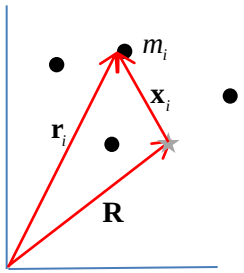
Consider a system of masses with position vectors $\mathbf{r}_i$ and velocities $\dot{\mathbf{r}}_i$



$$\boxed{M = \sum_i m_i}$$

Now write the position vectors and velocities in terms of a vector $\mathbf{R}$ and relative position vector $\mathbf{x}_i$

$$\mathbf{r}_i = \mathbf{R} + \mathbf{x}_i$$

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{x}}_i$$

The total *angular momentum* in the system is

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$$

$$\mathbf{L} = \sum_i m_i \left(\mathbf{R} + \mathbf{x}_i\right) \times \left(\dot{\mathbf{R}} + \dot{\mathbf{x}}_i\right)$$

$$\mathbf{L} = \sum_i m_i \left(\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{x}_i \times \dot{\mathbf{R}} + \mathbf{R} \times \dot{\mathbf{x}}_i + \mathbf{x}_i \times \dot{\mathbf{x}}_i\right)$$

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i + \sum_i m_i \left(\mathbf{x}_i \times \dot{\mathbf{R}}\right) + \sum_i m_i \mathbf{R} \times \dot{\mathbf{x}}_i$$

Now let us suggest a useful form for $\mathbf{R}$, that is the mass-weighted average position vector

$$\boxed{\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}} \qquad \therefore \mathbf{R} = \frac{\sum_i m_i \left(\mathbf{R} + \mathbf{x}_i\right)}{M} \Rightarrow \sum_i m_i \mathbf{x}_i = \mathbf{0}$$

For a continuous distribution of mass of density $\rho(\mathbf{r})$

$$\mathbf{R} = \tfrac{1}{M} \int \mathbf{r} \rho(\mathbf{r}) dV$$

volume element

Therefore $\sum_i m_i \left(\mathbf{x}_i \times \dot{\mathbf{R}}\right) + \sum_i m_i \mathbf{R} \times \dot{\mathbf{x}}_i$

$$= \left(\sum_i m_i \mathbf{x}_i\right) \times \dot{\mathbf{R}} + \mathbf{R} \times \frac{d}{dt}\left(\sum_i m_i \mathbf{x}_i\right)$$

since it is implied here that the masses *do not* vary with time

$$= \mathbf{0}$$

Therefore the total angular momentum is

$$\boxed{\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i}$$

Sum of angular momenta of masses about $\mathbf{R}$

Angular momentum of a single mass M at position $\mathbf{R}$

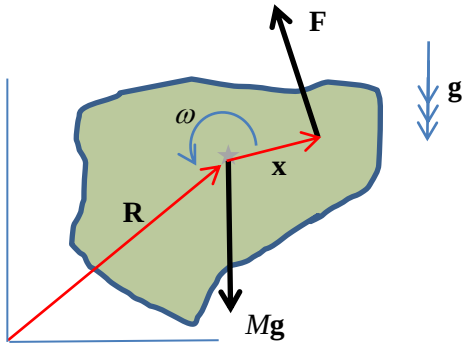
Hence: $\dot{\mathbf{L}} = M \frac{d}{dt}\left(\mathbf{R} \times \dot{\mathbf{R}}\right) + \sum_i m_i \frac{d}{dt}\left(\mathbf{x}_i \times \dot{\mathbf{x}}_i\right)$

$$\dot{\mathbf{L}} = M\left(\dot{\mathbf{R}} \times \dot{\mathbf{R}} + \mathbf{R} \times \ddot{\mathbf{R}}\right) + \sum_i m_i \left(\dot{\mathbf{x}}_i \times \dot{\mathbf{x}}_i + \mathbf{x}_i \times \ddot{\mathbf{x}}_i\right)$$

$$\boxed{\dot{\mathbf{L}} = \mathbf{R} \times M\ddot{\mathbf{R}} + \sum_i \mathbf{x}_i \times m_i \ddot{\mathbf{x}}_i}$$

i.e. **rate of change of angular momentum equals net *torque*,** since Newton's Second Law equates mass x acceleration with net force.

