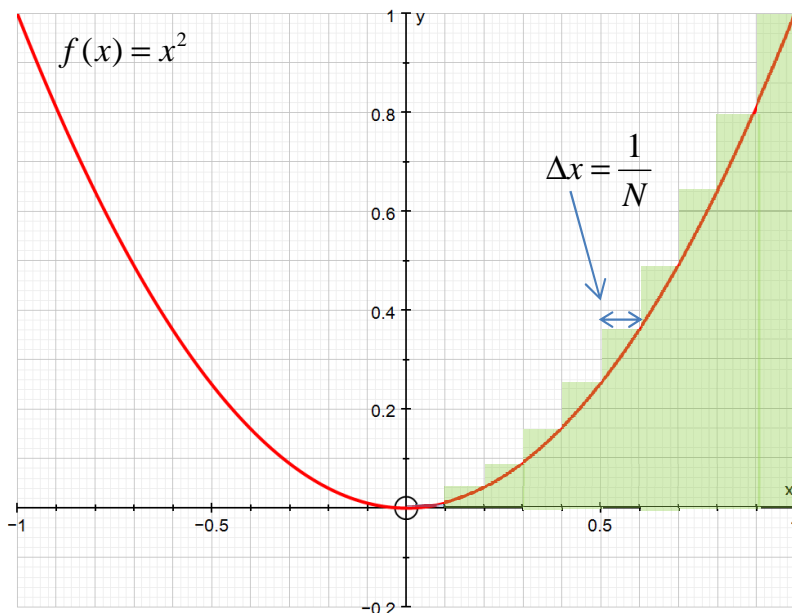


Introduction to integration



The area between a curve $f(x)$ and the x axis is defined as the *integral* of the *integrand* $f(x)$

$$A = \lim_{\Delta x \rightarrow 0} \left\{ \sum_{n=1}^N f(n\Delta x) \Delta x \right\} = \int_a^b f(x) dx$$

In the example above the limits would be $a = 0, b = 1$

The Old English 'S' means "sum rectangular strips of height $f(x)$ and width Δx , in the limit when Δx tends to zero." Note the dx has the same meaning as dx in the **gradient** df/dx

Now, for polynomials: $y = ax^{n+1} \quad \therefore \frac{dy}{dx} = (n+1)ax^n \quad \therefore \frac{dy}{n+1} = ax^n dx$

$$\therefore \frac{1}{n+1} \int dy = \int ax^n dx$$

$$\frac{1}{n+1} y + c = \int ax^n dx$$

$$\therefore \int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

Note we have omitted *limits* here. This integral is said to be 'indefinite' (whereas an integral with limits is 'definite'). An indefinite integral must always have a constant of integration added to account for the fact the limits have not been stated.

Hence, using this rule for integrating polynomials: $\int_0^1 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^1 = \left(\frac{1}{3} \right) - (0) = \frac{1}{3}$

What is the area A between a mathematically defined curve and the x axis? We can approximate this by summing a series of rectangular strips.

$$A \approx \sum_{n=1}^N f(n\Delta x) \Delta x$$

In the example on the left $f(x) = x^2$ and the limits of our area are $0 \leq x \leq 1$

The (fixed) strip width is therefore $\Delta x = \frac{1}{N}$

Clearly, in order to get a better estimate, we must reduce the size of the strip width, and thus increase the number of strips to cover the interval $[0,1]$. The exact area is therefore:

$$A = \lim_{\Delta x \rightarrow 0} \left\{ \sum_{n=1}^N f(n\Delta x) \Delta x \right\} \leftarrow f(n\Delta x) = \left(\frac{n}{N} \right)^2$$

$$A = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \left(\frac{n}{N} \right)^2 \frac{1}{N} \right\} = \lim_{N \rightarrow \infty} \left\{ \frac{1}{N^3} \sum_{n=1}^N n^2 \right\} \leftarrow \sum_{n=1}^N n^2 = \frac{1}{6} N(N+1)(2N+1)$$

$$A = \lim_{N \rightarrow \infty} \left\{ \frac{1}{N^3} \frac{1}{6} N(N+1)(2N+1) \right\} = \lim_{N \rightarrow \infty} \left\{ \frac{1}{N^3} \frac{1}{6} (2N^3 + 3N^2 + N) \right\}$$

