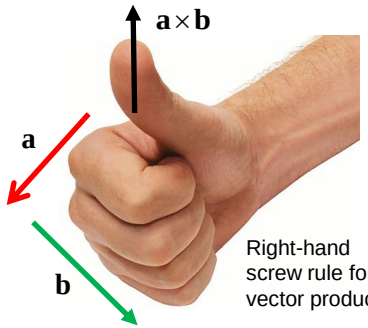
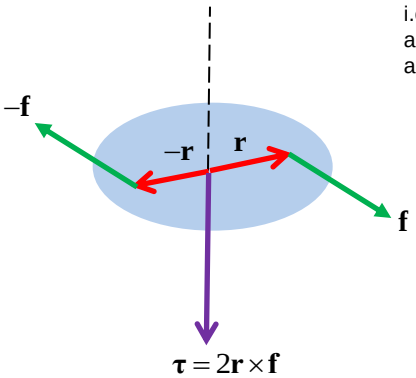
The **moment of a force**, or **torque**, is the *vector cross product* of a displacement $\mathbf{r}$ and a force $\mathbf{f}$. $\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}}$ If $\mathbf{r}$ is the displacement from the centre of mass of a rigid body then a net applied torque will result in a rotation of the body about an axis through its centre of mass. Note unless torque is applied via two equal and opposite direction forces (this is called a **couple**) the centre of mass will also accelerate due to the net force upon the body.

If $\mathbf{r}$ and $\mathbf{f}$ are perpendicular, then the magnitude of the torque (or **turning moment**) is **force x distance from the rotation axis.**



Right-hand screw rule for vector products

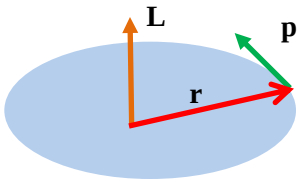


A **couple** applied to a rigid body creates torque $\boldsymbol{\tau} = 2\mathbf{r} \times \mathbf{f}$

This will not cause any translational movement of the centre of mass of the body but will cause it to rotate.

Angular momentum is defined as $\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$

i.e. the cross product between the momentum of a particle and a displacement vector from an axis of rotation



For a rigid body $\boxed{\mathbf{L} = \mathbf{I}\boldsymbol{\omega}}$

$\mathbf{I}$ is the **moment of inertia tensor** and $\boldsymbol{\omega}$ is the angular velocity vector which describes the *instantaneous rate of rotation* about an axis parallel to $\boldsymbol{\omega}$.

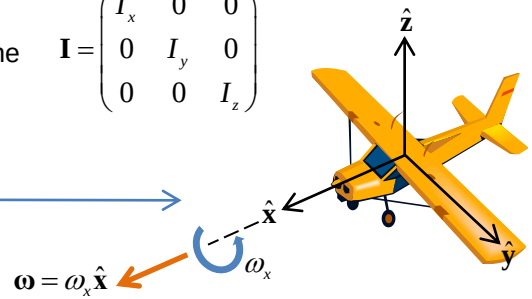
With respect to Cartesian *x,y,z* coordinates:

$$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2)\,dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2 + z^2)\,dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2 + y^2)\,dm \end{pmatrix}$$

The inertia tensor can often be simplified if one defines Cartesian coordinates to match the **symmetry axes** of a rigid body

$$\mathbf{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

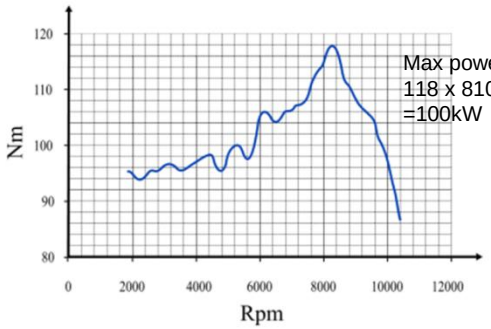
In this example, the aircraft is executing a *roll* (and not a *pitch* or *yaw*)



Most engines comprise of a rotating shaft which is then used to drive wheels etc. The **power** is given by $\boxed{P = \boldsymbol{\tau} \cdot \boldsymbol{\omega}}$

and the rotational kinetic energy

$$\boxed{E = \tfrac{1}{2}(\mathbf{I}\boldsymbol{\omega}) \cdot \boldsymbol{\omega} = \tfrac{1}{2} I_x \omega_x^2 + \tfrac{1}{2} I_y \omega_y^2 + \tfrac{1}{2} I_z \omega_z^2}$$



