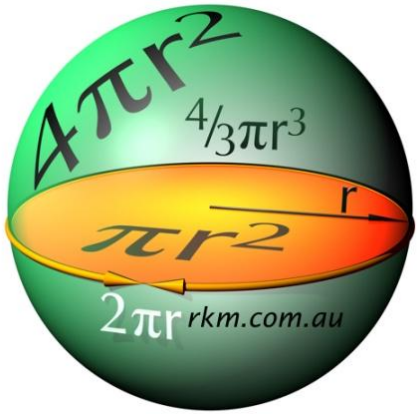
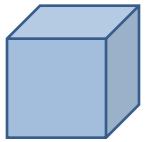
Surface areas of basic solids

Sphere of radius *r*

$$A = 4\pi r^2$$



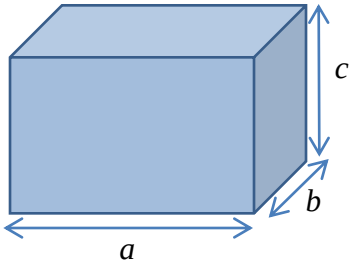
Cube of side *a*

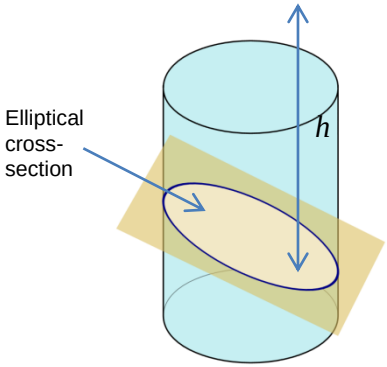


$$A = 6a^2$$

Cuboid

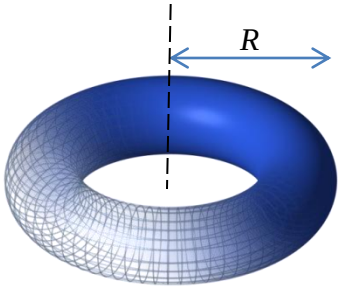
$$A = 2(ab + bc + ca)$$





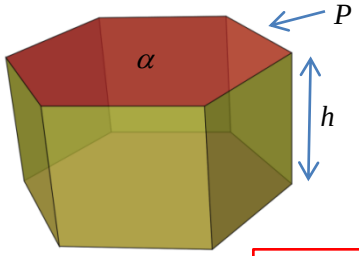
Cylinder, with radius *r* and height *h*

$$A = 2\pi r^2 + 2\pi rh$$



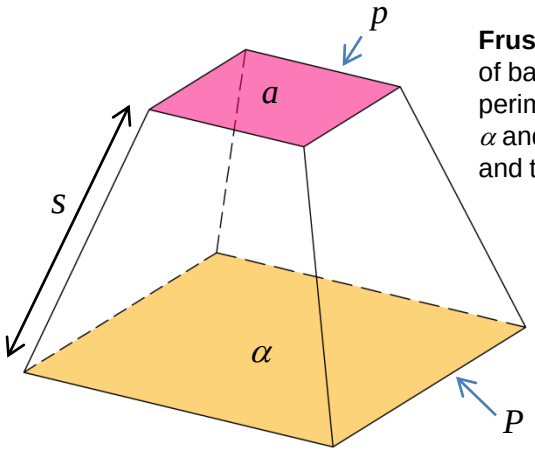
Torus, with a circular cross section of radius r

$$A = 2\pi r \times 2\pi R = 4\pi^2 rR$$



$$A = 2\alpha + Ph$$

Prism, of length *h* with uniform cross sectional area α with perimeter *P*



Frustum, (a truncated pyramid) of base perimeter *P*, top area perimeter *p* and slant height *s*. α and *a* are the respective base and top areas

$$A = a + \alpha + \tfrac{1}{2}s(p + P)$$

Assume all slants are the same length

Otherwise we have an *oblique* frustum or pyramid and these formulae will not hold