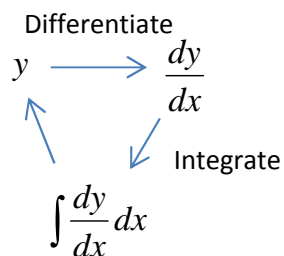
$$A = \lim_{N \rightarrow \infty} \left\{ \frac{1}{6} \left(2 + \frac{3}{N} + \frac{1}{N^2} \right) \right\} = \boxed{\frac{1}{3}}$$

$$\int_0^1 x^2 dx = \int_{-\infty}^1 x^2 dx - \int_{-\infty}^0 x^2 dx$$

i.e. we work out *definite integrals* by finding the difference in curve areas between the limits, relative to minus infinity. The constant of integration is therefore the same for each, and hence cancels in the difference.

Integration is *anti-differentiation*

$$\int \frac{dy}{dx} dx = \int dy = y + c$$



The caveat is that integration *creates information* (i.e. the constant of integration)

i.e. $y = x^3 + 2$

$$\frac{dy}{dx} = 3x^2$$

$$\int 3x^2 dx = x^3 + c$$

Regardless of what the constant added to x^3 is, differentiating it will yield the *same answer*.

Integrating the result will restore x^3 , but *any constant is possible*.

More information e.g. $y = 3$ when $x = 1$ is needed to define what the constant is.

Since integration is essentially a summation of terms:

$$\sum \{f(n) + g(n)\} = \sum f(n) + \sum g(n)$$

$$\sum af(n) = a \sum f(n)$$

$$\int \{f(x) + g(x)\} dx = \int f(x) dx + \int g(x) dx$$

$$\int af(x) dx = a \int f(x) dx$$

e.g. $\int \{2x^2 + \sqrt{x}\} dx = 2 \int x^2 dx + \int x^{\frac{1}{2}} dx = \frac{2}{3} x^3 + \frac{2}{3} x^{\frac{3}{2}} + c$

We can use the idea of integration as anti-differentiation to work out the integrals of many functions, using what we know about their derivatives

Polynomials

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

Exponentials

$$y = e^{ax}$$

$$\frac{dy}{dx} = ae^{ax} \quad \therefore \frac{1}{a} dy = e^{ax} dx$$

$$\therefore \int e^{ax} dx = \frac{1}{a} \int dy = \frac{1}{a} y + c$$

$$\therefore \int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

Trigonometric

$$y = \sin ax$$

$$\frac{dy}{dx} = a \cos ax \quad \therefore \frac{1}{a} dy = \cos ax dx$$

$$\therefore \int \cos ax dx = \frac{1}{a} \int dy = \frac{1}{a} y + c$$

$$\therefore \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$y = \cos ax$$

$$\frac{dy}{dx} = -a \sin ax \quad \therefore -\frac{1}{a} dy = \sin ax dx$$

$$\therefore \int \sin ax dx = -\frac{1}{a} \int dy = -\frac{1}{a} y + c$$

$$\therefore \int \sin ax dx = -\frac{1}{a} \cos ax + c$$

$$y = \tan ax$$

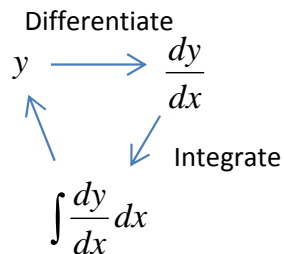
$$\frac{dy}{dx} = a \sec^2 ax \quad \therefore \frac{1}{a} dy = \sec^2 ax dx$$

$$\therefore \int \sec^2 ax dx = \frac{1}{a} \int dy = \frac{1}{a} y + c$$

$$\therefore \int \sec^2 ax dx = \frac{1}{a} \tan ax + c$$

Note this process is somewhat restrictive, for example it does not tell us what the integral of $\tan x$ is, or indeed logarithms of x . To compute these results we need *other techniques* such as *integration by parts* and *integration by substitution*.

