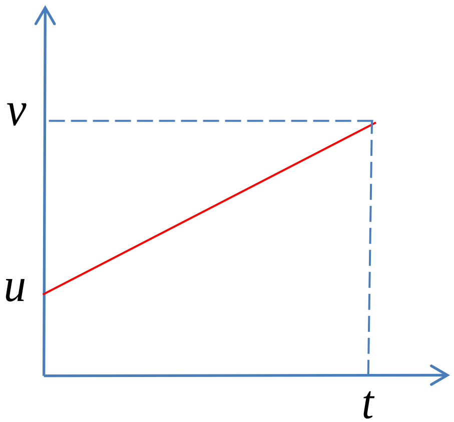
Velocity is the *rate of change of displacement*. If velocity is in the same direction as displacement, it is the gradient of a *(t,x)* graph.



Acceleration is the *rate of change of velocity*. If acceleration is in the same direction as velocity, it is the gradient of a *(t,v)* graph.

Constant acceleration motion

It is almost *always* a good idea to start with a *(t,v)* graph.
Let velocity increase at the same rate *a* from *u* to *v* in *t* seconds.



The acceleration is the gradient: $a = \dfrac{v-u}{t} \quad \therefore \quad \boxed{v = u + at}$

The area under the graph is the displacement.
Since this a trapezium shape: $\boxed{x = \tfrac{1}{2}(u+v)t}$

We can work out other useful relationships for constant acceleration motion

$$x = \tfrac{1}{2}(u + u + at)t$$

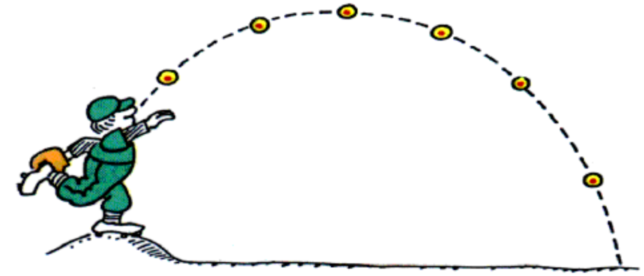
$$\boxed{x = ut + \tfrac{1}{2}at^2}$$

$$x = ut + \tfrac{1}{2}at^2$$

$$2ax = 2uat + a^2t^2$$

$$v^2 = (u + at)^2 = u^2 + 2uat + a^2t^2$$

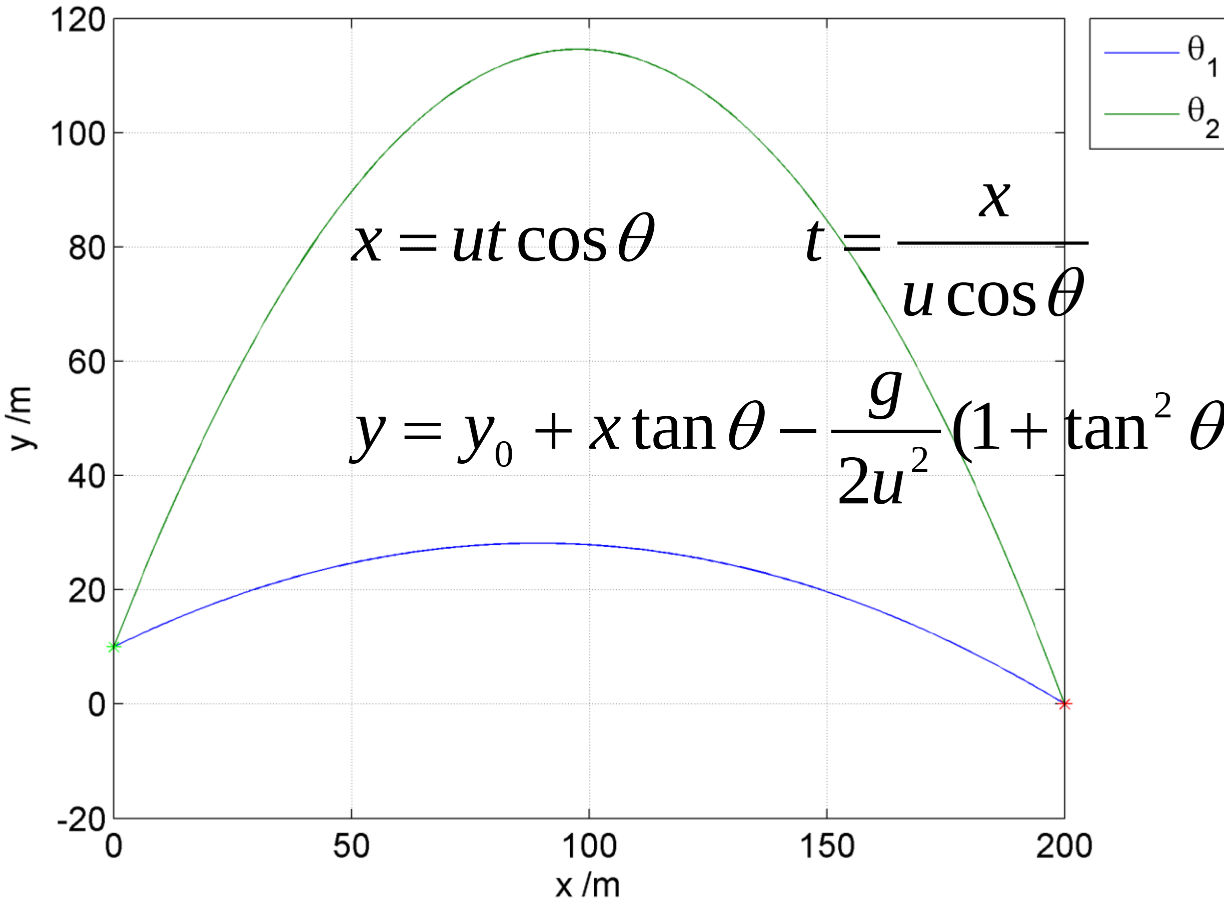
$$\therefore \boxed{v^2 = u^2 + 2ax}$$



PROJECTILE: $u = 50.0\,\mathrm{ms}^{-1}$
$\theta_1 = 22.2^\circ$, $T_1 = 4.3\,\mathrm{s}$. $v_1 = 51.9\,\mathrm{ms}^{-1}$
$\theta_2 = 65.0^\circ$, $T_2 = 9.5\,\mathrm{s}$. $v_2 = 51.9\,\mathrm{ms}^{-1}$