We can therefore decompose a system of masses **into bulk motion of the centre of mass** plus **internal motion (e.g. rotation etc) about the centre of mass**



We can analyse the dynamics of a irregular, *but rigid, mass* in terms of:

(i) **Acceleration** of the centre of mass

$$M\ddot{\mathbf{R}} = M\mathbf{g} + \mathbf{F}$$

(ii) **Rotation** about the centre of mass

$$\mathbf{x} \times \mathbf{F} = \frac{d\mathbf{L}}{dt} = \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt}$$

Torque about centre of mass

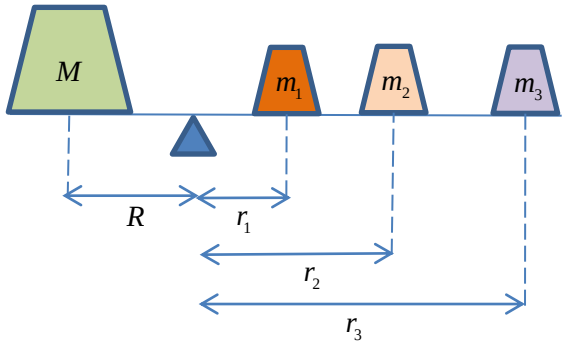
Inertia tensor x rate of change of angular velocity

Note forces passing through the centre of mass *only* effect the motion of the mass centre. They *cannot* cause any rotation of the rigid body.

In this case, the angular momentum L is relative to the centre of mass

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} = \sum_i \mathbf{x}_i \times m_i \ddot{\mathbf{x}}_i$$

A simple 1D example of centre of mass



In order for the above system to be in equilibrium, the total moments about the pivot must be zero.

$$MR = m_1 r_1 + m_2 r_2 + m_3 r_3$$

$$\therefore \boxed{R = \frac{\sum_i m_i r_i}{M}}$$

If M is the sum of all the masses, then we can think of the three-mass system *being equivalent* to a single mass M a distance R from the fulcrum.

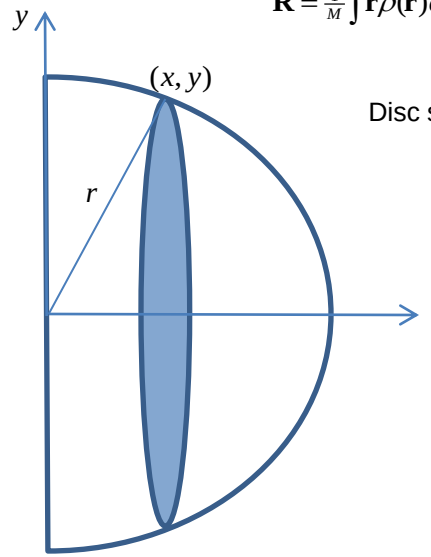
torque i.e. the 'moment' of a force $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$

Solid	Centre of mass	From
Solid hemisphere, radius *r*	$d=\frac{3r}{8}$	sphere centre
Hemispherical shell, radius *r*	$d=\frac{r}{2}$	sphere centre
Sector of disk, radius *r*, angle 2θ	$d=\frac{2}{3}r\frac{\sin\theta}{\theta}$	disk centre
Arc of circle, radius *r*, angle 2θ	$d=r\frac{\sin\theta}{\theta}$	circle centre
Arbitrary triangular lamina, height *h* (perpendicular distance between base and apex)	$d=\frac{h}{3}$	perpendicular from base
Solid cone or pyramid, height *h*	$d=\frac{h}{4}$	perpendicular from base
Solid spherical cap, height *h*, sphere radius *r*	$d=\frac{3}{4}\frac{(2r-h)^2}{3r-h}$	sphere centre
Spherical cap shell, height *h*, sphere radius *r*	$d=r-\frac{h}{2}$	sphere centre
Semi-elliptical lamina, height *h*	$d=\frac{4h}{3\pi}$	from base

Example: Finding the centre of mass of a solid hemisphere

$$\mathbf{R}=\tfrac{1}{M}\int\mathbf{r}\rho(\mathbf{r})dV$$

By symmetry, centre of mass must be along the x axis



Disc shape volume element $dV=\pi y^2dx$

$$\rho=\frac{M}{\frac{2}{3}\pi r^3}$$

$$d=\tfrac{1}{M}\int_{x=0}^{r}x\rho dV$$

$$d=\tfrac{1}{M}\frac{M\pi}{\frac{2}{3}\pi r^3}\int_0^r xy^2dx$$

$$d=\frac{1}{\frac{2}{3}r^3}\int_0^r x\left(r^2-x^2\right)dx$$

$$d=\frac{3}{2r^3}\left[\tfrac{1}{2}x^2r^2-\tfrac{1}{4}x^4\right]_0^r$$

$$d=\frac{3r^4}{2r^3}\left(\tfrac{1}{2}-\tfrac{1}{4}\right)$$

$$\boxed{d=\frac{3r}{8}}$$