Projectiles are typically modelled as point masses (i.e. 'particles') falling under gravity. In other words, internal motion and rotation is ignored and only the centre of mass of the projectile is considered. Air resistance is often ignored to enable analysis to proceed without a computer. Note this assumption may be significantly invalid for many real projectiles! Hence this system reduces to a two dimensional kinematics problem, where acceleration is constant.

Let the coordinates of the projectile be *(x,y)* on a Cartesian grid. Let the initial velocity be *u* at an elevation of θ and let the projectile be launched from (x_0,y_0)
Since acceleration is constant:

$$\boxed{\begin{aligned} v_x &= u\cos\theta \\ v_y &= u\sin\theta - gt \\ v_y^2 &= u^2\sin^2\theta - 2g(y-y_0) \\ x &= x_0 + ut\cos\theta \\ y &= y_0 + ut\sin\theta - \tfrac{1}{2}gt^2 \end{aligned}}$$

Note this means the *x* direction velocity is always constant throughout the motion!

We can therefore combine these equations to find various properties of the projectile's trajectory

$$x = x_0 + ut\cos\theta$$

$$\therefore \boxed{t = \frac{x - x_0}{u\cos\theta}}$$

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$\therefore \boxed{y = y_0 + (x - x_0)\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)(x - x_0)^2}$$

i.e. a projectile trajectory is an *inverted parabola*

If the projectile is required to pass through (or collide with!) a particular coordinate *(X,Y)*, we can solve the quadratic trajectory equation to determine the elevation angle, given speed *u* is known. This calculation relates to models of all ball sports, gunnery (ballistics) etc.

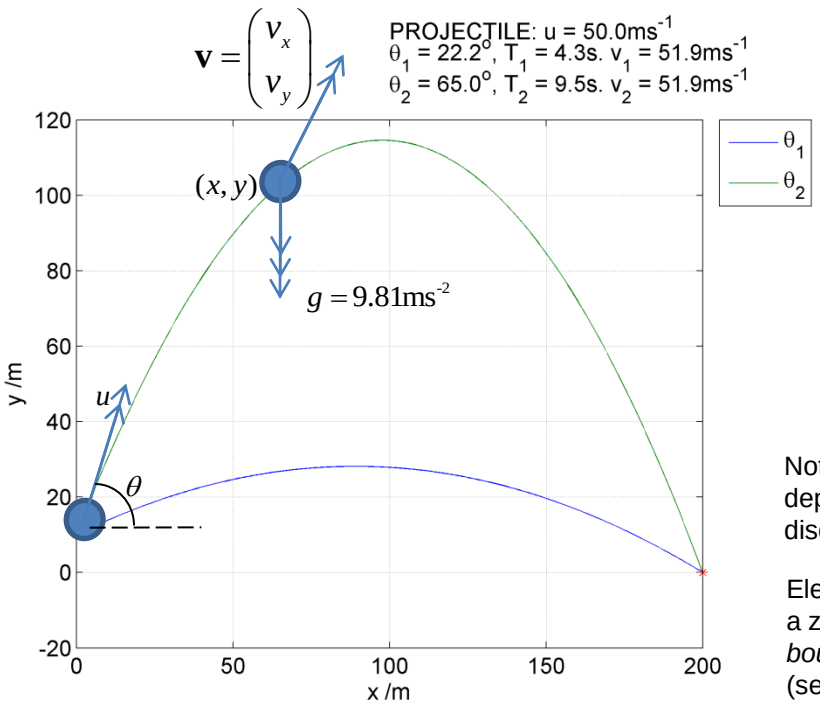
$$Y = y_0 + (X - x_0)\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)(X - x_0)^2$$

$$a\tan^2\theta + b\tan\theta + c = 0$$

$$\boxed{\begin{aligned} a &= -\frac{g}{2u^2}(X - x_0)^2 \\ b &= X - x_0 \\ c &= y_0 - Y - \frac{g}{2u^2}(X - x_0)^2 \\ \theta &= \tan^{-1}\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) \end{aligned}}$$

Note *multiple solutions* are possible, depending on the sign of the discriminant $b^2 - 4ac$

Elevation angles which give rise to a zero discriminant define the *bounding parabola* for the projectile (see next page).