Projectile motion

$$g = 9.81\,\mathrm{ms}^{-2}$$



$$x = ut\cos\theta \qquad t = \frac{x}{u\cos\theta}$$

$$y = y_0 + x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

Concepts to reflect on
(Lots today! Don't worry you will meet these again and again and again .....)

Work done = Force x distance

Gravitational force (weight) = mass x gravitational acceleration

Work done by gravity = weight x height

Elasticity

Air resistance

Moments

Centre of mass

Vectors

Kinematics

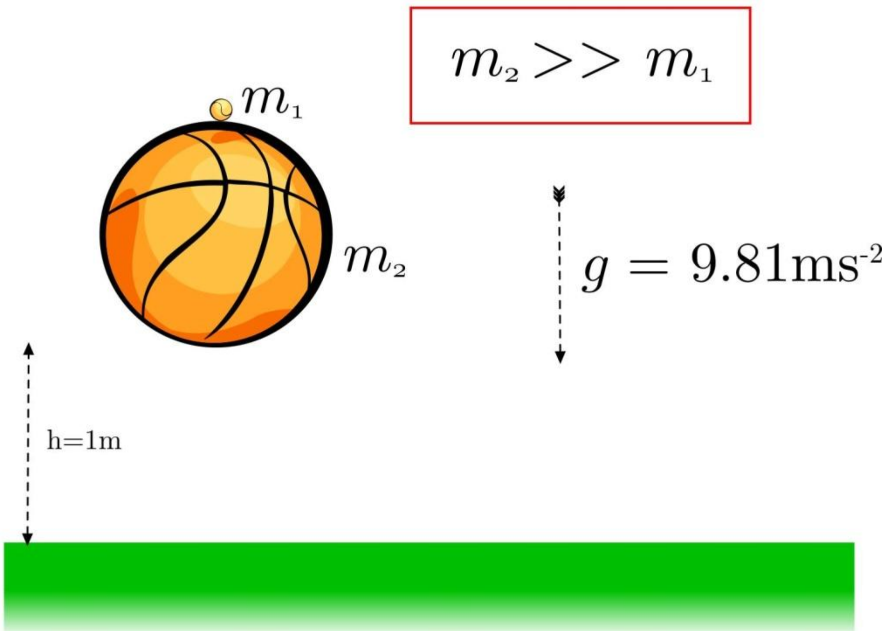
Constant acceleration motion

Projectiles

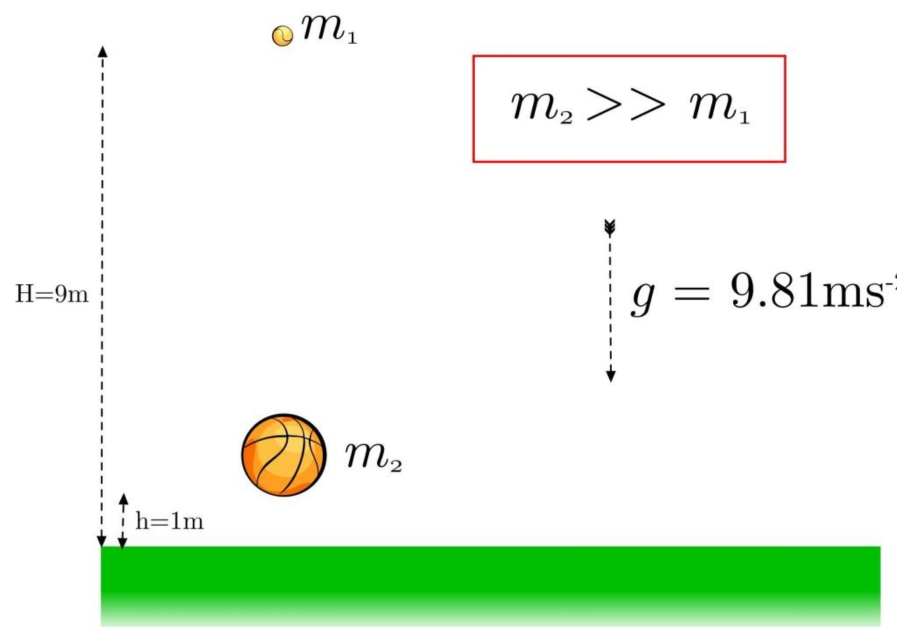
Depending on your course, we may not cover all of these. Review the topics you did meet. If you have time to spare, read on!

Velocity amplifying elastic collider – simple setup, surprising result!

Balls are dropped from rest



Following collision, the smaller mass rises up to *nine times* the distance fallen!



$$\boxed{H = h\left(\frac{3 - m_1/m_2}{1 + m_1/m_2}\right)^2}$$

$$m_1/m_2 \ll 1$$

$$H \to 9h$$

To work this out requires *three* principles: conservation of **momentum**, **restitution** and conservation of **energy**