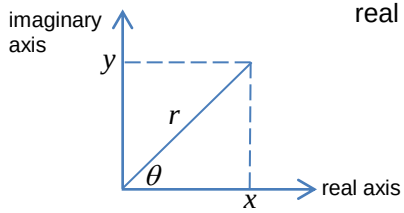
Complex numbers are a 'geometric' extension to the number system. If a real number can be represented on a horizontal number line, a complex number has an *additional vertical part*. This is known as the *imaginary axis*. A general complex number *z* can therefore be represented as an *(x,y)* coordinate on the *Complex Plane*, or *Argand Diagram.*



The fundamental difference between a complex number and a two dimensional vector is *the algebraic relationship* between the real and imaginary parts of *z*

$$\boxed{\begin{aligned} z &= x + iy \\ i^2 &= -1 \end{aligned}}$$

Note the algebra of complex numbers is the same as real numbers. They are not *non-commutative* like matrices

$z_1 \times z_2 = z_2 \times z_1$ Commutation

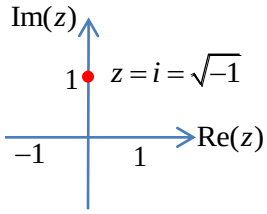
$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$ Distributive

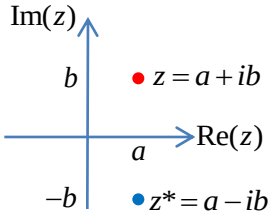
This definition enables complex numbers to represent *generalizations* of a *huge variety* of mathematical objects. The most obvious is the ability to geometrically represent the square root of minus 1.

Definitions:

$$\boxed{\begin{aligned} & z = x + iy \\ & i^2 = -1 \\ & x = \mathrm{Re}(z) \\ & y = \mathrm{Im}(z) \\ & r = \sqrt{x^2 + y^2} = |z| \\ & \theta = \tan^{-1}\tfrac{y}{x} = \arg(z) \\ & \mathrm{conj}(z) = z^* = \bar{z} = x - iy \end{aligned}}$$

Conjugate. Reverse the sign of the imaginary part.





The conjugate has useful properties

$$zz^* = (a+ib)(a-ib)$$

$$zz^* = a^2 + iab - iab - i^2b^2$$

$$zz^* = a^2 + b^2$$

$$\boxed{zz^* = |z|^2}$$

$$\boxed{\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}}$$

$$\therefore \boxed{\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}}$$

e.g. $\dfrac{1}{4-3i} = \dfrac{4+3i}{4^2+3^2} = \boxed{\tfrac{1}{25}(4+3i)}$

This means we can now represent the general solution to a quadratic equation even when the *discriminant* $b^2 - 4ac < 0$

$$az^2 + bz + c = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Assume all coefficients *a,b,c* are real numbers

$$b^2 - 4ac < 0$$

$$\Delta = -(b^2 - 4ac) \quad \therefore \Delta > 0$$

$$\therefore z = \frac{-b \pm \sqrt{\Delta}\sqrt{-1}}{2a} \Rightarrow \boxed{z = \frac{-b \pm i\sqrt{\Delta}}{2a}}$$

So if the discriminant is zero then *roots of the quadratic come in conjugate pairs*.

This is a special case of *The Fundamental Theorem of Algebra* that is, the roots of an nth order polynomial come in *n*/2 *conjugate pairs* if *n* is even, and *conjugate pairs and one real root* if *n* is odd.

Note for the quadratic, if the discriminant is greater than zero, the conjugate pair simply means a +/- pair of real roots.

Complex numbers can enable any (integer) polynomial to be written in factorized form. Consider a cubic with roots