The *apogee* of the trajectory is when v_y=0

$$v_y = u\sin\theta - gt \quad \therefore \quad v_y = 0 \Rightarrow \boxed{t_a = \frac{u\sin\theta}{g}}$$

$$v_y^2 = u^2\sin^2\theta - 2g(y - y_0) \quad \therefore \quad v_y = 0 \Rightarrow \boxed{y_a = y_0 + \frac{u^2\sin^2\theta}{2g}}$$

$$x_a = x_0 + ut_a\cos\theta \quad \therefore \quad \boxed{x_a = x_0 + \frac{u^2\sin\theta\cos\theta}{g}}$$

The speed *v* of the projectile is

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta - 2g(y - y_0)}$$

$$\boxed{v = \sqrt{u^2 - 2g(y - y_0)}}$$

Compute angle of velocity using

$$\boxed{\phi = \tan^{-1}\left(\frac{v_y}{v_x}\right)}$$

Possible values for *u* and the bounding parabola

$$\boxed{Y = y_0 + (X - x_0)\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)(X - x_0)^2}$$

Trajectory equation

$$a\tan^2\theta + b\tan\theta + c = 0$$

$$a = -\frac{g}{2u^2}(X - x_0)^2$$

$$b = X - x_0$$

$$c = y_0 - Y - \frac{g}{2u^2}(X - x_0)^2$$

$$\boxed{\theta = \tan^{-1}\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)}$$

For real values of θ $\quad b^2 - 4ac \geq 0$

Without loss of generality, set a coordinate system such that

i.e. very the target coordinates X,Y instead

$$x_0 = 0, \quad y_0 = 0$$

$$X^2 - 4\left(-\frac{gX^2}{2u^2}\right)\left(-Y - \frac{g}{2u^2}X^2\right) \geq 0$$

$$2u^4X^2 - 2gX^2\left(2Yu^2 + gX^2\right) \geq 0$$

$$u^4 - 2Ygu^2 - g^2X^2 \geq 0$$

$$\left(u^2 - Yg\right)^2 - Y^2g^2 - g^2X^2 \geq 0$$

$$u^2 \geq Yg + g\sqrt{X^2 + Y^2}$$

$$u^2 \leq Yg - g\sqrt{X^2 + Y^2}$$

Non physical, since u is real and positive

$$\therefore \boxed{u \geq \sqrt{g}\sqrt{Y + \sqrt{X^2 + Y^2}}}$$

The **minimum** u **parabola** is defined by the trajectory corresponding to the minimum velocity required to generate a projectile trajectory which intersects with (X,Y).

$$\boxed{u^2 = g\left(Y + \sqrt{X^2 + Y^2}\right)}$$

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$a\tan^2\theta + b\tan\theta + c = 0$$

$$a = -\frac{g}{2u^2}X^2, \quad b = X, \quad c = -Y - \frac{g}{2u^2}X^2$$

$$b^2 - 4ac = 0$$

$$\therefore \theta = \tan^{-1}\left(\frac{-b}{2a}\right)$$

$$\theta = \tan^{-1}\left(\frac{X}{\frac{g}{u^2}X^2}\right)$$

$$\theta = \tan^{-1}\left(\frac{u^2}{gX}\right)$$

$$\boxed{\theta = \tan^{-1}\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right)}$$

minimum u **parabola elevation angle**

$$\tan\theta = \frac{Y + \sqrt{X^2 + Y^2}}{X}$$

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$y = x\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right) - \frac{g}{2g\left(Y + \sqrt{X^2 + Y^2}\right)}\left(1 + \frac{\left(Y + \sqrt{X^2 + Y^2}\right)^2}{X^2}\right)x^2$$

