Defining sequences

term in sequence

$$u_8 = 42$$

term number or index

Iterative form $\boxed{u_{n+1} = f(u_n, n)}$

$u_{n+1} = u_n^2 + 1$

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = 1, 2, 5, 26, 677, 458330$

Term form $\boxed{u_n = g(n)}$

$u_n = n^3$

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = 1, 8, 27, 64, 125, 216, 343, 512$

Summing sequences

$S_n = u_1 + u_2 + \ldots\ldots + u_n$

$$\sum_{m=1}^{n} u_m = u_1 + u_2 + \ldots\ldots + u_n$$

A useful shorthand for describing a sum of *n* terms is to use *sigma notation*

Arithmetic Progression (AP)

A sequence where the **difference** between subsequent terms is a **constant**

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = n$

$$S_{100} = \boxed{1} + \boxed{2} + \ldots\ldots + \boxed{99} + \boxed{100}$$
$$S_{100} = \boxed{100} + \boxed{99} + \ldots\ldots + \boxed{2} + \boxed{1}$$
$$2S_{100} = 101 + 101 + \ldots\ldots + 101 + 101$$
$$\therefore S_{100} = \tfrac{1}{2} 101 \times 100 = \boxed{5050}$$

Reverse order of an arithmetic sequence and sum corresponding terms. The sum will always be the *same number*. Hence this number times the number of terms equals twice the sum of the sequence

$$\boxed{u_n = a + (n-1)d}$$

a is the first term
d is the *common difference*

$u_1 = a$

$u_2 = a + d$

$u_3 = a + 2d$

$u_4 = a + 3d$

$\ldots\ldots$

$$S_n = \boxed{a} + \boxed{a+d} + \ldots\ldots\ldots + \boxed{a+(n-1)d}$$
$$S_n = \boxed{a+(n-1)d} + \boxed{a+(n-2)d} + \ldots\ldots\ldots + \boxed{a}$$
$$2S_n = 2a + (n-1)d + 2a + (n-1)d + \ldots\ldots\ldots + 2a + (n-1)d$$

$$\therefore \boxed{\begin{aligned} S_n &= \tfrac{1}{2} n\left(2a + (n-1)d\right) \\ S_n &= \tfrac{1}{2} n\left(u_1 + u_n\right) \end{aligned}}$$

Sum of an Arithmetic Progression is the **mean average** of the first and last terms **multiplied by the number of terms**.

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = 10, 8, 6, 4, 2, 0, -2, -4$

$u_n = 10 - 2(n-1)$

$u_n = 12 - 2n$

$S_n = \tfrac{1}{2} n\left(u_1 + u_n\right)$

$S_n = \tfrac{1}{2} n\left(10 + 12 - 2n\right)$

$S_n = n\left(11 - n\right)$

In this case:

$u_n < 0$

$12 - 2n < 0$

$\boxed{6 < n}$

$S_n < 0 \;;\; n > 0$

$n\left(11 - n\right) < 0$

$\boxed{11 < n}$

So find *n* when S_n is > a constant you will need to solve a *quadratic* in *n* and then round to the nearest *n* as appropriate

Geometric Progression (GP)

$$\boxed{u_n = ar^{n-1}}$$

r is the *common ratio*

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$u_4 = ar^3$$
$$\ldots\ldots$$

$$\boxed{S_n = \sum_{m=1}^{n} u_m = a\frac{1-r^n}{1-r}}$$

Proof of formula for the sum of a geometric sequence

$$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$$
$$rS_n = \quad ar + ar^2 + \ldots\ldots + ar^{n-1} + ar^n$$
$$rS_n - S_n = ar^n - a$$
$$S_n = a\frac{r^n - 1}{r-1} = \boxed{a\frac{1-r^n}{1-r}}$$

Special case:

$$\lim_{n\to\infty}\left(r^n\right) = 0 \quad \text{if} \quad |r| < 1$$
$$\therefore \text{if} \quad |r| < 1$$
$$\boxed{\sum_{m=1}^{\infty} ar^{m-1} = \frac{a}{1-r}}$$

This is a *convergent* sequence

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\ldots + \frac{1}{2^{n-1}}$$
$$S_n = \frac{1-2^{-n}}{1-\frac{1}{2}} = 2\left(1-2^{-n}\right)$$
$$S_\infty = 2$$