$x = a, b+ic, b-ic$ i.e. with only one real root

$$y = (x-a)(x-b-ic)(x-b+ic) = x^3 - Ax^2 + Bx - C$$

a,b,c are defined to be real numbers i.e. $a,b,c \in \mathbb{R}$

	x	$-b$	$-ic$
x	x^2	$-bx$	$-icx$
$-b$	$-bx$	b^2	ibc
ic	icx	$-ibc$	$-i^2c^2$

$$\therefore y = (x-a)(x^2 + b^2 + c^2 - 2bx)$$

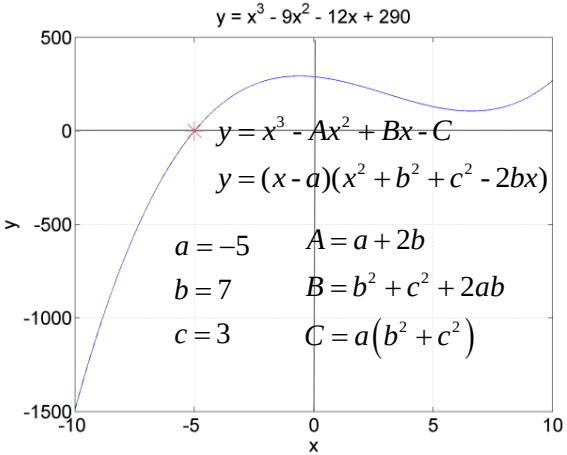
$$y = x^3 - x^2(a+2b) + x(b^2 + c^2 + 2ab) - a(b^2 + c^2)$$

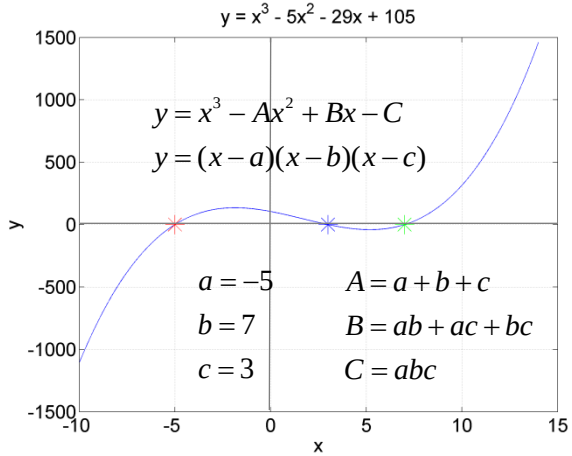
So *all coefficients A,B,C of the polynomial are real* if two of the roots are in conjugate pairs. Can we find *a,b,c* in terms of *A,B,C* ?
If so we have a *cubic formula for solving y = 0.*

$$\boxed{\begin{aligned} A &= a + 2b \\ B &= b^2 + c^2 + 2ab \\ C &= a(b^2 + c^2) \end{aligned}}$$

Unfortunately to solve these equations requires firstly solving a cubic in one of the unknowns *a,b,c*, so this *does not represent a good method for solving a cubic*. We also need to know *a priori* that the cubic only has one real root! However, if we *know a,b,c* then we can *construct* a cubic with one real root.

[Continued on next page...]





Constructing a cubic with only one real root

$$y=(x-a)(x-b-ic)(x-b+ic)$$
$$y=x^3-Ax^2+Bx-C$$

So if c = 0 we have a *two real roots, with one 'repeated'*

$$A=a+2b$$
$$B=b^2+2ab$$
$$C=ab^2$$
$$y=(x-a)(x-b)^2$$
$$y=x^3-(a+2b)x^2+(b^2+2ab)x-ab^2$$

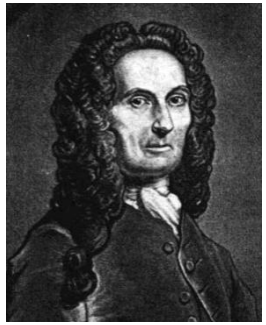
For *three real roots a,b,c* we need an alternative formulation

$$y=(x-a)(x-b)(x-c)$$
$$y=(x-a)(x^2-(b+c)x+bc)$$
$$y=x^3-x^2(a+b+c)+x(ab+ac+bc)-abc$$

i.e. $y=x^3-Ax^2+Bx-C$

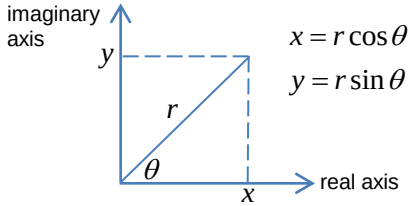
$$A=a+b+c$$
$$B=ab+ac+bc$$
$$C=abc$$

De Moivre's Theorem, powers of complex numbers, trigonometry & exponentials



Abraham de Moivre
1667-1754

Consider writing a complex number z in terms of the polar angle θ and range r in the Argand Diagram. Note the polar angle is in *radians*.



$$x=r\cos\theta$$
$$y=r\sin\theta$$

$$z=x+iy$$
$$\therefore z=r(\cos\theta+i\sin\theta)$$

Now consider the following *Maclaurin Expansions* of cosine and sine functions

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\ldots$$
$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\ldots$$