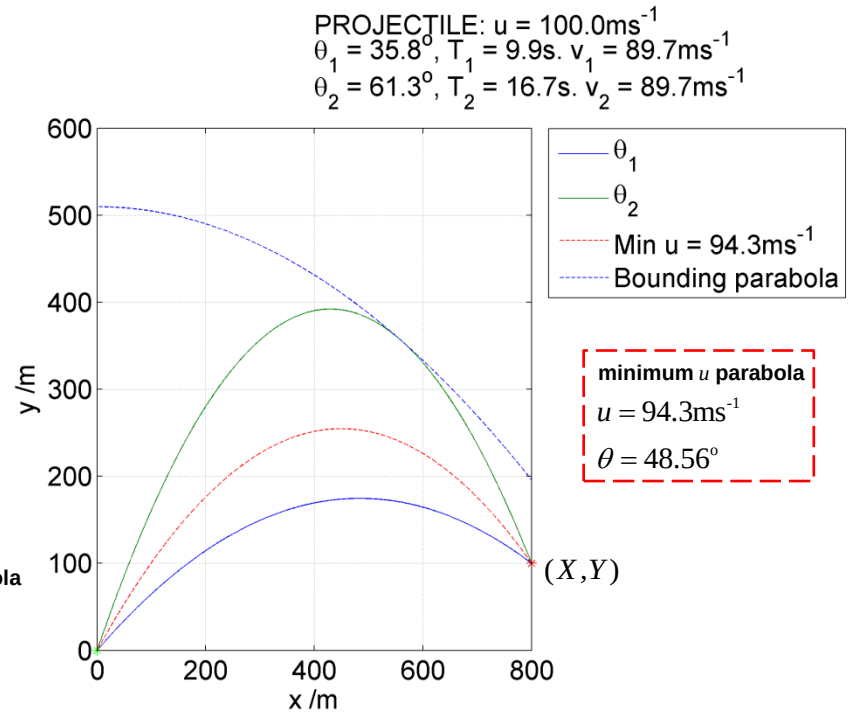
$$y = x\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right) - \frac{g}{2g\left(Y + \sqrt{X^2 + Y^2}\right)}\left(\frac{X^2 + Y^2 + 2Y\sqrt{X^2 + Y^2} + X^2 + Y^2}{X^2}\right)x^2$$

$$y = x\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right) - \frac{1}{Y + \sqrt{X^2 + Y^2}}\left(\frac{X^2 + Y^2 + Y\sqrt{X^2 + Y^2}}{X^2}\right)x^2$$

$$y = x\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right) - \frac{\sqrt{X^2 + Y^2}}{Y + \sqrt{X^2 + Y^2}}\left(\frac{\sqrt{X^2 + Y^2} + Y}{X^2}\right)x^2$$

$$\boxed{y = x\left(\frac{Y + \sqrt{X^2 + Y^2}}{X}\right) - \frac{\sqrt{X^2 + Y^2}}{X^2}x^2}$$

minimum u **parabola. Only *one* value of** θ **is possible, since the trajectory equation discriminant is zero.**



minimum u **parabola**

$u = 94.3\text{ms}^{-1}$

$\theta = 48.56^\circ$

The **bounding parabola** is slightly different – this bounds the possible set of trajectories *given* a value of u

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$2u^2y = 2u^2x\tan\theta - gx^2 - gx^2\tan^2\theta$$

$$gx^2\tan^2\theta - 2u^2x\tan\theta + 2u^2y + gx^2 = 0$$

For positive discriminant

$$4u^4x^2 - 4gx^2\left(2u^2y + gx^2\right) \geq 0$$

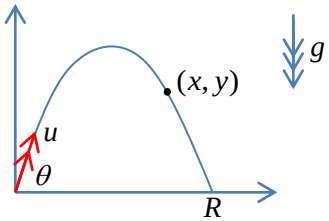
$$\frac{u^4}{g} \geq 2u^2y + gx^2$$

$$y \leq \frac{u^2}{2g} - \frac{g}{2u^2}x^2$$

bounding parabola

$$\boxed{y = \frac{u^2}{2g} - \frac{g}{2u^2}x^2}$$

The maximum range problem



Given a fixed projectile launch speed what angle maximises range?

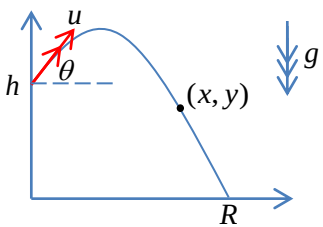
$$x = ut\cos\theta$$
$$y = ut\sin\theta - \tfrac{1}{2}gt^2$$

$$x = R, y = 0$$
$$\therefore 0 = t\left(u\sin\theta - \tfrac{1}{2}gt\right)$$
$$t > 0 \Rightarrow u\sin\theta - \tfrac{1}{2}gt = 0$$
$$\therefore t = \frac{2u\sin\theta}{g}$$
$$R = ut\cos\theta$$
$$\therefore R = \frac{2u^2}{g}\sin\theta\cos\theta$$
$$\boxed{R = \frac{u^2}{g}\sin 2\theta}$$