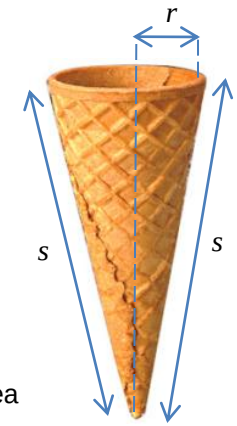
Pyramid, with base perimeter *P* and slant height *s*

$$A = \tfrac{1}{2}Ps$$

Cone, with base radius *r* and slant height *s*

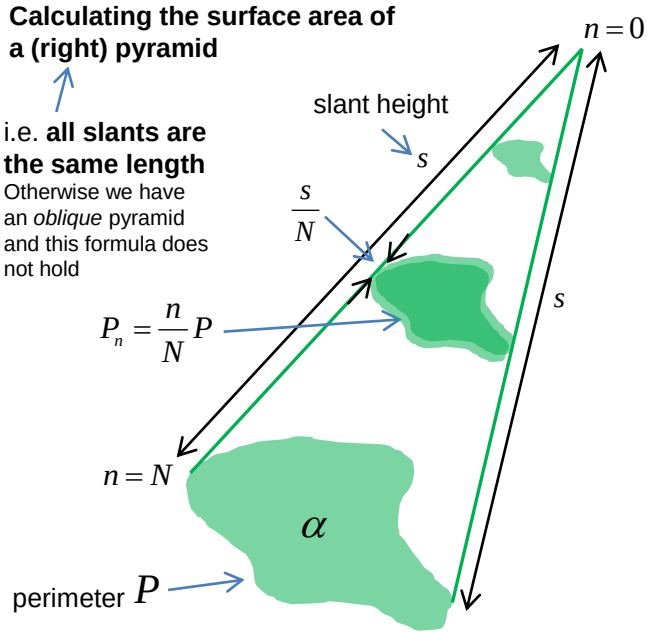
$$A = \pi rs$$

This is the *curved* surface area i.e. an 'open' cone



Calculating the surface area of a (right) pyramid

i.e. **all slants are the same length**
Otherwise we have an *oblique* pyramid and this formula does not hold



A pyramid of perpendicular slant height *s* and base perimeter *P* can be thought of being composed of an (infinitely) large number of **similar** *laminae*.

The *enlargement factor* of each lamina *is in direct proportion to the slant distance from the apex of the pyramid.* This is true since the sides of the pyramid are straight lines with a constant gradient.

Let there be *N* laminae of thickness *s/N*. Since all laminae are *similar*, the perimeters scale as *n/N* where *n* is the lamina number from the apex. The exterior area of each lamina is:

$$A_n = P_n \times \frac{s}{N}$$

$$A_n = \frac{n}{N}P \times \frac{s}{N}$$

$$A_n = \frac{Ps}{N^2}n$$

Therefore the total exterior of the pyramid is the sum of the exterior areas of the *N* laminae

$$A_N = \frac{Ps}{N^2}1 + \frac{Ps}{N^2}2 + ... + \frac{Ps}{N^2}N$$

$$A_N = \frac{Ps}{N^2}(1+2+....+N)$$

Now the sum of the first *N* integers is

$$1+2+....+N = \tfrac{1}{2}N(N+1)$$

Hence:

$$A_N = \frac{Ps}{N^2}\tfrac{1}{2}N(N+1)$$

$$A_N = \tfrac{1}{2}Ps\frac{N^2+N}{N^2}$$

$$A_N = \tfrac{1}{2}Ps\left(1+\tfrac{1}{N}\right)$$

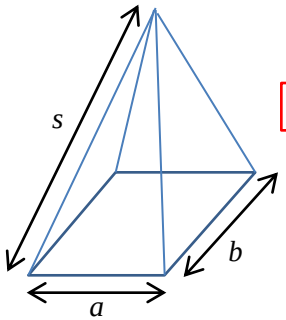
Therefore as *N* becomes infinite, the area tend towards

$$\boxed{A = \tfrac{1}{2}Ps}$$

Add the base area α to make the total surface area

i.e. *"the total area of a pyramid is the base plus half the product of the base perimeter and the slant height"*