Integration of polynomials, exponentials and trigonometric functions – spotting the pattern from knowledge of the derivative



Since 'integration is anti-differentiation' and

$$\int af'(x)dx = a \int f'(x)dx$$

for the following examples we can work out what the integral expression by inspection of what we need to transform the derivative to get back to the original function.

Polynomials

$y = x^2$	$y = x^3$	$y = x^5$	$y = x^{n+1}$
$\frac{dy}{dx} = 2x$	$\frac{dy}{dx} = 3x^2$	$\frac{dy}{dx} = 5x^4$	$\frac{dy}{dx} = (n+1)x^n$
$\int x dx = \frac{1}{2}x^2 + c$	$\int x^2 dx = \frac{1}{3}x^3 + c$	$\int x^4 dx = \frac{1}{5}x^5 + c$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

Sine

$y = \sin x$	$y = \sin 2x$	$y = \sin \frac{1}{2}x$	$y = \sin ax$
$\frac{dy}{dx} = \cos x$	$\frac{dy}{dx} = 2 \cos 2x$	$\frac{dy}{dx} = \frac{1}{2} \cos \frac{1}{2}x$	$\frac{dy}{dx} = a \cos ax$
$\int \cos x dx = \sin x + c$	$\int \cos 2x dx = \frac{1}{2} \sin 2x + c$	$\int \cos \frac{1}{2}x dx = 2 \sin \frac{1}{2}x + c$	$\int \cos ax dx = \frac{1}{a} \sin ax + c$

$$y = x^{n+1}$$

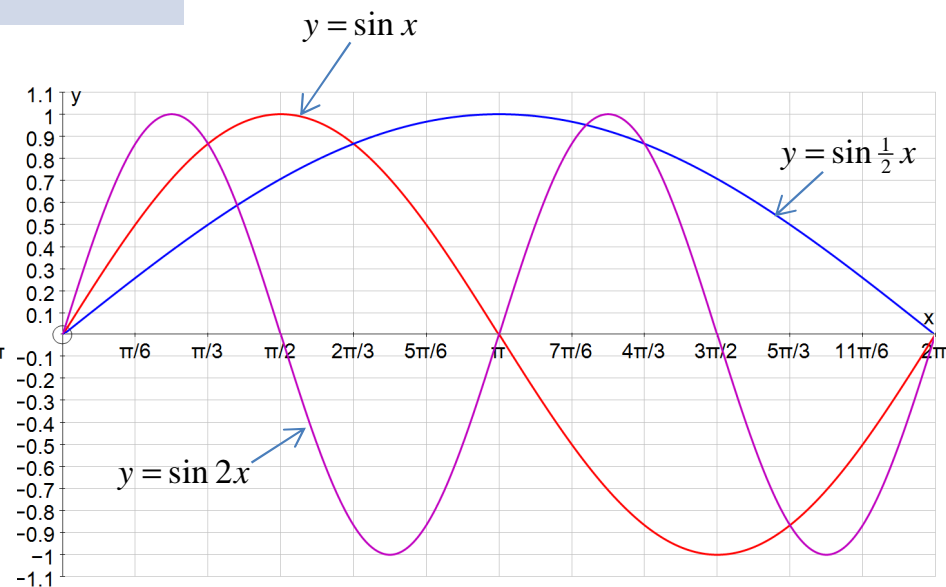
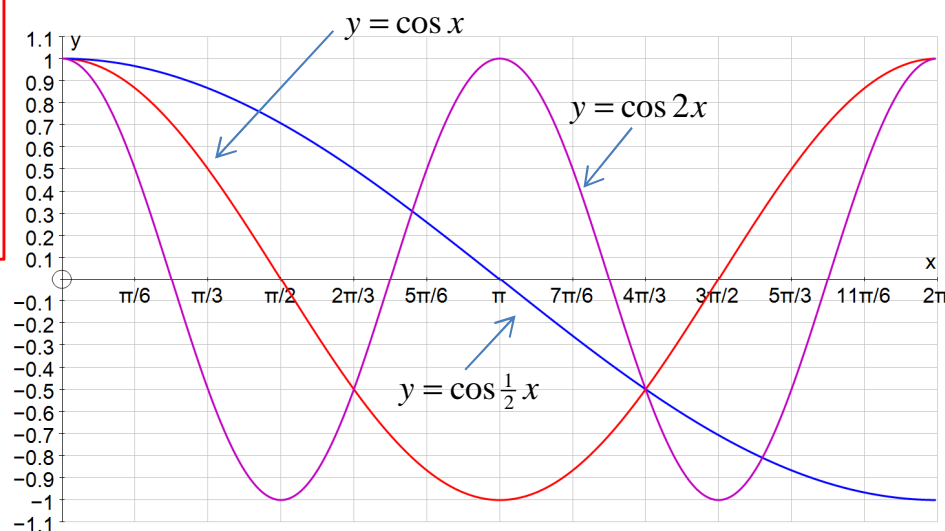
$$\frac{dy}{dx} = (n+1)x^n$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