Max power is 118 x 8100 x 0.105 =100kW

Analogous to Newton's Second Law, **applied torque equates to the rate of change of angular momentum**

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

Newton's Second Law describes how the centre of mass of a body responds to a force, of the total mass is constant.

$$\boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt}}$$

Torque equates to a rate of change of **Angular Momentum.**

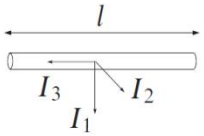
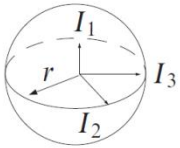
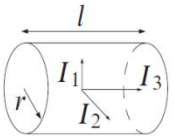
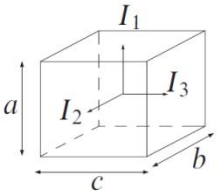
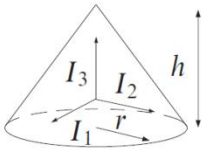
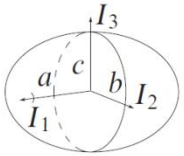
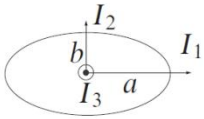
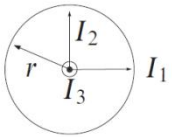
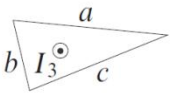
Note $\mathbf{I}$ must be constant

Typically we are interested in situations when $\mathbf{r}$, $\mathbf{f}$ and $\mathbf{t}$ are all mutually perpendicular.

In the top left figure, if the body is a disk of radius *a* and mass *m*

$$I = \tfrac{1}{2}Ma^2 \quad \therefore 2rF = \tfrac{1}{2}Ma^2\frac{d\omega}{dt} \quad \therefore \boxed{\omega = \frac{4rF}{Ma^2}t + \omega_0}$$

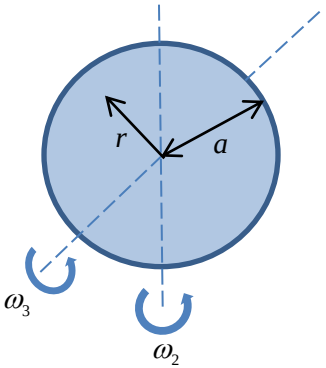
1 rpm = $\pi/30$ = 0.105 rads^{-1}

Thin rod, length l	$I_1 = I_2 = \frac{ml^2}{12}$ $I_3 \simeq 0$	
Solid sphere, radius r	$I_1 = I_2 = I_3 = \frac{2}{5}mr^2$	
Spherical shell, radius r	$I_1 = I_2 = I_3 = \frac{2}{3}mr^2$	
Solid cylinder, radius r, length l	$I_1 = I_2 = \frac{m}{4}\left(r^2 + \frac{l^2}{3}\right)$ $I_3 = \frac{1}{2}mr^2$	
Solid cuboid, sides a,b,c	$I_1 = m(b^2+c^2)/12$ $I_2 = m(c^2+a^2)/12$ $I_3 = m(a^2+b^2)/12$	
Solid circular cone, base radius r, height h^b	$I_1 = I_2 = \frac{3}{20}m\left(r^2 + \frac{h^2}{4}\right)$ $I_3 = \frac{3}{10}mr^2$	
Solid ellipsoid, semi-axes a,b,c	$I_1 = m(b^2+c^2)/5$ $I_2 = m(c^2+a^2)/5$ $I_3 = m(a^2+b^2)/5$	
Elliptical lamina, semi-axes a,b	$I_1 = mb^2/4$ $I_2 = ma^2/4$ $I_3 = m(a^2+b^2)/4$	
Disk, radius r	$I_1 = I_2 = mr^2/4$ $I_3 = mr^2/2$	
Triangular plate[c]	$I_3 = \frac{m}{36}(a^2+b^2+c^2)$	

Moments of Inertia about principal axes for various basic shapes
From Woan, *The Cambridge Handbook of Physics Formulas.*