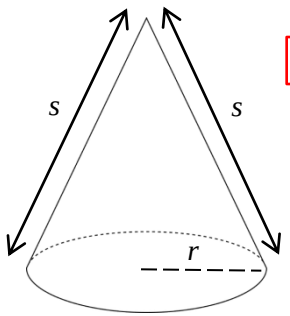
Area of a *right* rectangular based pyramid



$$A = ab + \tfrac{1}{2}s(2a+2b)$$

$$\boxed{A = ab + s(a+b)}$$

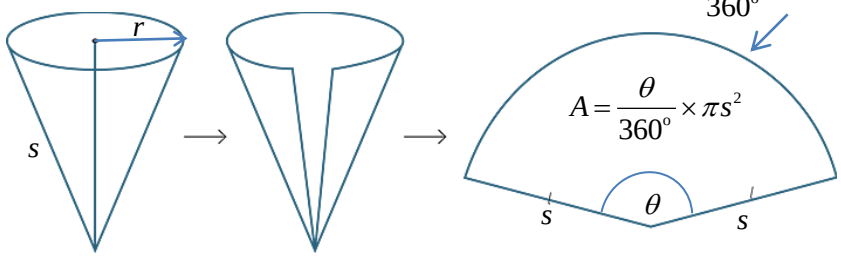
Area of a *right* cone



$$A = \pi r^2 + \tfrac{1}{2}2\pi rs$$

$$\boxed{A = \pi r^2 + \pi rs}$$

Note we can find the **curved area of a cone** by noting how it can be constructed from a circle with a sector missing



$$\therefore \frac{\theta}{360^\circ} = \frac{r}{s}$$

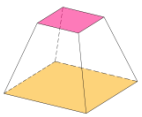
$$\therefore A = \frac{r}{s} \times \pi s^2$$

$$\therefore \boxed{A = \pi rs}$$

A *frustum* is a *truncated pyramid*

Curved area of entire pyramid (assume NOT oblique)

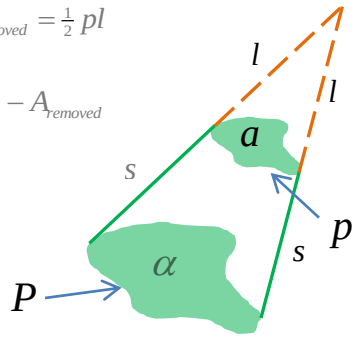
$$A_{entire} = \tfrac{1}{2}P(s+l)$$



Since cap is *similar* to entire pyramid, curved of removed 'cap' pyramid is: $A_{removed} = \tfrac{1}{2}pl$

Therefore frustum curved area is $A = A_{enitre} - A_{removed}$

$$A = \tfrac{1}{2}P(s+l) - \tfrac{1}{2}pl$$



Now since the top and base of the frustum are *similar*

$$\frac{P}{p} = \frac{s+l}{l}$$

$$\frac{P}{p} = \frac{s}{l} + 1$$

$$\frac{P}{p} - 1 = \frac{s}{l}$$

$$\frac{P-p}{p} = \frac{s}{l}$$

$$l = \frac{ps}{P-p}$$

Hence:

$$A = \tfrac{1}{2}P(s+l) - \tfrac{1}{2}pl$$

$$A = \tfrac{1}{2}Ps + \tfrac{1}{2}l(P-p)$$

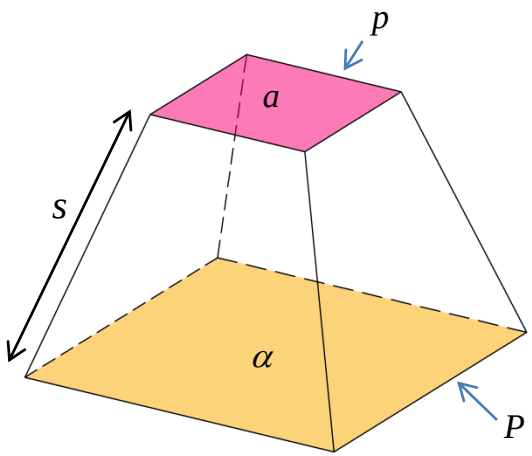
$$A = \tfrac{1}{2}Ps + \tfrac{1}{2}\frac{ps}{P-p}(P-p)$$

$$\boxed{A = \tfrac{1}{2}s(P+p)}$$

Therefore total surface area of the frustum is:

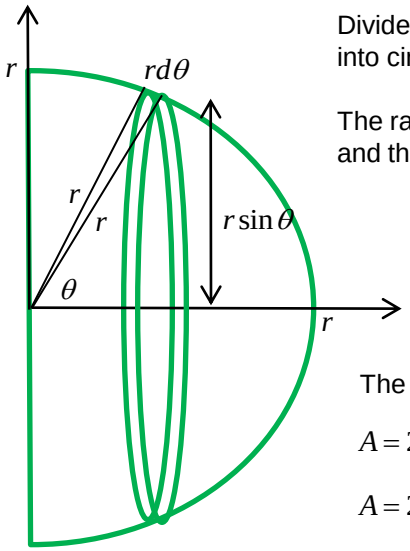
$$\boxed{A = a + \alpha + \tfrac{1}{2}s(p+P)}$$

This is only true for non oblique frustums!



Calculating the area of a sphere using calculus

Note all angles are in *radians* i.e. π radians = 180º.



Divide up the surface of a hemisphere into circular ribbons

The radius of each ribbon is $r\sin\theta$ and the width is the arc length $rd\theta$

The area of each ribbon is

$$dA = 2\pi \times r\sin\theta \times rd\theta = 2\pi r^2 \sin\theta d\theta$$

The total area of the *sphere* is therefore

$$A = 2 \times \int dA$$

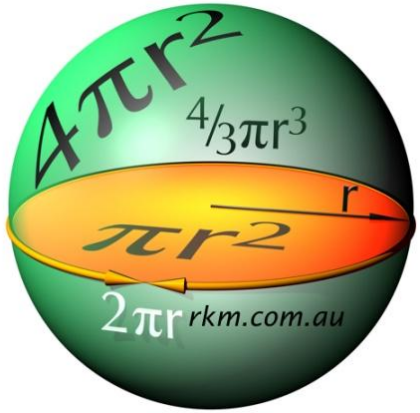
$$A = 2 \times 2\pi r^2 \int_0^{\frac{1}{2}\pi} \sin\theta d\theta$$

$$A = 4\pi r^2 \left[-\cos\theta\right]_0^{\frac{1}{2}\pi}$$

$$A = 4\pi r^2 \left\{\left(-\cos\tfrac{1}{2}\pi\right) - \left(-\cos(0)\right)\right\}$$

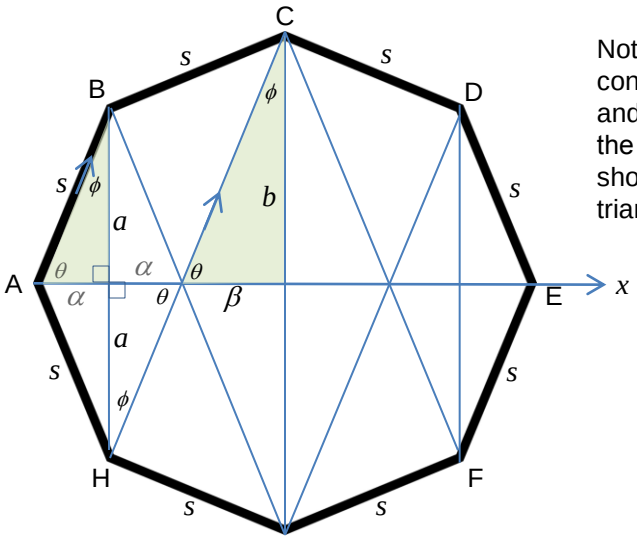
$$A = 4\pi r^2 \left\{0 - (-1)\right\}$$