GP Example 1

Consider the sequence: 3, 6, 12, 24, 48, ... Find S_{10}

$$u_n = 3\times 2^{n-1}$$
$$S_n = 3\frac{2^n - 1}{2-1}$$
$$S_n = 3\left(2^n - 1\right)$$
$$\therefore S_{10} = \boxed{3069}$$

GP Example 2

Consider the sequence: 1024, -512, 256, -128, 32, ... Find the sum to infinity

$$u_n = 1024\times\left(-\tfrac{1}{2}\right)^{n-1}$$
$$S_n = 1024\frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}$$
$$S_n = \frac{2048}{3}\left(1 - \frac{(-1)^n}{2^n}\right)$$
$$\therefore S_\infty = \frac{2048}{3} = \boxed{682\tfrac{2}{3}}$$

GP Example 3

$$u_3 = 63$$
$$u_5 = 567$$
$$S_{10} = ?$$

$$u_n = ar^{n-1}$$
$$63 = ar^2$$
$$567 = ar^4$$
$$\therefore \frac{567}{63} = 9 = r^2 \Rightarrow r = 3$$

$$63 = ar^2$$
$$63 = 9a$$
$$7 = a$$
$$\therefore u_n = 7\times 3^{n-1}$$
$$\therefore S_n = \tfrac{7}{2}\left(3^n - 1\right)$$
$$\therefore S_{10} = \boxed{206,668}$$

Defining sequences

Iterative form $\boxed{u_{n+1} = f(u_n, n)}$

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$u_n = 1, 2, 5, 26, 677, 458330$

These are *divergent* sequences

Term form $\boxed{u_n = g(n)}$

$u_n = n^3$

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$u_n = 1, 8, 27, 64, 125, 216, 343, 512$

Summing sequences: using sigma notation

$$\sum_{n=1}^{4} g(n) = g(1) + g(2) + g(3) + g(4)$$

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{m=1}^{n} (m+1)^3 = (1+1)^3 + (2+2)^3 + (3+2)^3 + \ldots + (n+1)^3$$

Some useful properties of sigma notation

$$\sum_n Ag(n) = A\sum_n g(n)$$

$$\sum_n \{f(n) + g(n)\} = \sum_n f(n) + \sum_n g(n)$$

$$\sum_n (n+1)^2 = \sum_n (n^2 + 2n + 1) = \sum_n n^2 + 2\sum_n n + \sum_n 1$$

$$= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{6}{6}n(n+1) + \tfrac{6}{6}n$$

$$= \tfrac{1}{6}n(n+1)(2n+7) + \tfrac{6}{6}n$$

$$= \tfrac{1}{6}n\left\{2n^2 + 9n + 8\right\}$$

Difference formula $\boxed{g(n) = f(n+1) - f(n)}$

$g(1) = f(2) - f(1)$

$g(2) = f(3) - f(2)$

$g(3) = f(4) - f(3)$

.

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$g(m-1) = f(m) - f(m-1)$

$g(m) = f(m+1) - f(m)$

$$\sum_{m=1}^{n} g(m) = f(n+1) - f(1)$$

$$\boxed{\sum_{m=1}^{n} \{f(m+1) - f(m)\} = f(n+1) - f(1)}$$

$$f(m+1) - f(m) = (m+1)^2 - m^2 = 2m + 1$$

$$(n+1)^2 - 1 = n^2 + 2n$$

$$\therefore \sum_{m=1}^{n} (2m+1) = n^2 + 2n$$

$$2\sum_{m=1}^{n} m + n = n^2 + 2n$$

$$\therefore \boxed{\sum_{m=1}^{n} m = \tfrac{1}{2}n(n+1)}$$

Sums of integer powers

These can all be derived by application of an appropriate difference formula

$$\sum_{m=1}^{n} 1 = n$$

$$\sum_{m=1}^{n} m = 1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1)$$

$$\sum_{m=1}^{n} m^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{m=1}^{n} m^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

Faulhaber's formula can be used to generate a closed form expression for the sum of any integer powers of integers. This involves *Bernoulli numbers.*

Also by the difference formula

$$\sum_{m=1}^{n} \frac{1}{m(m+1)} = \sum_{m=1}^{n} \left\{\frac{1}{m} - \frac{1}{m+1}\right\} = \frac{n}{n+1}$$