Hence maximum range is

$$\boxed{R_{\max} = \frac{u^2}{g}, \quad \theta = 45^\circ}$$

Let us now extend the problem to a starting height which is *not* at ground level



$$x = ut\cos\theta$$
$$y = ut\sin\theta - \tfrac{1}{2}gt^2 + h$$

$$x = R, y = 0$$
$$\therefore 0 = ut\sin\theta - \tfrac{1}{2}gt^2 + h$$
$$t^2 - \frac{2ut}{g}\sin\theta - \frac{2h}{g} = 0$$
$$\left(t - \frac{u\sin\theta}{g}\right)^2 - \frac{u^2\sin^2\theta}{g^2} - \frac{2gh}{g^2} = 0$$
$$t = \frac{u\sin\theta}{g} + \frac{u}{g}\sqrt{\sin^2\theta + \frac{2gh}{u^2}}$$

positive root since $t > 0$

$$R = ut\cos\theta$$
$$\boxed{R = \frac{u^2}{g}\left(\sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \frac{2gh}{u^2}}\right)}$$

To maximize R we need to find θ such that

$$\frac{d}{d\theta}\left(\frac{Rg}{u^2}\right) = 0$$

For brevity define $\boxed{\alpha = \frac{2gh}{u^2}}$

Note: $\alpha = \frac{2gh}{u^2} = \frac{mgh}{\frac{1}{2}mu^2} = \frac{\text{GPE}}{\text{KE}}$

$$\frac{d}{d\theta}\left(\sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}\right) = 0$$
$$\sin\theta(-\sin\theta) + \cos\theta(\cos\theta) + \frac{\frac{1}{2}\cos\theta}{\sqrt{\sin^2\theta + \alpha}}(2\sin\theta\cos\theta) - \sin\theta\sqrt{\sin^2\theta + \alpha} = 0$$
$$-\sin^2\theta + \cos^2\theta = \sin\theta\sqrt{\sin^2\theta + \alpha} - \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + \alpha}}$$
$$\sqrt{\sin^2\theta + \alpha}\left(1 - 2\sin^2\theta\right) = \sin\theta\left(\sin^2\theta + \alpha\right) - \sin\theta\left(1 - \sin^2\theta\right) \qquad \cos^2\theta = 1 - \sin^2\theta$$
$$\sqrt{\sin^2\theta + \alpha} = \frac{2\sin^3\theta + (\alpha - 1)\sin\theta}{1 - 2\sin^2\theta}$$
$$\sin^2\theta + \alpha = \frac{4\sin^6\theta + 4(\alpha - 1)\sin^4\theta + (\alpha - 1)^2\sin^2\theta}{1 - 4\sin^2\theta + 4\sin^4\theta}$$
$$\left(1 - 4\sin^2\theta + 4\sin^4\theta\right)\left(\sin^2\theta + \alpha\right) = 4\sin^6\theta + 4(\alpha - 1)\sin^4\theta + (\alpha - 1)^2\sin^2\theta$$
$$\sin^2\theta + \alpha - 4\sin^4\theta - 4\alpha\sin^2\theta + 4\sin^6\theta + 4\alpha\sin^4\theta =$$
$$4\sin^6\theta + 4\alpha\sin^4\theta - 4\sin^4\theta + (\alpha - 1)^2\sin^2\theta$$

$$\alpha + (1 - 4\alpha)\sin^2\theta = (\alpha - 1)^2\sin^2\theta$$
$$\alpha = \left(\alpha^2 - 2\alpha + 1 - 1 + 4\alpha\right)\sin^2\theta$$
$$\alpha = \left(\alpha^2 + 2\alpha\right)\sin^2\theta$$
$$\frac{1}{\alpha + 2} = \sin^2\theta$$
$$\boxed{\sin\theta = \frac{1}{\sqrt{2 + \alpha}}}$$
$$\cos\theta = \sqrt{1 - \tfrac{1}{2+\alpha}}$$
$$\boxed{\cos\theta = \sqrt{\frac{1 + \alpha}{2 + \alpha}}}$$

The range maximizing angle is therefore:

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{\sqrt{2 + \alpha}}\right)}$$