Note the convention I_1, I_2, I_3 has the same meaning as I_x, I_y, I_z

Example: Calculate the moment of inertia of a disc about its symmetry axes



Out of the plane

$$\rho = \frac{M}{\pi a^2}$$

$$I_3 = \int r^2 dm$$

dm is a hoop of width *dr* and radius *r*, or density ρ

$$I_3 = \int_{r=0}^{a} r^2 \times \rho \times 2\pi r dr$$

$$I_3 = \frac{2M}{a^2}\int_{r=0}^{a} r^3 dr$$

$$I_3 = \frac{2M}{4a^2}a^4$$

$$\boxed{I_3 = \tfrac{1}{2}Ma^2}$$

Perpendicular axis theorem
This works for *laminae only*
i.e. with no 'z' extent

$$I_1 = \int\left(y^2 + z^2\right)dm = \int y^2 dm$$

$$I_2 = \int\left(x^2 + z^2\right)dm = \int x^2 dm$$

Lateral or longitudinal flip

$$I_3 = \int\left(x^2 + y^2\right)dm = \int x^2 dm + \int y^2 dm$$

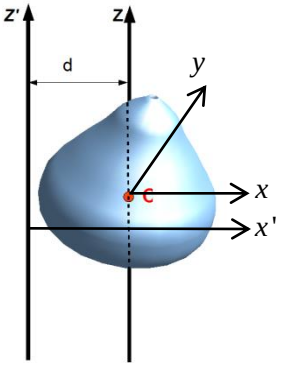
$$\therefore I_3 = I_2 + I_1$$

$$I_1 = I_2$$

Since a disc and symmetric in x,y plane

$$\therefore I_2 = \tfrac{1}{2}I_3$$

$$\boxed{I_2 = \tfrac{1}{4}Ma^2}$$



Parallel axis theorem

$$I_z = \int\left(x^2 + y^2\right)dm$$

$$I_{z'} = \int\left(x'^2 + y^2\right)dm = \int\left(\left(x+d\right)^2 + y^2\right)dm$$

$$I_{z'} = \int\left(x^2 + y^2\right)dm + 2d\int x dm + d^2\int dm$$

$$\int x dm = 0$$

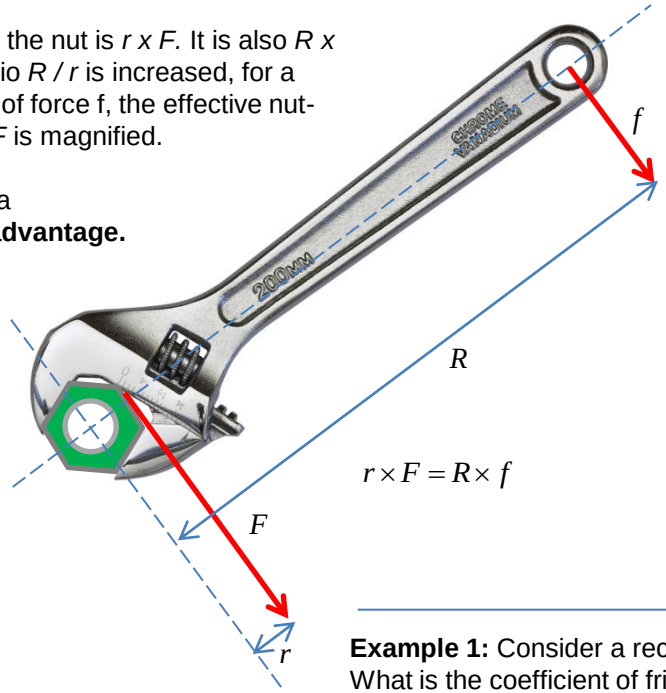
Since C is the centre of mass

$$\boxed{I_{z'} = I_z + Md^2}$$

A tool like a screwdriver or a wrench can deliver the required turning moment to tighten a screw or nut with less force.

The torque on the nut is *r x F.* It is also *R x f.* So if the ratio *R / r* is increased, for a given amount of force f, the effective nut-turning force *F* is magnified.

