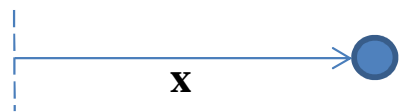


Kinematics, from the Greek κίνημα, (*kinema*) literally means the study of motion. This is done via analysis of the mathematical relationships between *time*, *displacement*, *velocity* and *acceleration*. For a kinematic analysis, the internal motion of a body (e.g. deformation, rotation etc) is ignored; only the bulk motion of the centre of mass is considered. i.e. a *particle* model.



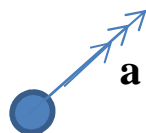
Displacement, velocity, acceleration are *vector* quantities. i.e. they have both magnitude and direction



Displacement is the vector between a fixed origin and the point of interest. If an object is moving, the displacement will vary with time t



Velocity is the *rate of change of displacement*. If velocity is in the same direction as displacement, it is the gradient of a (t,x) graph.



Acceleration is the *rate of change of velocity*. If acceleration is in the same direction as velocity, it is the gradient of a (t,v) graph.

In calculus notation:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} \quad \mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}$$

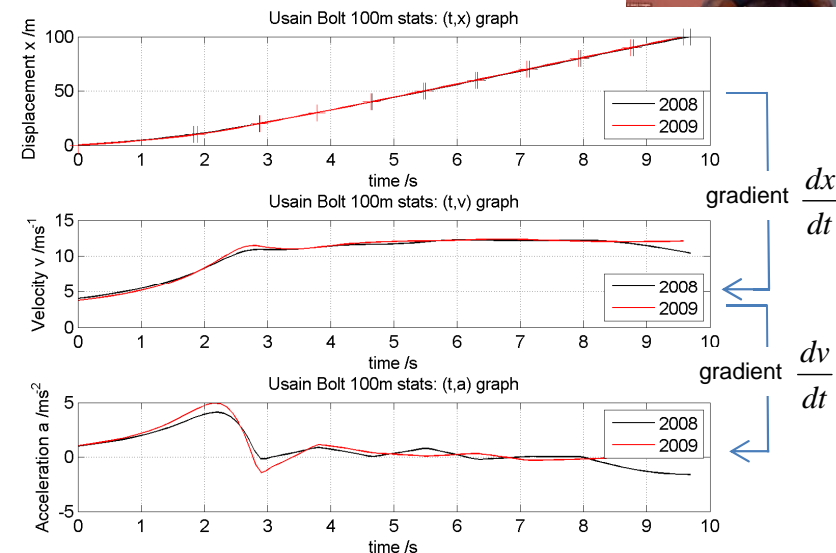
In many situations displacement, velocity and acceleration will all be *parallel*. We can therefore use *scalar* quantities. Direction (i.e. forwards or backwards) is indicated by a positive or negative sign.

$$v = \frac{dx}{dt} \quad a = \frac{dv}{dt} \quad a = \frac{d^2x}{dt^2} \quad a = v \frac{dv}{dx}$$

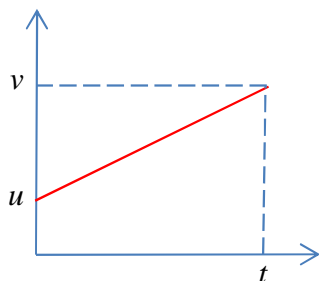
Displacement is therefore the *anti-derivative* of velocity and velocity is the anti-derivative of acceleration. In other words, the area under a (t,v) graph is the displacement and the area under a (t,a) graph is velocity.

$$x = \int v dt \quad v = \int a dt$$

i.e. $v = 1 - 2t + 3t^2 \Rightarrow x = x_0 + t - t^2 + t^3$



Constant acceleration motion. It is almost *always* a good idea to start with a (t,v) graph. Let velocity increase at the same rate a from u to v in t seconds.



The acceleration is the gradient: $a = \frac{v-u}{t} \therefore v = u + at$

The area under the graph is the displacement. Since this a trapezium shape:

$$x = \frac{1}{2}(u + v)t$$

We can work out other useful relationships for constant acceleration motion

$$x = \frac{1}{2}(u + u + at)t$$

$$x = ut + \frac{1}{2}at^2$$

$$x = ut + \frac{1}{2}at^2$$

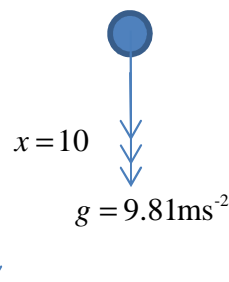
$$2ax = 2uat + a^2t^2$$

$$v^2 = (u + at)^2 = u^2 + 2uat + a^2t^2$$

$$\therefore v^2 = u^2 + 2ax$$

These equations are ONLY VALID when acceleration a is CONSTANT

Example 1: Falling due to gravity
If we ignore the effect of air resistance, what speed does a diver strike the water after a 10m jump? How long is the diver in the air for?