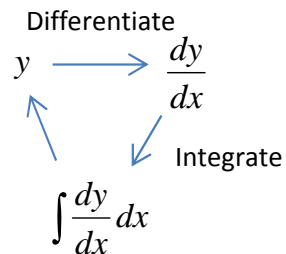
$$y = \sin ax$$

$$\frac{dy}{dx} = a \cos ax$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + c$$



Integration of polynomials, exponentials and trigonometric functions – spotting the pattern from knowledge of the derivative



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Exponentials

$y = e^x$	$y = e^{2x}$	$y = e^{-3x}$	$y = e^{\frac{1}{2}x}$	$y = e^{ax}$
$\frac{dy}{dx} = e^x$	$\frac{dy}{dx} = 2e^{2x}$	$\frac{dy}{dx} = -3e^{-3x}$	$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$	$\frac{dy}{dx} = ae^{ax}$
$\int e^x dx = e^x + c$	$\int e^{2x} dx = \frac{1}{2}e^{2x} + c$	$\int e^{-3x} dx = -\frac{1}{3}e^{-3x} + c$	$\int e^{\frac{1}{2}x} dx = 2e^{\frac{1}{2}x} + c$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

Cosine

$y = \cos x$	$y = \cos 2x$	$y = \cos \frac{1}{2}x$	$y = \cos ax$
$\frac{dy}{dx} = -\sin x$	$\frac{dy}{dx} = -2\sin 2x$	$\frac{dy}{dx} = -\frac{1}{2}\sin \frac{1}{2}x$	$\frac{dy}{dx} = -a\sin ax$
$\int \sin x dx = -\cos x + c$	$\int \sin 2x dx = -\frac{1}{2}\cos 2x + c$	$\int \sin \frac{1}{2}x dx = -2\cos \frac{1}{2}x + c$	$\int \sin ax dx = -\frac{1}{a}\cos ax + c$

