Moment Generating Functions (MGF)

An elegant way to determine the *mean*, *variance* (and indeed measures such as *skew* and *kurtosis*) of a probability distribution is via a MGF $M_x(t)$, which is particular to the probability distribution of a random variable x.

Firstly define the *expectation* $E[x]$, i.e. the mean x value of a distribution

$$E[x]=\mu=\sum_x xP(x) \qquad E[x]=\mu=\int_{-\infty}^{\infty} xp(x)dx$$

Discrete e.g. *Binomial Distribution*

$$x=0.....N$$

$$P(x)=\frac{N!}{(N-x)!x!}p^x(1-p)^{N-x}$$

Continuous e.g. *Normal Distribution*

$$p(x)=\frac{\exp\left(-\frac{(x-\mu)^2}{4\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}$$

Define the MGF, where *only* x is the random variable

$$\boxed{M_x(t)=E[e^{xt}]}$$

$$M_x(t)=E[1+tx+\tfrac{1}{2!}(tx)^2+\tfrac{1}{3!}(tx)^3+..]$$

$$M_x(t)=1+tE[x]+\tfrac{1}{2!}t^2E[x^2]+....$$

Using a Maclaurin Expansion

Hence:

$$\boxed{E[x^n]=\left.\frac{\partial^n M_x(t)}{\partial t^n}\right|_{t=0}}$$

i.e. not in terms of a sum or integral

If we can compute a closed, and t differentiable, form of $M_x(t)$, then this representation makes the computation of *variance*, *skew* and *kurtosis* statistical measures *significantly* easier than more direct methods!

The *Moment Generating Function* for the Binomial Distribution is:

$$M_x(t)=E[e^{tx}]$$

$$M_x(t)=\sum_{n=0}^{N}e^{tx}\binom{N}{x}p^x(1-p)^{N-x}$$

$$M_x(t)=\sum_{n=0}^{N}\binom{N}{x}\left(pe^t\right)^x(1-p)^{N-x}$$

$$\boxed{M_x(t)=\left(pe^t+1-p\right)^N}$$

Binomial expansion

The MGF allows us to calculate the expectation and variance of the Binomial Distribution rather efficiently

$$\boxed{E[x]=\left.\frac{\partial M}{\partial t}\right|_{t=0}=Np}$$

$$\boxed{V[x]=\left.\frac{\partial^2 M}{\partial t^2}\right|_{t=0}-\left(E[x]\right)^2=Np(1-p)}$$

The method generalizes for other distributions:

- Substitute the probability distribution into the definition of $M_x(t)=E[e^{xt}]$
- Attempt to find a closed form expression (i.e. use a summation formula or evaluate an integral depending if the distribution is discrete or continuous)

Binomial

$$P(x\,|\,N,p)=\frac{N!}{(N-x)!x!}p^x(1-p)^{N-x}$$

$$M(x,t)=\left(pe^t+1-p\right)^N$$

$$E[x]=Np$$

$$V[x]=Np(1-p)$$

Geometric

$$P(x)=(1-p)^{x-1}p$$

$$M_x(t)=\frac{pe^t}{1-e^t(1-p)}$$

$$E[x]=\frac{1}{p}$$

$$V[x]=\frac{1-p}{p^2}$$

Poisson

$$P(x\,|\,\lambda)=\frac{\lambda^x}{x!}e^{-\lambda}$$

$$M_x(t)=\exp\left(\lambda\left(e^t-1\right)\right)$$

$$E[x]=\lambda$$

$$V[x]=\lambda$$

Normal

$$p(x\,|\,\mu,\sigma)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{4\sigma^2}\right)$$

$$M_x(t)=\exp\left(\mu t+\tfrac{1}{2}\sigma^2t^2\right)$$

$$E[x]=\mu$$

$$V[x]=\sigma^2$$

Variance

$$V[x]=\sigma^2=E[x^2]-\mu^2$$

"Spread about the mean"

"Asymmetry of a distribution about the mean"

Skew

$$\text{skew}[x]=\gamma=E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right]=\frac{E[x^3]-3\mu\sigma^2-\mu^3}{\sigma^3}$$

Kurtosis

"Fatness of tail" i.e. propensity for extreme values

$$\text{kurt}[x]=E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right]-3$$

$$\text{kurt}[x]=\frac{E[x^4]-4\mu E[x^3]+6\mu^2E[x^2]-4\mu^3E[x]+\mu^4}{\sigma^4}-3$$

$$\text{kurt}[x]=\frac{E[x^4]-4\mu E[x^3]+6\mu^2E[x^2]-3\mu^4}{\sigma^4}-3$$