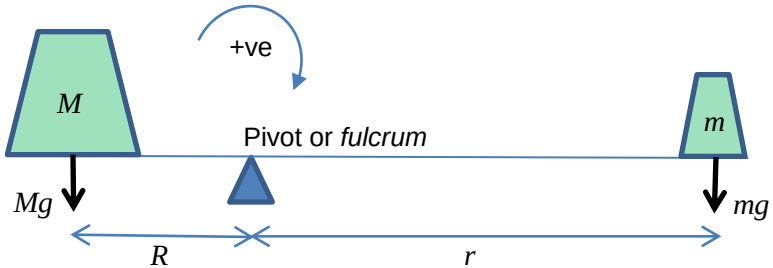
This is called a **mechanical advantage.**



$$r \times F = R \times f$$

If a body is not rotating, or rotating with a constant angular velocity, then the sum of moments must equate to zero. This is very useful in calculating forces in equilibrium problems involving rigid bodies, since it doesn't matter in this case which point 'we take moments about.' (They must always sum to zero).

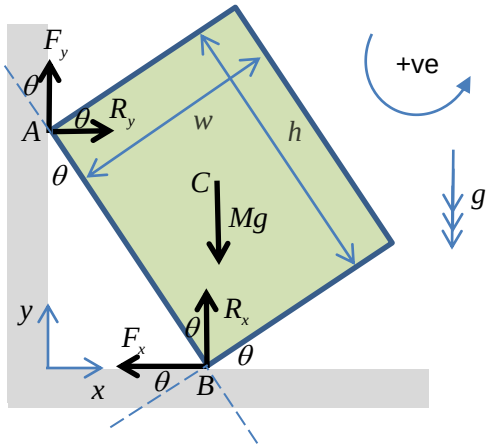
Only when there is net torque is it a good idea to choose the centre of mass of an object, and use symmetry axis to simplify the moment of inertia tensor.



Using the convention that clockwise moments are positive (i.e. a torque vector into the page is the positive direction), the above pivot system is in rotational equilibrium if

$$-MgR + mgr = 0 \quad \therefore \quad mr = MR$$

Example 1: Consider a rectangular block on the verge of sliding in the *x* and –ve *y* directions when it is inclined at angle θ. What is the coefficient of friction μ between the block and the floor? Assume wall is smooth and floor is rough.



$F_y = 0$ Smooth wall

$F_x = \mu R_x$ About to slide on rough floor

$0 = R_y - F_x$ Newton's Second Law in x direction

$0 = R_x - Mg$ Newton's Second Law in y direction

$$-R_x \cos\theta \times \tfrac{1}{2} w + R_x \sin\theta \times \tfrac{1}{2} h + \ldots$$
$$-F_x \cos\theta \times \tfrac{1}{2} h + F_x \sin\theta \times \tfrac{1}{2} w + \ldots$$
$$-R_y \cos\theta \times \tfrac{1}{2} h + R_y \sin\theta \times \tfrac{1}{2} w$$
$$= 0$$

Moments about centre of mass C

$$Mg\left(-w\cos\theta + h\sin\theta\right) +$$
$$+\mu Mg\left(-h\cos\theta + w\sin\theta\right) + \ldots$$
$$+\mu Mg\left(-h\cos\theta + w\sin\theta\right) = 0$$

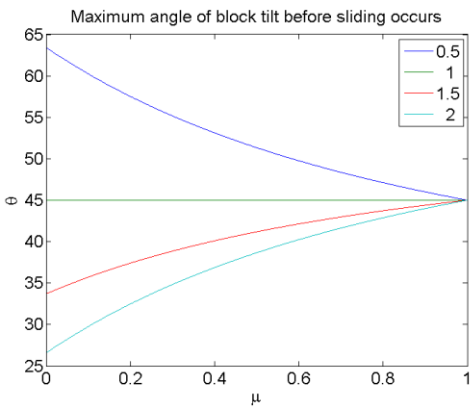
Substituting into the moments equation

$$-w\cos\theta + h\sin\theta + \ldots$$
$$+\mu\left(-h\cos\theta + w\sin\theta\right) = 0$$

$$\mu = \frac{\cos\theta - \frac{h}{w}\sin\theta}{\sin\theta - \frac{h}{w}\cos\theta} = \frac{1 - \frac{h}{w}\tan\theta}{\tan\theta - \frac{h}{w}}$$

