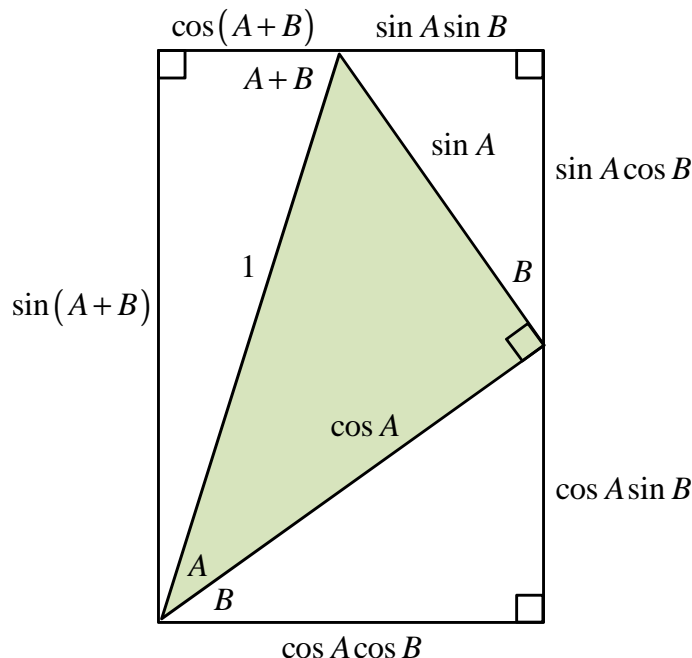


Trigonometric identities

Useful algebraic relationships can be proven between the basic trigonometric functions of sine, cosine and tangent.

Geometric proof of the addition formulae



$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(\pm A) = \cos A$$

$$\sin(\pm A) = \pm \sin A$$

odd and even relationship

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

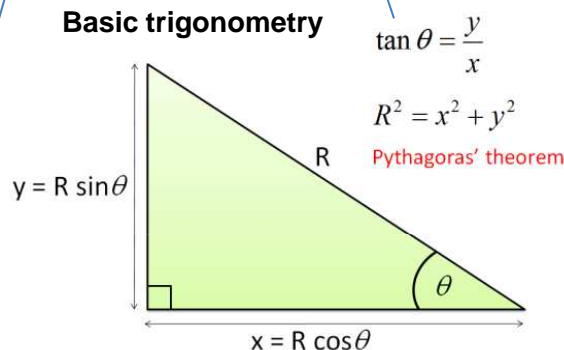
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Addition formulae for tan

$$\tan(A \pm B) = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$

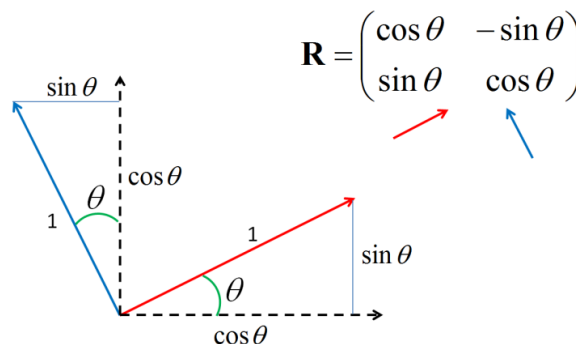
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Basic trigonometry



$$\sin^2 \theta + \cos^2 \theta = 1$$

Rotation anticlockwise about (0,0) by angle θ



$$\mathbf{R} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Use of rotation matrices to derive sine and cosine addition formulae

$$\mathbf{R}_{A+B} = \mathbf{R}_{\pm B} \mathbf{R}_A$$

$$\begin{pmatrix} \cos(A \pm B) & -\sin(A \pm B) \\ \sin(A \pm B) & \cos(A \pm B) \end{pmatrix} = \begin{pmatrix} \cos B & \mp \sin B \\ \pm \sin B & \cos B \end{pmatrix} \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}$$

\therefore

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Double angle formulae

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2 \sin^2 A$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Squares of sine and cosine

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$\cos 2A = 2 \cos^2 A - 1$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

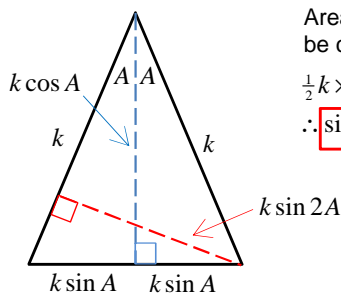
Formulae for cubes of sine and cosine

$$\sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$$

$$\cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$



Area of an isosceles triangle can be calculated in two ways:

$$\frac{1}{2} k \times k \sin 2A = 2 \times \frac{1}{2} k \sin A \times k \cos A$$

$$\therefore \sin 2A = 2 \sin A \cos A$$

$$\sin 3A = 2 \sin A \cos A \cos A + (\cos^2 A - \sin^2 A) \sin A$$

$$\sin 3A = 2 \sin A \cos^2 A + \cos^2 A \sin A - \sin^3 A$$

$$\sin 3A = 3 \sin A \cos^2 A - \sin^3 A$$

$$\sin 3A = 3 \sin A (1 - \sin^2 A) - \sin^3 A$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\sin^3 A = \frac{1}{4} (3 \sin A - \sin 3A)$$

$$\cos 3A = (\cos^2 A - \sin^2 A) \cos A - 2 \sin A \cos A \sin A$$

$$\cos 3A = (\cos^2 A - (1 - \cos^2 A)) \cos A - 2 \sin A \cos A \sin A$$

$$\cos 3A = (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$$

$$\cos 3A = 2 \cos^3 A - \cos A - 2(1 - \cos^2 A) \cos A$$

$$\cos 3A = 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\cos^3 A = \frac{1}{4} (3 \cos A + \cos 3A)$$

Addition formulae for inverses

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$A \pm B = \sin^{-1}(\sin A \sqrt{1 - \sin^2 B} \pm \sqrt{1 - \sin^2 A} \sin B)$$

$$a = \sin A$$

$$b = \sin B$$

$$\sin^{-1} a \pm \sin^{-1} b = \sin^{-1}(a \sqrt{1 - b^2} \pm b \sqrt{1 - a^2})$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$A \pm B = \cos^{-1}(\cos A \cos B \mp \sqrt{1 - \cos^2 A} \sqrt{1 - \cos^2 B})$$

$$a = \cos A$$

$$b = \cos B$$

$$\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}(ab \mp \sqrt{1 - a^2} \sqrt{1 - b^2})$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$A \pm B = \tan^{-1}\left(\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}\right)$$

$$a = \tan A$$

$$b = \tan B$$

$$\tan^{-1} a \pm \tan^{-1} b = \tan^{-1}\left(\frac{a \pm b}{1 \mp ab}\right)$$

Addition of mixed sine and cosine

$$a \sin A \pm b \cos A$$

$$R \sin(A \pm \alpha) = R \sin A \cos \alpha \pm R \cos A \sin \alpha$$

$$R \cos \alpha = a$$

$$R \sin \alpha = b$$

$$\tan \alpha = \frac{b}{a}$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = a^2 + b^2$$

$$R = \sqrt{a^2 + b^2}$$

$$a \sin A \pm b \cos A = \sqrt{a^2 + b^2} \sin\left(A \pm \tan^{-1} \frac{b}{a}\right)$$

Other useful identities provable from addition formulae

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$