Integration by substitution

In order to evaluate many integrals, we may need to change the variable using a suitable **substitution**. A substitution *may* be sufficient to transform the integral into one you already know how to do. Note this aspect of Mathematics is not entirely prescriptive! A substitution *may not always* make the integral easier to evaluate. Sometimes *more than one* substitution may be required.

Remember **to change the limits** of a definite integral if you are using a substitution.

Integral includes	Try substitution
$(ax+b)^n$	$u=ax+b$
$\sqrt[n]{ax+b}$	$u^n=ax+b$
$a-bx^2$	$a\sin^2 u=bx^2$
$a+bx^2$	$a\tan^2 u=bx^2$
bx^2-a	$a\sec^2 u=bx^2$
e^x	$u=e^x$
$\ln(ax+b)$	$e^u=ax+b$

Example 1

$$I=\int\frac{1}{1+x^2}dx$$

$$x=\tan u \quad \therefore u=\tan^{-1}x$$

$$\frac{dx}{du}=\sec^2 u \quad \therefore\ dx=\sec^2 u\,du$$

$$1+x^2=1+\tan^2 u=\sec^2 u$$

$$I=\int\frac{1}{1+x^2}dx=\int\frac{\sec^2 u}{\sec^2 u}du$$

$$I=\int du=u+c$$

$$\boxed{I=\tan^{-1}x+c}$$

Example 2

$$I=\int\sin^6 x\cos x\,dx$$

$$u=\sin x$$ Define substitution

$$\frac{du}{dx}=\cos x \quad \therefore\ dx=\frac{du}{\cos x}$$ Differentiate substitution and find out what *dx* turns into

$$I=\int u^6\cos x\frac{du}{\cos x}$$

$$I=\int u^6 du=\tfrac{1}{7}u^7+c$$ Replace terms in integral and hope any functions in *x* cancel (leaving only functions in *u* that can then be integrated).

$$\boxed{I=\tfrac{1}{7}\sin^7 x+c}$$ Replace *u* by its function of *x* to yield the final answer.

Example 3

$$I=\int_0^1\sqrt[3]{1+2x}\,dx$$

$$u=1+2x$$

$$\frac{du}{dx}=2 \quad \therefore dx=\tfrac{1}{2}du$$

$$I=\tfrac{1}{2}\int_{u=1}^{3}u^{\frac{1}{3}}du$$

$$I=\tfrac{1}{2}\left[\tfrac{3}{4}u^{\frac{4}{3}}\right]_1^3$$

$$\boxed{I=\tfrac{3}{8}\left(3^{\frac{4}{3}}-1\right)}$$

Change the limits to be those of the new variable

Example 4

$$I=\int_{x=0}^{\frac{1}{6}}\frac{1}{\sqrt{1-9x^2}}dx$$

$$3x=\sin u \quad \therefore\ u=\sin^{-1}(3x)$$

$$3\frac{dx}{du}=\cos u \quad \therefore\ dx=\tfrac{1}{3}\cos u\,du$$

$$I=\int_{u=\sin^{-1}0}^{\sin^{-1}\frac{1}{2}}\frac{1}{\sqrt{1-\sin^2 u}}\tfrac{1}{3}\cos u\,du$$

$$I=\tfrac{1}{3}\int_0^{\frac{1}{6}\pi}\frac{1}{\cos u}\cos u\,du$$

$$I=\tfrac{1}{3}\int_0^{\frac{1}{6}\pi}du$$

$$I=\tfrac{1}{3}[u]_0^{\frac{1}{6}\pi}$$

$$\boxed{I=\tfrac{1}{18}\pi}$$

Example 5

$$I=\int\frac{1}{1-\sqrt{x}}dx$$

$$u=1-\sqrt{x} \quad \therefore \sqrt{x}=1-u$$

$$\frac{du}{dx}=-\frac{1}{2\sqrt{x}} \quad \therefore dx=-2\sqrt{x}du=2(u-1)du$$

$$I=\int\frac{1}{u}2(u-1)du$$

$$I=2\int du-2\int\frac{1}{u}du$$

$$I=2\left(u-\ln|u|\right)+c$$

$$\boxed{I=2\left(1-\sqrt{x}-\ln\left|1-\sqrt{x}\right|\right)+c}$$