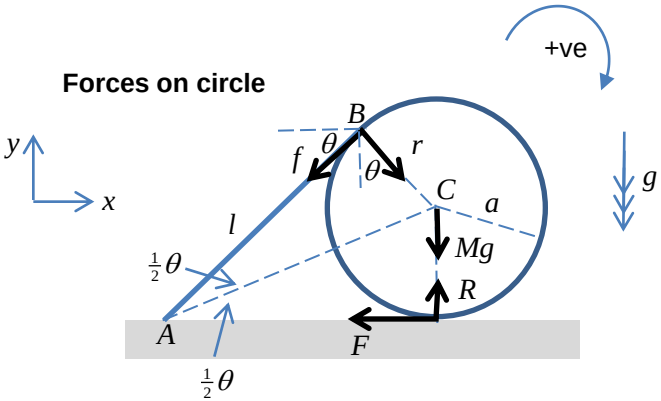
$$\therefore \left(\mu + \tfrac{h}{w}\right)\tan\theta = 1 + \tfrac{h}{w}\mu$$

$$\boxed{\tan\theta = \frac{1 + \frac{h}{w}\mu}{\mu + \frac{h}{w}}}$$



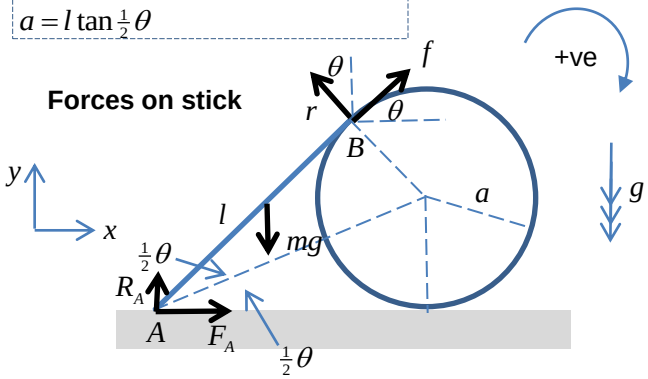
Example 2: Stick on a circle problem (Morin, Introduction to Classical Mechanics pp38)
What are the coefficients of friction between (i) the stick and a circle and (ii) between the circle and the floor necessary to maintain equilibrium?

Forces on circle



Since stick is on a circle tangent
$a = l\tan\frac{1}{2}\theta$

Forces on stick



Stick: Moments about A

$$0 = \tfrac{1}{2} l\cos\theta \times mg - rl \qquad \therefore mg = \frac{2r}{\cos\theta}$$

$$R_A = \frac{2r}{\cos\theta} - \frac{r\sin^2\theta}{1+\cos\theta} - r\cos\theta$$

$$R_A = r\left\{\frac{2}{\cos\theta} - \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{1+\cos\theta}\right\}$$

$$R_A = r\left\{\frac{2}{\cos\theta} - 1\right\}$$

$$R_A = \frac{r(2-\cos\theta)}{\cos\theta}$$

Circle: Newton II

$$x:\ 0 = -F + r\sin\theta - f\cos\theta$$

$$y:\ 0 = R - Mg - r\cos\theta - f\sin\theta$$

Circle: Moments about centre C

$$f = F \qquad \therefore f = \frac{r\sin\theta}{1+\cos\theta} = r\tan\tfrac{1}{2}\theta$$

