Work, Energy & Power

The *First Law of Thermodynamics* states that the **energy of the universe is conserved.** However, it can take many forms. e.g:

- Mechanical work done via a force
- Work done by a electric, magnetic or gravitational field upon a charge (or mass) moving through it
- The random motion of molecules – i.e. heat
- 'Potential energy' stored in a spring, capacitor, battery ....
- Energy stored in chemical bonds between molecules
- Energy stored between the sub-atomic particles within an atomic nucleus ..
- Electromagnetic waves
- Pressure waves in liquids and gases
- Longitudinal and transverse waves in solids

Energy is a *scalar* quantity, and therefore is often a good starting point for calculations in physics. Unnecessary complexity can result from using forces (e.g. Newton's Second Law) when an energy calculation will do.

Mechanical work is, in words, the product of ***force x distance*** that the force is applied. Since both force and the trajectory that it is applied can vary with position, we can write this most generally as a *line integral*

$$\boxed{W = \int \mathbf{f} \cdot d\mathbf{r}}$$

f is a force and d**r** is vector line element

$$d\mathbf{r} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$$

If the force causes a mass *m* to move (and no mass is lost)

$$\mathbf{f} = m\frac{d\mathbf{v}}{dt} \qquad \therefore W = m\int \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r}$$

i.e. Newton's Second Law

Now $\frac{1}{2}\frac{d}{dt}v^2 = \frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}$

and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$

$$\therefore d\mathbf{r} = \mathbf{v}dt$$

Hence

$$W = m\int \frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dt$$

$$W = m\int \frac{1}{2}\frac{dv^2}{dt}dt$$

$$W = \frac{1}{2}m\int_u^v dv^2$$

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2}$$

$\boxed{E = \frac{1}{2}mv^2}$ is called **kinetic energy**

The mechanical work done on a mass causes a change in its kinetic energy, as long as no energy loss occurs.

Power is literally the *work rate.*

$$P = \frac{dW}{dt}$$

$$dW = \mathbf{f}\cdot d\mathbf{r}$$

$$\therefore P = \mathbf{f}\cdot\frac{d\mathbf{r}}{dt}$$

$$\boxed{P = \mathbf{f}\cdot\mathbf{v}}$$

So **work rate is force x velocity**

A selection of energy formulae associated with Mechanics

$\boxed{\frac{1}{2}mv^2}$ Kinetic energy of mass m moving with speed v << c
c is the speed of light c = 2.998 x 108 ms-1. Close to the speed of light, the equation for kinetic energy changes from this form to

$$\boxed{E = (\gamma - 1)mc^2} \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$\boxed{\frac{1}{2}I\omega^2}$ Rotational energy of a rigid body with moment of inertia I and angular velocity ω

$\boxed{mgh}$ Potential energy in a uniform gravitational field .
Mass *m*, gravitational field strength *g* and height *h*

$\boxed{-\frac{GMm}{R}}$ Potential energy of mass *m* in a spherically symmetric gravitational field.
Mass within radius *R* is *M*. *G* = 6.67 x 10^{-11} $m^3kg^{-1}s^{-2}$

$\boxed{\frac{1}{2}kx^2}$ Energy stored in a spring of spring constant *k*, extended from natural length by *x*

$\boxed{\frac{1}{2}mc^2}$ Energy equivalence to mass. In a nuclear reaction, the enormous amount of energy released is due to mass being converted into electromagnetic waves i.e. 'pure energy'.

Example 1: A coin is dropped down a well of depth 10 metres. What is the impact speed when it reaches the bottom of the well, assuming negligible work has been done against air resistance?

Energy is conserved so gravitational potential energy must be converted into the kinetic energy of the coin. We also ignore any rotation or heating!

$$\frac{1}{2}mv^2 = mgh \therefore v = \sqrt{2gh}$$

$$v = \sqrt{2\times 9.81\times 10} \approx \boxed{14\text{ms}^{-1}}$$

Example 2: A boy of mass 60kg leaps to a height of 50cm and lands on a trampoline, which sags to a maximum extent of 20cm. What is the effective spring constant of the trampoline?

