

Calculus toolbox

Definition of the derivative \rightarrow

The **differential** (or 'derivative') of a function $y = f(x)$ is the gradient of the curve $y = f(x)$ at the point (x, y) .

$$y = f(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

Alternative forms of derivatives

$$f'(x) \equiv \frac{df}{dx} \quad f''(x) \equiv \frac{d^2 f}{dx^2} \quad f^{(n)}(x) \equiv \frac{d^n f}{dx^n}$$

$$\dot{x} \equiv \frac{dx}{dt} \quad \ddot{x} \equiv \frac{d^2 x}{dt^2}$$

$$\left. \frac{dy}{dx} \right|_{\text{stationary point}} = 0$$

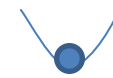


$$y = ax^2, \quad a < 0$$

Maxima

$$\left. \frac{d^2 y}{dx^2} \right|_{\text{stationary point}} < 0$$

Stationary points
i.e. zero gradient



$$y = ax^2, \quad a > 0$$

Minima

$$\left. \frac{d^2 y}{dx^2} \right|_{\text{stationary point}} > 0$$

$$y = u(x)v(x)$$

$$y = u \frac{dv}{dx} + v \frac{du}{dx}$$

Product Rule

$$y = \frac{u(x)}{v(x)}$$

$$y = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Quotient Rule

$$y = f(z)$$

$$z = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

Chain Rule

$$\frac{d}{dx} f(y) = \frac{df}{dy} \times \frac{dy}{dx}$$

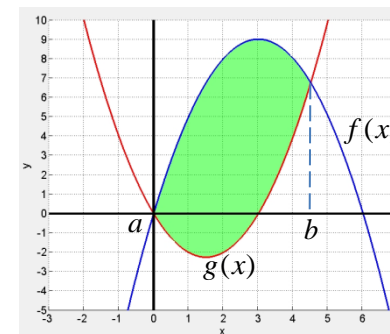
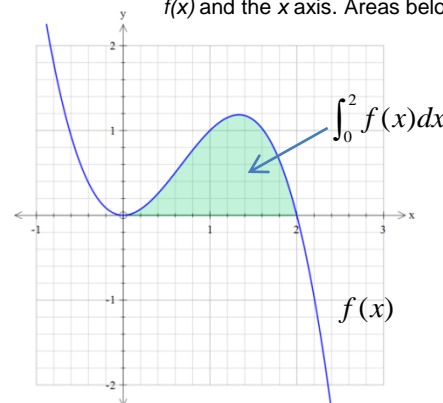
$$\text{e.g. } \frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}$$

Implicit differentiation

$$\frac{dy}{dx} = \left(\frac{dx}{dy} \right)^{-1}$$

Reciprocity

A **definite integral** can be used to find the area between a curve $y = f(x)$ and the x axis. Areas below the x axis are *negative*.



Integration is anti-differentiation

Differentiate

$$y \rightarrow \frac{dy}{dx}$$

Integrate

$$\int \frac{dy}{dx} dx$$

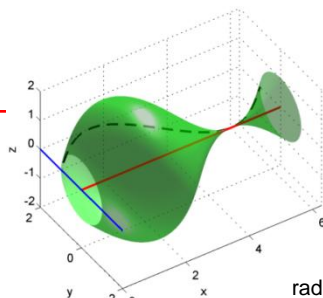
$$\int f'(x) dx = f(x) + c$$

$$\frac{dF}{dx} = f(x) \quad F(x) \text{ is the anti-differential of } f(x)$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{Definite integral}$$

$$\int f(x) dx = F(x) + c \quad \text{Indefinite integral}$$

Integrals mean 'sum rectangular strips of height $f(x)$ and width Δx ' (in the limit $\Delta x \rightarrow 0$). This means they have the same properties as sums of discrete numbers.



Note integration *creates information* (i.e. the constant of integration)

Volumes of revolution

$$V_x = \int_{x=a}^b \pi y^2 dx$$

$$V_y = \int_{y=a}^b \pi x^2 dy$$

$$A_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \quad \leftarrow \text{surface area}$$

$$\int \{f(x) + g(x)\} dx = \int f(x) dx + \int g(x) dx$$

$$\int a f(x) dx = a \int f(x) dx$$

Length along a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

radius of curvature

$$R = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}} \left(\frac{d^2 y}{dx^2} \right)^{-1}$$