$f \le \mu_B r$ For no sliding at point B

$$\therefore \mu_B \ge \frac{f}{r}$$

$$\boxed{\mu_B \ge \frac{\sin\theta}{1+\cos\theta} = \tan\tfrac{1}{2}\theta}$$

Stick: Newton II

$$x:\ 0 = F_A + f\cos\theta - r\sin\theta$$

$$y:\ 0 = R_A - mg + f\sin\theta + r\cos\theta$$

$$\therefore R_A = mg - \frac{r\sin^2\theta}{1+\cos\theta} - r\cos\theta$$

$$F_A = r\sin\theta - \frac{r\sin\theta\cos\theta}{1+\cos\theta}$$

$$F_A = r\sin\theta\left(1 - \frac{\cos\theta}{1+\cos\theta}\right)$$

$$\therefore F_A = \frac{r\sin\theta}{1+\cos\theta} = f$$

$F_A \le \mu_A R_A$ For no sliding at point A

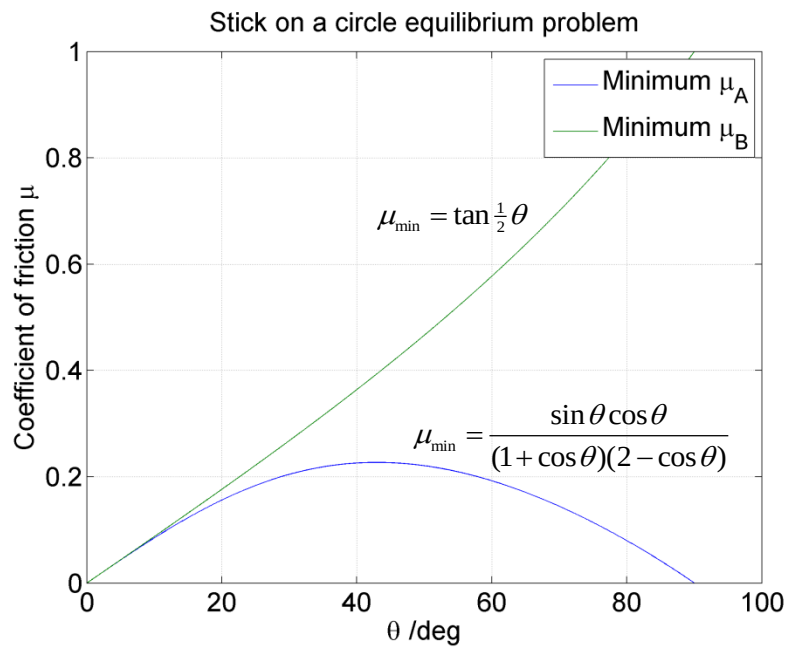
$$\therefore \mu_A \ge \frac{F_A}{R_A}$$

$$\boxed{\mu_A \ge \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}}$$

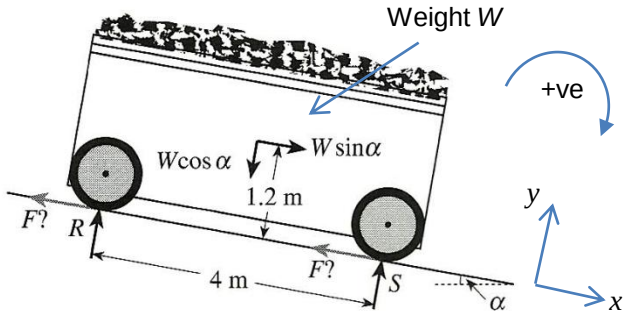
Note if there is no sliding at the stick and floor interface (A) *and* at the stick and circle interface (B), this is a *sufficient condition* for the system to be in equilibrium.

An additional no-slip condition at the floor-cylinder junction does not need to be obeyed, so therefore cannot be generally true.

i.e. $F \le \mu_D R$ may *not* be generally true



Example 4: Quarry truck problem
(Quadling, Mechanics 2, pp96)



A quarry truck has wheels 4 metres apart. When fully loaded with stone, the centre of mass is 1.2 metres from the tracks and midway between the wheels. A brake can be used to lock one of the pairs of wheels when the truck is on a slope. Is it better for the brake to be on the lower or the upper wheels? If the coefficient of friction between the wheels and the rails is 0.4, what is the steepest slope on which the truck can stand when fully loaded?

Moments about lower wheel: $4R + 1.2W\sin\alpha - 2W\cos\alpha = 0$

$$\therefore \boxed{R = W\left(0.5\cos\alpha - 0.3\sin\alpha\right)}$$

Moments about upper wheel: $-4S + 1.2W\sin\alpha + 2W\cos\alpha = 0$

$$\therefore \boxed{S = W\left(0.5\cos\alpha + 0.3\sin\alpha\right)}$$

Therefore $S > R$ for any α chosen. Since break force F is proportional to S or R, get a larger F if you break with the **lower wheels.**

For no sliding down the slope $F < 0.4S$

Newton II in x direction: $0 = W\sin\alpha - F \quad \therefore F = W\sin\alpha$

$$F < 0.4S$$
$$W\sin\alpha < 0.4W\left(0.5\cos\alpha + 0.3\sin\alpha\right)$$
$$\sin\alpha < 0.2\cos\alpha + 0.12\sin\alpha$$
$$0.88\tan\alpha < 0.2$$
$$\alpha < \tan^{-1}\tfrac{5}{22}$$
$$\boxed{\alpha < 12.8^\circ}$$