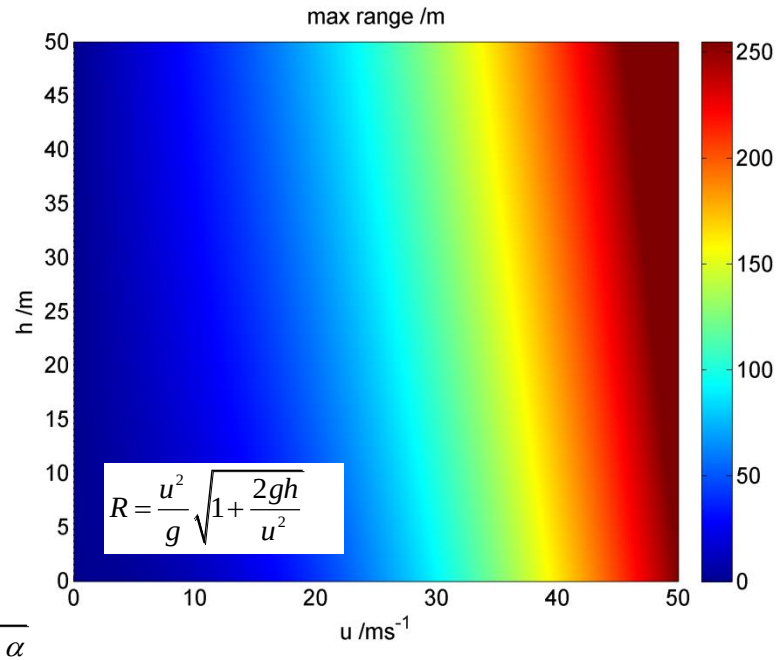
$$\frac{Rg}{u^2} = \sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}$$
$$\frac{Rg}{u^2} = \frac{1}{\sqrt{2 + \alpha}}\sqrt{\frac{1 + \alpha}{2 + \alpha}} + \sqrt{\frac{1 + \alpha}{2 + \alpha}}\sqrt{\frac{1}{2 + \alpha} + \alpha}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha} + \sqrt{\frac{1 + \alpha}{2 + \alpha}}\sqrt{\frac{1 + 2\alpha + \alpha^2}{2 + \alpha}}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha} + \frac{\sqrt{1 + \alpha}}{2 + \alpha}\sqrt{(1 + \alpha)^2}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha}(1 + 1 + \alpha) = \frac{\sqrt{1 + \alpha}}{2 + \alpha}(2 + \alpha)$$
$$\frac{Rg}{u^2} = \sqrt{1 + \alpha}$$

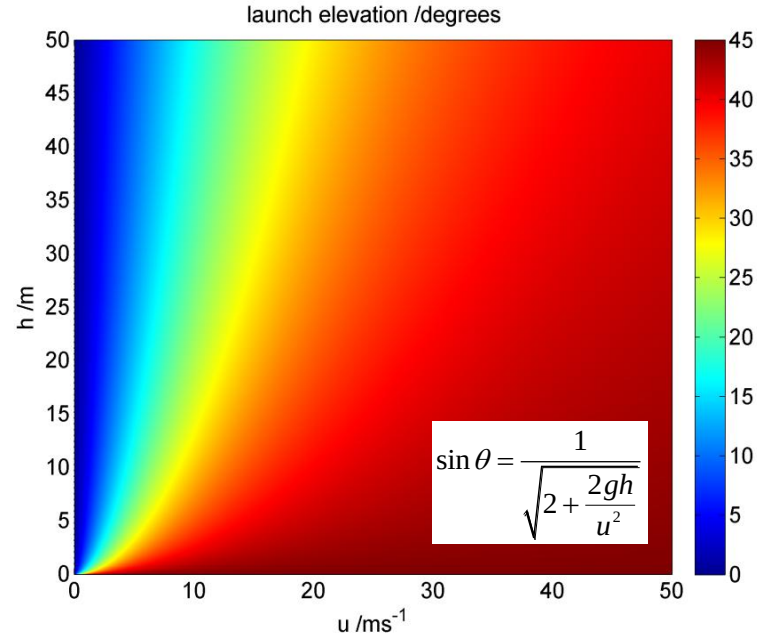
$$\boxed{R = \frac{u^2}{g}\sqrt{1 + \frac{2gh}{u^2}}, \quad \sin\theta = \frac{1}{\sqrt{2 + \frac{2gh}{u^2}}}}$$

Given the maximum range problem involves *two parameters*, to visualize possible solutions we need to plot a *surface* graph.