Noting these results, consider the Maclaurin Expansion of $e^{i\theta}$

$$e^{i\theta}=1+(i\theta)+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{3!}+\frac{(i\theta)^5}{3!}\ldots$$
$$e^{i\theta}=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{2!}-\ldots\right)$$
$$\therefore e^{i\theta}=\cos\theta+i\sin\theta$$

Hence we can write *any* complex number in *exponential* form. Note the *Euler Relation* when $\theta=\pi$

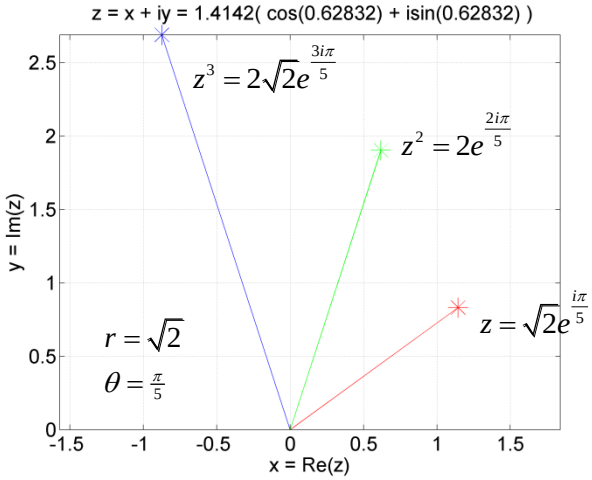
$$e^{i\pi}+1=0$$

$$z=x+iy$$
$$z=re^{i\theta}$$
$$r=|z|$$
$$\theta=\arg(z)$$

Exponential form means we can readily evaluate (and geometrically interpret) **powers** of complex numbers

$$z^n=r^ne^{in\theta}=r^n(\cos n\theta+i\sin n\theta)$$

Raising a complex number by power n therefore increases its *magnitude* |z| by power n, and *multiplies* its *polar angle* θ by n



Complex roots

Exponential form enables us to generalize the meaning of the nth roots of positive real numbers.

i.e. solutions to the equation $\boxed{z^n = k}$
where $n, k \in \mathbb{R}^+$

To fully realize this, let us note that the *periodicity* of sine and cosine functions mean we can write any complex number in the form

$$z = re^{i(\theta + 2N\pi)}$$

$r = |z|$ So r is positive

$\theta = \arg z$

$N \in \mathbb{Z}$ N is any integer

Since, if $N \in \mathbb{Z}$

$$\cos(\theta + 2\pi N) = \cos\theta$$

$$\sin(\theta + 2\pi N) = \sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore \boxed{e^{i(i\theta + 2\pi N)} = e^{i\theta}}$$

Hence:

$$\boxed{\sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{r}e^{i\left(\frac{\theta + 2\pi N}{n}\right)}}$$

The ***n*th root of unity** is a special case of this multiplicity of roots. i.e. solutions to the equation

$$\boxed{z^n = 1}$$

$$z = re^{i\theta}$$

$$z^n = 1$$

$$1 = e^{2\pi i N} \quad ; \quad N \in \mathbb{Z}$$

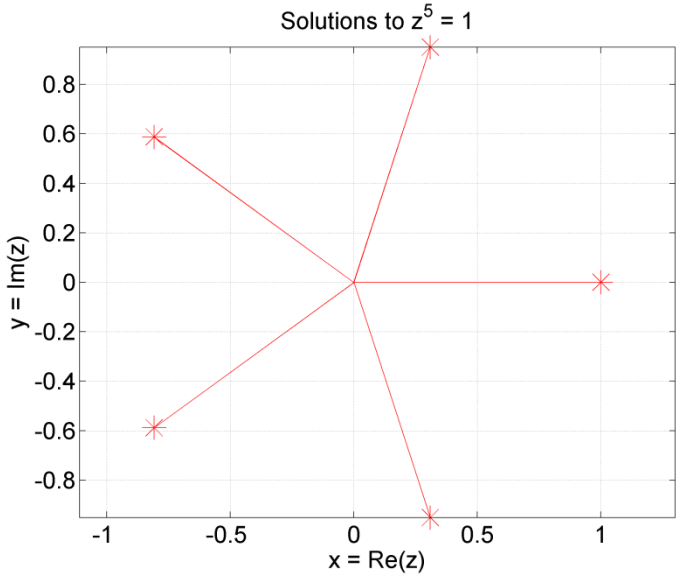
$$\therefore re^{i\theta} = e^{\frac{2\pi i N}{n}}$$

$$\boxed{\begin{aligned} &\therefore r = 1 \\ &\therefore \theta = 2\pi \frac{N}{n} \end{aligned}}$$