$$\boxed{A = 4\pi r^2}$$



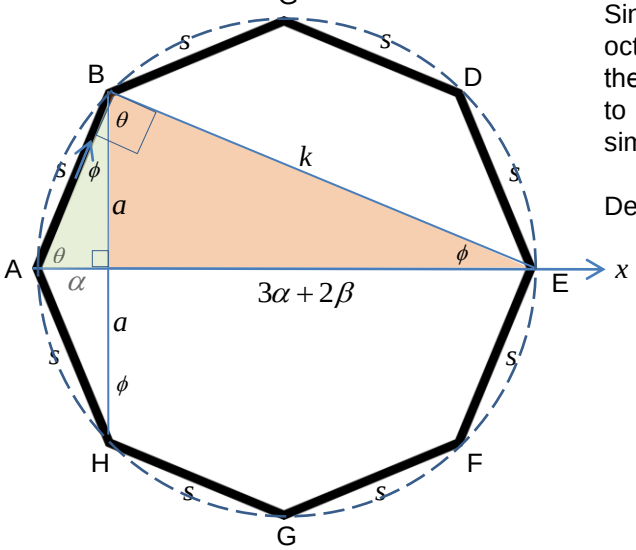
Archimedes' method for calculating the area of a sphere

Firstly this involves finding a formula for the surface area of a *volume of revolution* of a regular octagon about the *x* axis. Generalizing for regular even-number sided polygons of increasing large numbers of sides, we anticipate the resulting surface area to be that of a sphere.



Noting the parallel lines connecting vertices AB and HC, and by application of the 'Z-angle' theorem, one can show that the green right angled triangles highlighted are *similar*

Hence: $\boxed{\dfrac{b}{\beta}=\dfrac{a}{\alpha}}$



Since we can *circumscribe* the octagon with a circle, we can use the 'right angle diameter theorem' to show the orange circle is also similar to the green triangles.

Define k = BE

$$\therefore \frac{k}{s}=\frac{b}{\beta}\Rightarrow \boxed{b=\beta\frac{k}{s}}$$

$$\therefore \frac{k}{s}=\frac{a}{\alpha}\Rightarrow \boxed{a=\alpha\frac{k}{s}}$$

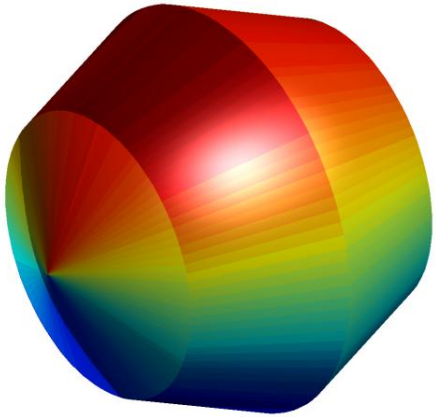
A volume of revolution of the octagon is the volume of **two cones** formed by the rotation of triangle ABH and **two frustums** formed by the rotation of trapezium BHGC

The total surface area is therefore:

$$A=2\pi as+2\times\tfrac{1}{2}s\left(2\pi a+2\pi b\right)$$

$$A=2\pi as+2\pi s\left(a+b\right)$$

$$A=2\pi s\left(2a+b\right)$$



Using the results above

$$A=2\pi k\left(2\alpha+\beta\right)$$

$$A=\pi k\times\left(4\alpha+2\beta\right)$$

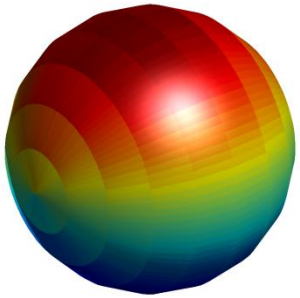
$$\boxed{A=\pi k\times \mathrm{AE}}$$

Now we would expect the same result *for any even polygon*, but as the number of sides increase, **we expect *k* to approach AE** as *s* shrinks

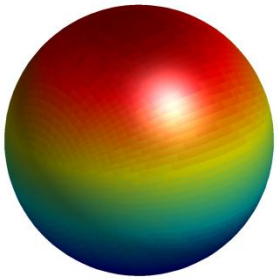
For a circle AE = 2*r*. Therefore if in this scenario *k* = AE

$$A=\pi 2r\times 2r$$

$$\boxed{A=4\pi r^2}$$



20 sided polygon of revolution



100 sided polygon of revolution – this looks very much like a sphere!