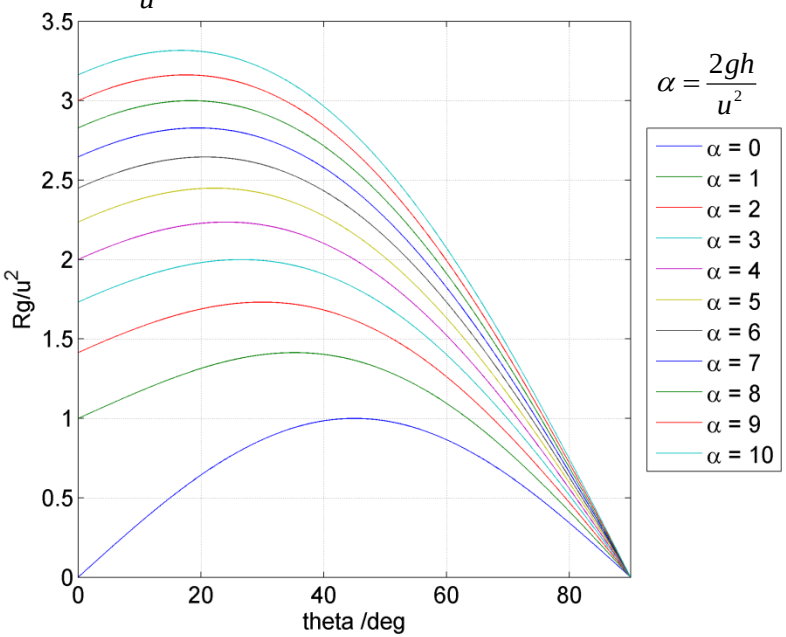
In the example plots, *colour* is used to indicate the height of the surface.

In all examples $g = 9.81\text{ms}^{-2}$





$$\frac{Rg}{u^2} = \sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}$$

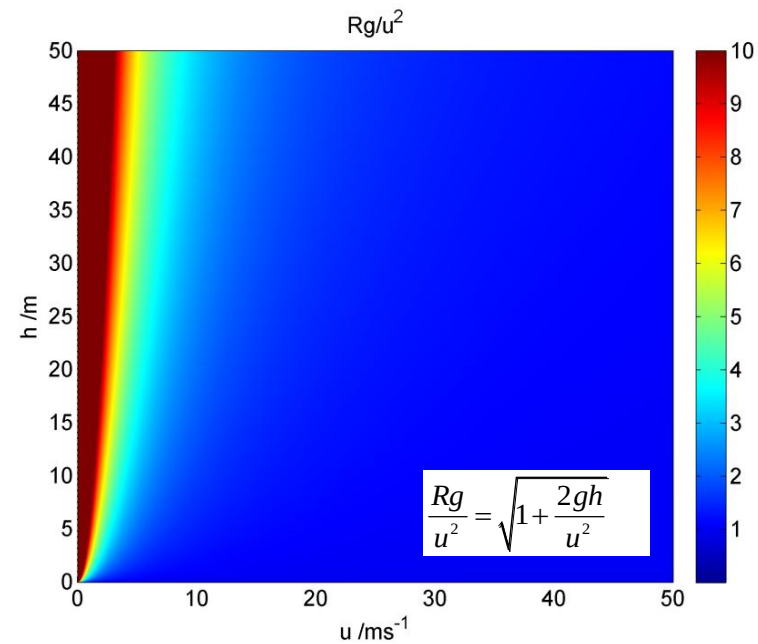


$$\alpha = \frac{2gh}{u^2}$$

This graph demonstrates that range has a maximum value as the launch elevation is varied.

The angle which results in the maximum range is given by

$$\sin\theta = \frac{1}{\sqrt{2+\alpha}}$$



Example 1: A projectile is fired from a canon.
The elevation is 45º and the muzzle velocity is 100ms^{-1} .
Assuming the projectile is fired on level ground from (0,0), calculate
(i) the apogee coordinates; (ii) the ground range; (iii) time of flight.
Assume air resistance is ignorable and set g = 9.81ms^{-2}

