Proof of the following standard integrals used in the *Maxwell Boltzmann* (molecular speed) and *Planck* (photon wavelength) distributions in **Statistical Physics.**

$$\boxed{\begin{aligned} I_n &= \int_0^\infty x^n e^{-ax^2}\,dx \\ I_n &= \frac{n-1}{2a} I_{n-2} \\ I_0 &= \tfrac{1}{2}\sqrt{\tfrac{\pi}{a}} \\ I_1 &= \tfrac{1}{2a} \\ I_2 &= \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}} \end{aligned}}$$

$$\boxed{\begin{aligned} I_n &= \int_0^\infty x^n e^{-ax^2}\,dx \\ I_1 &= \tfrac{1}{2a} \end{aligned}}$$

$$I = \int_0^\infty x e^{-ax^2}\,dx$$

$$u = ax^2 \therefore du = 2axdx$$

$$\therefore xdx = \tfrac{1}{2a}du$$

$$\therefore I = \tfrac{1}{2a}\int_0^\infty e^{-u}du = \tfrac{1}{2a}\left[-e^{-u}\right]_0^\infty = \tfrac{1}{2a}$$

$$\boxed{\begin{aligned} I_n &= \int_0^\infty x^n e^{-ax^2}\,dx \\ I_0 &= \tfrac{1}{2}\sqrt{\tfrac{\pi}{a}} \end{aligned}}$$

$$I = \int_0^\infty e^{-ax^2}dx = \tfrac{1}{2}\int_{-\infty}^\infty e^{-ax^2}dx$$

$$I^2 = \tfrac{1}{4}\int_{x=-\infty}^\infty e^{-ax^2}dx\int_{y=-\infty}^\infty e^{-ay^2}dy = \tfrac{1}{4}\int_{x=-\infty}^\infty\int_{y=-\infty}^\infty e^{-ax^2}e^{-ay^2}dxdy$$

$$I^2 = \tfrac{1}{4}\int_{x=0}^\infty\int_{y=0}^\infty e^{-a\left(x^2+y^2\right)}dxdy$$

$$r^2 = x^2 + y^2$$

$$\int_{x=-\infty}^\infty\int_{y=-\infty}^\infty dxdy = \int_{r=0}^\infty 2\pi r dr$$

i.e. think about integrating up rings of radius *r* and thickness *dr*. Note this is the same trick we use for computing centres of mass and moments of inertia for objects with circular symmetry.

$$\therefore I^2 = \tfrac{1}{4}\int_{r=0}^\infty 2\pi r e^{-ar^2}dr$$

$$u = ar^2 \therefore du = 2ardr$$

$$\therefore I^2 = \tfrac{1}{4}\int_{u=0}^\infty \frac{\pi}{a}e^{-u}du = \frac{\pi}{4a}\left[-e^{-u}\right]_0^\infty = \left(-0-(-1)\right)\frac{\pi}{4a} = \frac{\pi}{4a}$$

$$\therefore I = \tfrac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\boxed{\begin{aligned} I_n &= \int_0^\infty x^n e^{-ax^2}\,dx \\ I_n &= \frac{n-1}{2a} I_{n-2} \end{aligned}}$$

Integration by parts

$$\int uvdx = u\int vdx - \int\left\{\left(\int vdx\right)\frac{du}{dx}\right\}dx$$

$$I_n = \int_0^\infty x^n e^{-ax^2}dx$$

$$I_n = \int_0^\infty x x^{n-1} e^{-ax^2}dx$$

$$I_n = \left[\tfrac{1}{2}x^2 x^{n-1}e^{-ax^2}\right]_0^\infty - \int_0^\infty \tfrac{1}{2}x^2\left((n-1)x^{n-2}e^{-ax^2} - 2ae^{-ax^2}x^n\right)dx$$

$$\frac{d}{dx}\left(x^{n-1}e^{-ax^2}\right) = (n-1)x^{n-2}e^{-ax^2} - 2axe^{-ax^2}x^{n-1} = (n-1)x^{n-2}e^{-ax^2} - 2ae^{-ax^2}x^n$$

$$I_n = -\tfrac{1}{2}(n-1)\int_0^\infty x^n e^{-ax^2}dx + a\int_0^\infty x^{n+2}e^{-ax^2}dx$$

$$I_n = -\tfrac{1}{2}(n-1)I_n + aI_{n+2}$$

$$I_n\left(1+\tfrac{1}{2}(n-1)\right) = aI_{n+2}$$

$$\tfrac{1}{2}I_n\left(2+n-1\right) = aI_{n+2}$$

$$I_{n+2} = \frac{n+1}{2a}I_n$$

$$n \to n-2$$

$$\therefore I_n = \frac{n-1}{2a}I_{n-2}$$

$$\boxed{\begin{aligned} I_n &= \int_0^\infty x^n e^{-ax^2}\,dx \\ I_2 &= \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}} \end{aligned}}$$

$$I_2 = \int_0^\infty x^2 e^{-ax^2}dx$$

$$I_n = \frac{n-1}{2a}I_{n-2}$$

$$\therefore I_2 = \frac{2-1}{2a}I_0$$

$$\therefore I_2 = \frac{1}{2a}\tfrac{1}{2}\sqrt{\tfrac{\pi}{a}}$$

$$\therefore I_2 = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}$$