Since no motion at the apogee of the leap and at the depth of the trampoline sag, conservation of energy implies

$$mgh = \frac{1}{2}kx^2$$

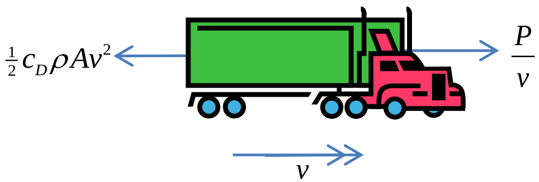
$$k = \frac{2mgh}{x^2} = \frac{2\times 60\times 9.81\times 0.7}{0.2^2} \approx \boxed{21{,}000\text{Nm}^{-1}}$$

Note total fall distance is 70cm

Example 3: A lorry is travelling a constant speed of 60 mph. If friction between the tyres and the road can be ignored at this speed, and internal losses such has heating etc can be ignored, the *driving force* of the engine is balanced by *air resistance*. If the cab has a cross section of 8 m^2, estimate the engine power P.

Since lorry is in equilibrium, driving force = air resistance

$$\tfrac{1}{2}c_D\rho A v^2 = \frac{P}{v}$$
$$\therefore P = \tfrac{1}{2}c_D\rho A v^3$$



Assume drag coefficient c_D = 1, density of air ρ = 1kgm-3
v = 60/2.34 = 25.64ms^{-1}

$$P = \tfrac{1}{2}\times 1\times 1\times 8\times 25.64^3$$
$$\boxed{P \approx 67.4\text{kW}}$$

Example 4: Calculate the minimum velocity required to escape the gravity of a spherical astronomical body of mass M and radius R

In order to escape, the total energy of the system must be positive at an infinite distance from the body. In other words, it will have some kinetic energy and will never be gravitationally attracted back towards the body.

For a mass m blasting off with velocity v, it will escape the gravitational influence of M if

$$\tfrac{1}{2}mv^2 - \frac{GMm}{R} > 0 \qquad \therefore \boxed{v > \sqrt{\frac{2GM}{R}}}$$

For Earth, the escape velocity is:

$$v_{escape} = \sqrt{\frac{2GM}{R}}$$
$$v_{escape} = \sqrt{\frac{2\times 6.67\times 10^{-11}\times 5.97\times 10^{24}}{6.38\times 10^6}} \approx \boxed{11.2\text{kms}^{-1}}$$

It is interesting to work out the radius of a star of mass M such that the escape velocity exceeds that of the speed of light. Since this is not possible, the star becomes a *Black Hole.*

$$\sqrt{\frac{2GM}{R}} > c$$
$$\frac{2GM}{R} > c^2$$
$$\boxed{R < \frac{2GM}{c^2}}$$

This inequality defines the maximum radius of a Black Hole, which is called the *Schwarzschild radius.* Alternatively, this is the *event horizon*, or 'point of no return' from the centre of a Black Hole.

For the Sun to become a Black Hole (M = 2 x 10^{30} kg, R = 6.96 x 10^8 m) its radius would have to shrink to less than $\boxed{\text{2.97 km.}}$

This is a mindblowing density of $\boxed{1.8\times 10^{19}\ \text{kgm}^{-3}}$

$$\rho_{Black\ hole} > \frac{M}{\tfrac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3}$$
$$\boxed{\rho_{Black\ hole} > \frac{3c^6}{32\pi G^3 M^2}}$$

As enormous as this sounds, it is not *entirely* outrageous given the density of the nucleus of a typical atom is approximately

$$\rho \approx \frac{2\times 10^{-27}}{\tfrac{4}{3}\pi\times\left(10^{-15}\right)^3} \approx \boxed{5\times 10^{17}\ \text{kgm}^{-3}}$$

In other words, compressing a typical nucleus* by only a factor of 36 or so would create an object of similar density to a Black Hole formed by a collapsing star.

*Even if a nucleus *could* be compressed to a density of 2 x 10^{19} kgm^{-3}, it is the *mass dependence* of the density which determines whether the object will become a Black Hole. For a nucleus to become a Black Hole, the density would have to be a whopping 1.8 x 10^{133} kgm^{-3}