The relevant equations are:

$$y_a = \frac{u^2 \sin^2\theta}{2g}$$

$$x_a = \frac{u^2 \sin\theta\cos\theta}{g}$$

i.e. $x = x_{\max}, y = 0$

$$0 = x_{\max}\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x_{\max}^2$$

$$\therefore x_{\max} = \frac{2u^2\tan\theta}{g(1+\tan^2\theta)} = \frac{2u^2\sin\theta\cos\theta}{g} = \boxed{\frac{u^2\sin 2\theta}{g}}$$

$$t = \frac{x_{\max}}{u\cos\theta}$$

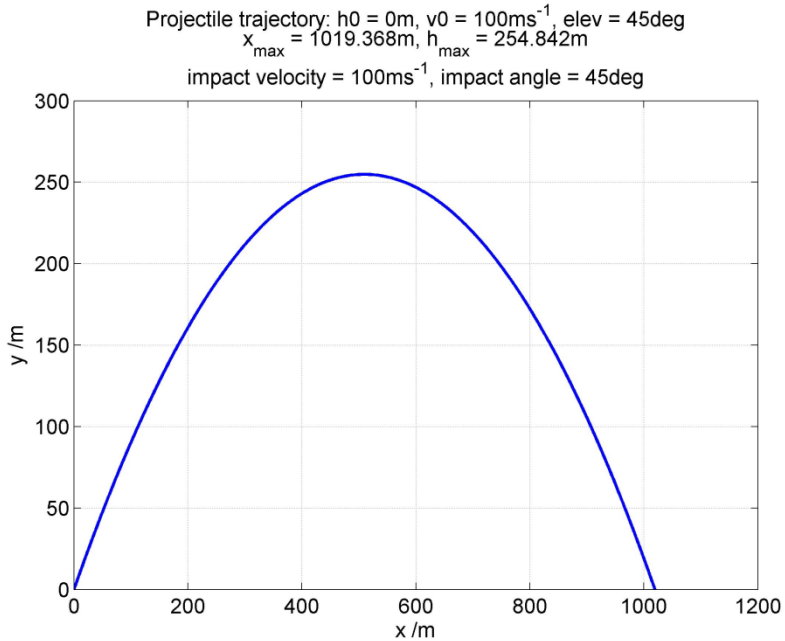
Hence:

$$y_a = \frac{100^2 \times \frac{1}{2}}{2\times 9.81} = \boxed{254.8\text{m}}$$

$$x_a = \frac{100^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}}{9.81} = \boxed{509.7\text{m}}$$

$$x_{\max} = \frac{100^2}{9.81} = \boxed{1019.4\text{m}}$$

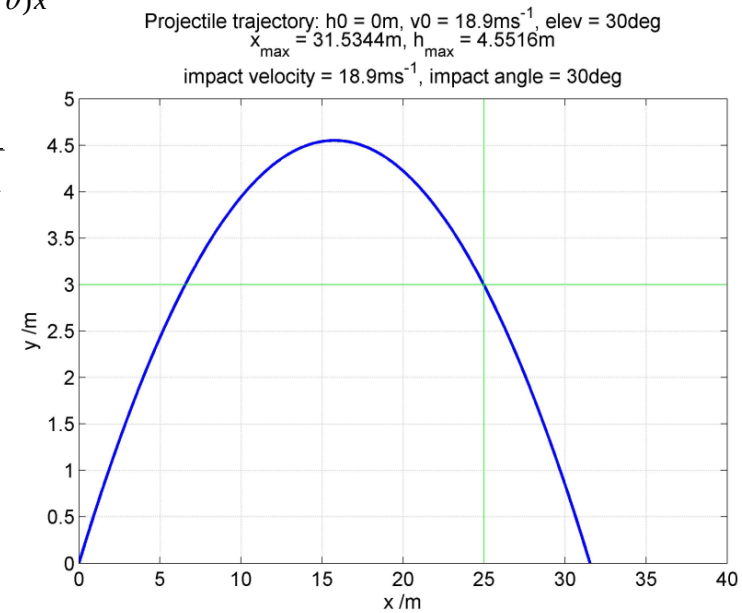
$$t = \frac{x_{\max}}{100\times\frac{1}{\sqrt{2}}} = \boxed{14.4\text{s}}$$



Example 2: In the 2003 Rugby Union World Cup final, Jonny Wilkinson scored the winning drop goal for England. If he struck the ball from ground level at an angle of thirty degrees, what velocity must he have kicked the ball, given he cleared the goals (height 3m) from a range of 25m? Ignore air resistance and set g = 9.81ms^{-2}

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$u = \sqrt{\frac{g(1+\tan^2\theta)x^2}{2(x\tan\theta - y)}}$$

$$\therefore u > \sqrt{\frac{9.81\times(1+\frac{1}{3})\times 25^2}{2\times\left(\frac{25}{\sqrt{3}}-3\right)}}$$

$$\therefore u > \boxed{18.9\text{ms}^{-1}}$$