$$y = e^{ax}$$

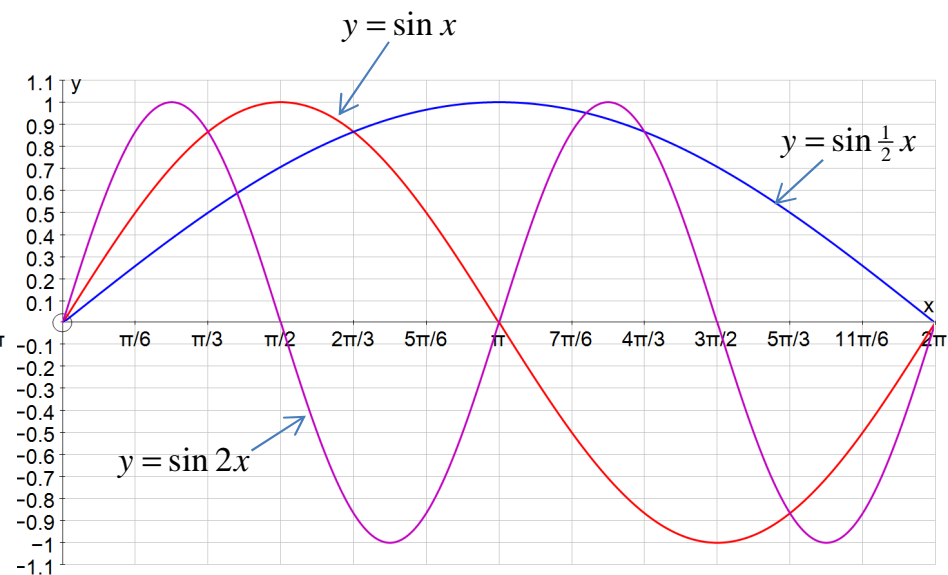
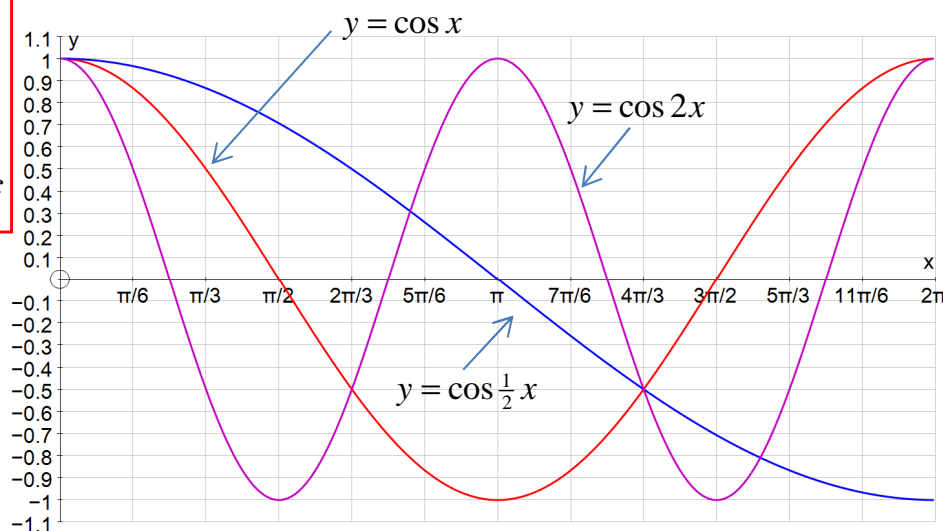
$$\frac{dy}{dx} = ae^{ax}$$

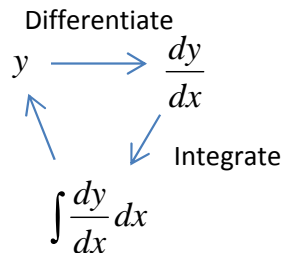
$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$y = \cos ax$$

$$\frac{dy}{dx} = -a\sin ax$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + c$$





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$y = x^2$	$y = x^3$	$y = x^5$	$y = x^{n+1}$
$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$
$\int x dx =$	$\int x^2 dx =$	$\int x^4 dx =$	$\int x^n dx =$

Sine

$y = \sin x$	$y = \sin 2x$	$y = \sin \frac{1}{2}x$	$y = \sin ax$
$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$
$\int \cos x dx =$	$\int \cos 2x dx =$	$\int \cos \frac{1}{2}x dx =$	$\int \cos ax dx =$