If n is an *integer* the **roots of unity** are:

$$\boxed{\begin{aligned} &z = \omega, \omega^2, \omega^3, \ldots, \omega^n \\ &\omega = e^{\frac{2\pi i}{n}} \end{aligned}}$$

Since powers of a complex number give rise to a *multiplier* of the argument (or polar angle), the integer n roots of unity create the vertices of a regular *n-gon* in the Argand Diagram



Logarithms of complex numbers are *multi-valued* for similar reasons

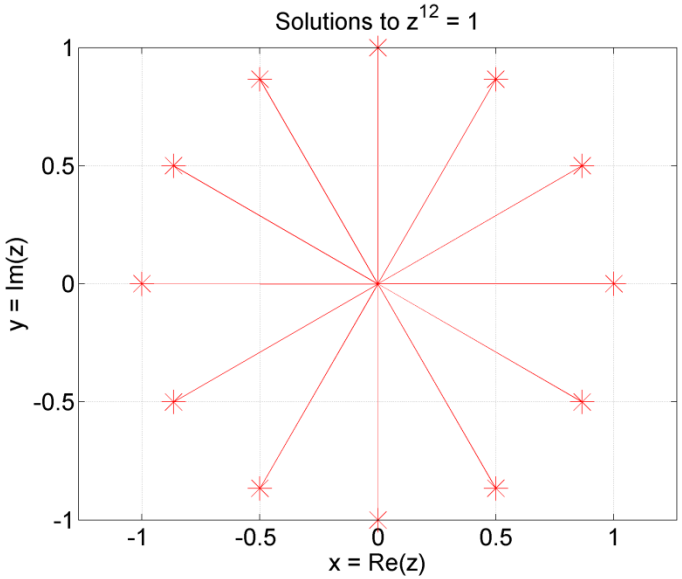
$$\boxed{\begin{aligned} &z = re^{i(\theta + 2N\pi)} \\ &\log z = \log r + i(\theta + 2N\pi) \end{aligned}}$$

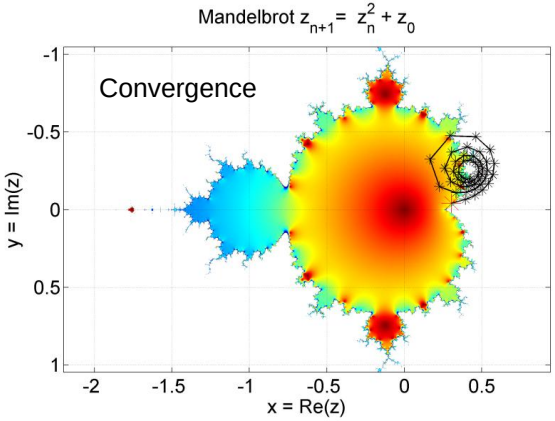
Example:

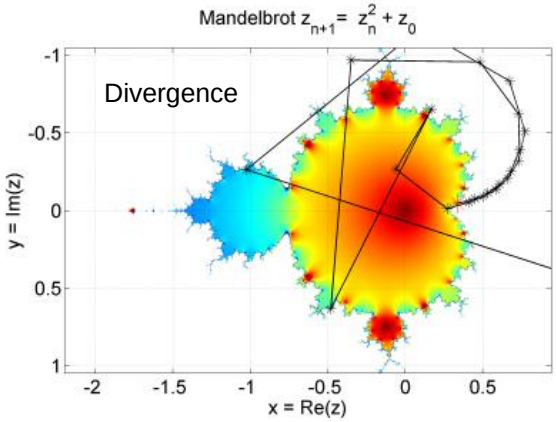
$$\log(2 + 3i) = \log\left(\sqrt{2^2 + 3^2}\, e^{i\left(\tan^{-1}\frac{3}{2} + 2\pi N\right)}\right)$$

$$\log(2 + 3i) = \log 13 + i\left(\tan^{-1}\tfrac{3}{2} + 2\pi N\right)$$

$$N \in \mathbb{Z}$$







The Mandlebrot Iteration

Let z_0 describe a particular point on the Argand Diagram. Let it be subject to the following iteration:

$$\boxed{z_{n+1} = z_n^2 + z_0}$$

Amazingly, this simple formula generates *unbelievable* complexity!