Since acceleration is constant $a = g$ and the diver starts from rest, i.e. $u = 0$

$$v^2 = u^2 + 2ax$$

$$\therefore v = \sqrt{2gx} \approx 4.43\sqrt{x}$$

$$\therefore v \approx 14 \text{ ms}^{-1}$$

$$v = u + at$$

$$\therefore t = \frac{\sqrt{2gx}}{g} = \sqrt{\frac{2x}{g}} \approx 0.45\sqrt{x}$$

$$\therefore t \approx 1.43 \text{ s}$$

Useful speed conversions:

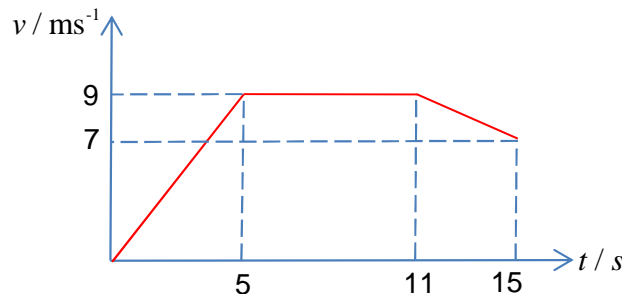
$$1 \text{ ms}^{-1} = 2.24 \text{ miles per hour}$$

$$1 \text{ ms}^{-1} = 3.6 \text{ km per hour}$$

Speed in mph	Time in minutes per 10 miles
10	60
20	30
30	20
40	15
50	12
60	10
70	8.57

$$t / \text{min} = 60 \times \frac{x / \text{miles}}{v / \text{mph}}$$

Example 2: What was the average speed of the following athlete?



Acceleration is constant in each 'phase' but not over the whole motion. We must treat each trapezium of the (t,v) graph separately.

Displacement is

$$x = \frac{1}{2}(5)(9) + (9)(6) + \frac{1}{2}(9+7)(4)$$

$$x = 108.5 \text{ m}$$

And therefore average speed is

$$\bar{v} = \frac{108.5 \text{ m}}{15 \text{ s}} = 7.23 \text{ ms}^{-1}$$

Example 3: A cyclist starts an uphill climb at 10 ms^{-1} . The hill causes him to lose speed at 0.5 ms^{-2} . How far could he cycle before he stops? How long will it take him to cycle 96 metres?

$$v = 10 - \frac{1}{2}t$$

$$x = 10t - \frac{1}{4}t^2$$

$$v = 0 \Rightarrow t_{\text{max}} = 20$$

$$\therefore x_{\text{max}} = 100$$

$$96 = 10t - \frac{1}{4}t^2$$

$$16 \times 6 \times 4 = 40t - t^2$$

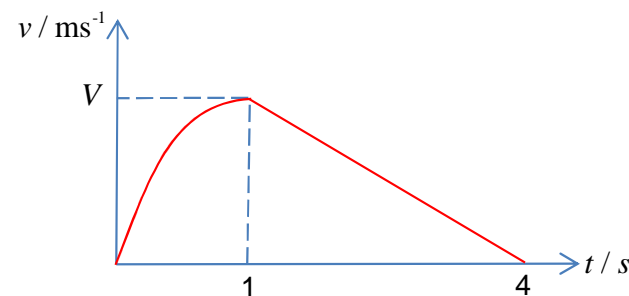
$$t^2 - 40t + 16 \times 24 = 0$$

$$(t-16)(t-24) = 0$$

$$\text{Since } t \leq 20, t = 16$$

$$t / \text{min} = \frac{60}{\frac{1}{10} v / \text{mph}}$$

Example 4: Between 0 and 1 seconds a cat accelerates to V metres per second. The acceleration of the cat diminishes linearly during this time, such that at exactly 1 second, it is moving at constant speed. After 1 second the cat decelerates at a constant rate until it stops, at 4 seconds. If the cat travels 13 metres in total, find V



Between $t=0$ and $t=1$ seconds, acceleration a is

$$a = a_0 - At \quad \text{Cat is moving at constant speed at } t=1$$

$$0 = a_0 - A \therefore a_0 = A$$

Hence in first three seconds

$$a = A - At \therefore v = At - \frac{1}{2}At^2 \therefore x = \frac{1}{2}At^2 - \frac{1}{6}At^3$$

$$\text{Which means } V = \frac{1}{2}A$$

The cat travels $\frac{3}{2}V$ metres in the last 3 seconds, hence

$$13 = \frac{1}{2}A - \frac{1}{6}A + \frac{3}{2}V$$

$$13 = \frac{1}{2}A - \frac{1}{6}A + \frac{3}{4}A$$

$$13 = \frac{13}{12}A \therefore A = 12$$

$$\therefore V = 6$$