$$y = x^{n+1}$$

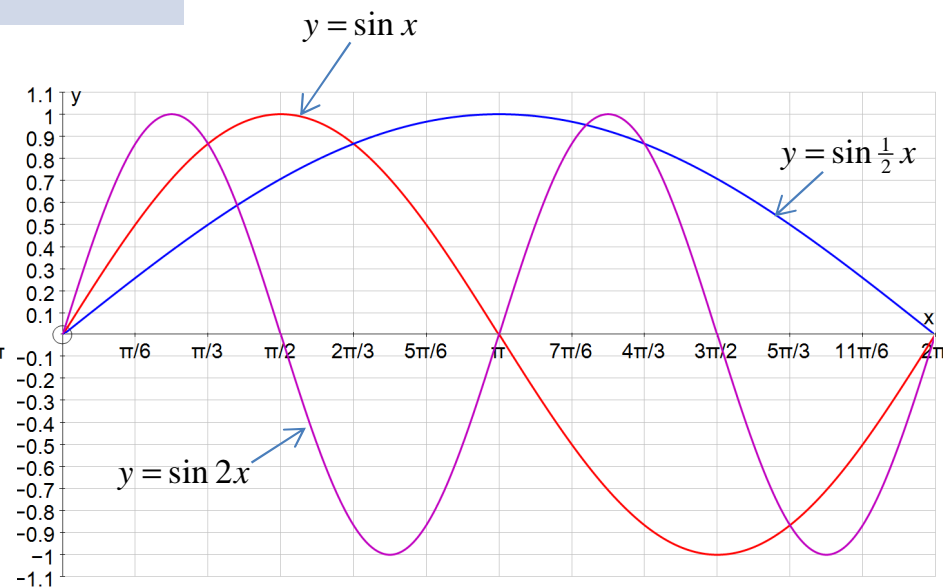
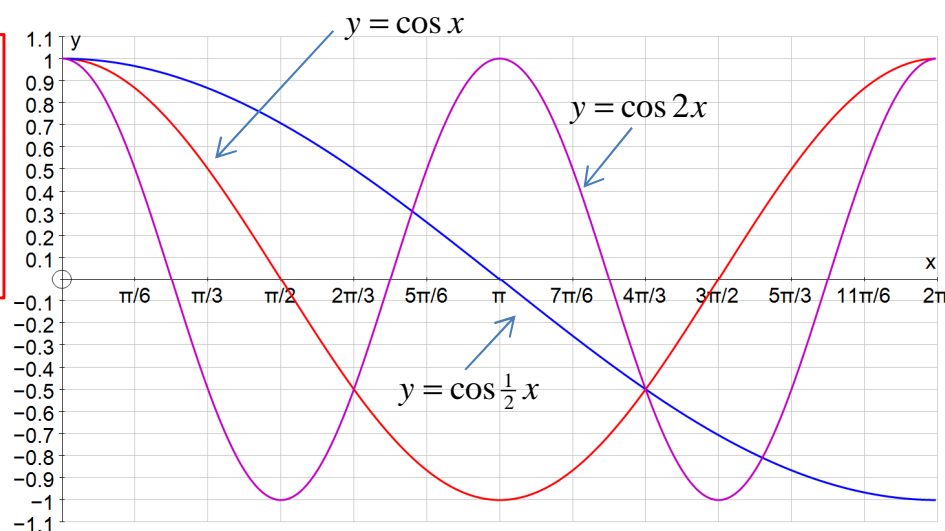
$$\frac{dy}{dx} =$$

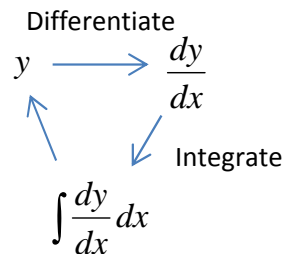
$$\int x^n dx =$$

$$y = \sin ax$$

$$\frac{dy}{dx} =$$

$$\int \cos ax dx =$$





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$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$
$\int e^x dx =$	$\int e^{2x} dx =$	$\int e^{-3x} dx =$	$\int e^{\frac{1}{2}x} dx =$	$\int e^{ax} dx =$

Cosine

$y = \cos x$	$y = \cos 2x$	$y = \cos \frac{1}{2}x$	$y = \cos ax$
$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$
$\int \sin x dx =$	$\int \sin 2x dx =$	$\int \sin \frac{1}{2}x dx =$	$\int \sin ax dx =$

$$y = e^{ax}$$

$$\frac{dy}{dx} =$$

$$\int e^{ax} dx =$$

$$y = \cos ax$$

$$\frac{dy}{dx} =$$

$$\int \sin ax dx =$$