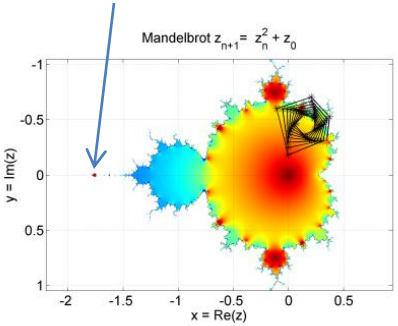
Benoit Mandelbrot
1924-2010

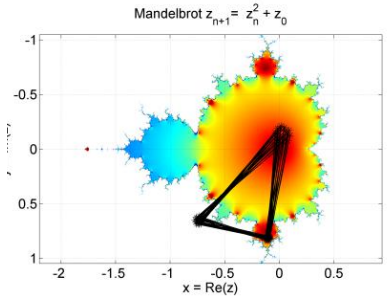
The colourful shape is produced by plotting a surface of height *h* based upon the function:

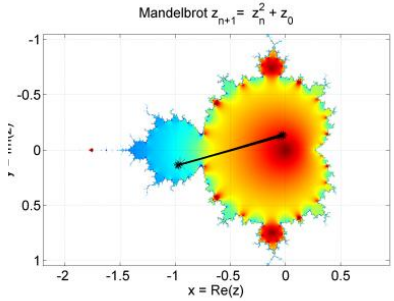
$$h = e^{-|z_{20}(x,y)|}$$
$$z_0 = x + iy$$
$$z_{n+1}(x,y) = z_n^2(x,y) + z_0(x,y)$$

The blue to red colours are based on the colour scale [0,1], with red corresponding to unity. The function has a very complicated boundary, beyond which the height drops away to essentially zero. (The white colour is used when $h < 0.001$).

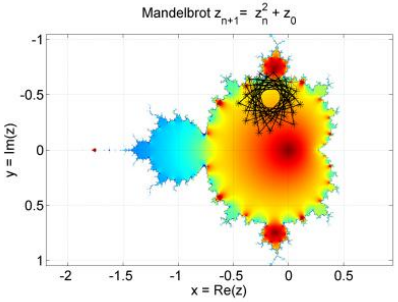
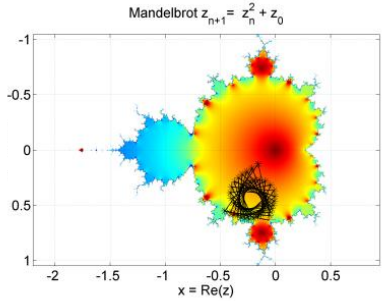
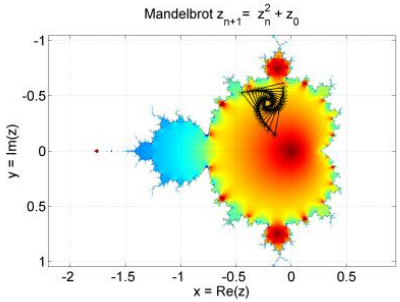
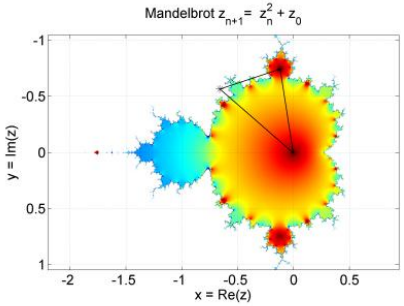
If you zoom in then *more iterations* are needed to reveal the extra structure







In all the images, a sequence of the *z* values corresponding to the first 50 terms of the Mandelbrot Iteration is overlaid as a series of black stars connected by black lines. Each plot corresponds to a different z_0.

If z_0 is outside the coloured region, the iterations diverge. Inside the coloured region ("the **Mandelbrot Set**") the iterations repeat in a pattern, always returning to the coloured region.

Amazingly, if one zooms in on the boundary region (which gives rise to the most complicated of patterns) *infinitely* complicated patterns are revealed, including exact miniature replicas of the original pattern! This *self similar* structure is known as a *fractal*.