Ratio of derivative to function

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

Integration by-parts

$$\int (uv) dx = u \int v dx - \int \left\{ \frac{du}{dx} \left(\int v dx \right) dx \right\}$$

$$\int_a^b (uv) dx = \left[u \int v dx \right]_a^b - \int_a^b \left\{ \frac{du}{dx} \left(\int v dx \right) dx \right\}$$

Calculus of Variations

$$I = \int_a^b f(y, y', x) dx, \quad y' = \frac{dy}{dx}$$

If I is *stationary* (i.e. maxima or minima)

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \quad \text{Euler-Lagrange Equation}$$

Derivatives and integrals of basic functions and some useful identities involving basic functions

For all trigonometric functions, x is in **radians**

$\frac{dy}{dx}$	$y = f(x)$	$\int f(x)dx$	$\frac{dy}{dx}$	$y = f(x)$	$\int f(x)dx$
$\frac{dy}{dx} = nx^{n-1}$	$y = x^n$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1$	$\frac{dy}{dx} = \sec x \tan x$	$y = \sec x$	$\int \sec x dx = \ln \sec x + \tan x + c$
$\frac{dy}{dx} = ae^{ax}$	$y = e^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$	$\frac{dy}{dx} = -\cot x \times \operatorname{cosec} x$	$y = \operatorname{cosec} x$	$\int \operatorname{cosec} x dx = -\ln \operatorname{cosec} x + \cot x + c$
$\frac{dy}{dx} = ab^{ax} \ln b$	$y = b^{ax} = (e^{\ln b})^{ax} = e^{ax \ln b}$	$\int b^{ax} dx = \frac{b^{ax}}{a \ln b} + c$	$\frac{dy}{dx} = -\operatorname{cosec}^2 x$	$y = \cot x$	$\int \cot x dx = \ln \sin x + c$
$\frac{dy}{dx} = \frac{1}{x}$	$y = \ln x $	$\int \ln x dx = x \ln x - x + c$	$\frac{dy}{dx} = \frac{a}{\sqrt{1-a^2 x^2}}$	$y = \sin^{-1} ax$	$\int \sin^{-1} x dx = \frac{1}{a} \sqrt{1-a^2 x^2} + x \sin^{-1} ax + c$
$\frac{dy}{dx} = a \cos ax$	$y = \sin ax$	$\int \sin ax dx = -\frac{1}{a} \cos ax + c$	$\frac{dy}{dx} = -\frac{a}{\sqrt{1-a^2 x^2}}$	$y = \cos^{-1} ax$	$\int \cos^{-1} x dx = x \cos^{-1} ax - \frac{1}{a} \sqrt{1-a^2 x^2} + c$
$\frac{dy}{dx} = -a \sin ax$	$y = \cos ax$	$\int \cos ax dx = \frac{1}{a} \sin ax + c$	$\frac{dy}{dx} = \frac{a}{1+a^2 x^2}$	$y = \tan^{-1} ax$	$\int \tan^{-1} ax dx = x \tan^{-1} ax - \frac{1}{2a} \ln(a^2 x^2 + 1) + c$
$\frac{dy}{dx} = a \sec^2 ax$	$y = \tan ax$	$\int \tan ax dx = -\frac{1}{a} \ln \cos ax + c$	<div> $\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ </div> <div> $\sec x = \frac{1}{\cos x}$ $\operatorname{cosec} x = \frac{1}{\sin x}$ $\cot x = \frac{1}{\tan x}$ </div>		

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\log_b x + \log_b y = \log_b xy$$

$$\log_b x - \log_b y = \log_b \frac{x}{y}$$

$$\log_b x^a = a \log_b x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$x = b^{\log_b x}$$

$$N^x = 10^{x \log_{10} N - \text{floor}(x \log_{10} N)} \times 10^{\text{floor}(x \log_{10} N)}$$

floor() means round down to the nearest integer. Use this result to evaluate very large (or very small) numbers of the form N^x in standard form.

More derivatives and integrals of trigonometric functions

$\frac{dy}{dx}$	$y = f(x)$	$\int f(x)dx$
$\frac{dy}{dx} = m \cos mx \cos nx - n \sin mx \sin nx$	$y = \sin mx \cos nx$	$\int \sin mx \cos nx dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad m^2 \neq n^2$
$\frac{dy}{dx} = m \cos mx \sin nx + n \sin mx \cos nx$	$y = \sin mx \sin nx$	$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad m^2 \neq n^2$
$\frac{dy}{dx} = -m \sin mx \cos nx - n \sin nx \cos mx$	$y = \cos mx \cos nx$	$\int \cos mx \cos nx dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad m^2 \neq n^2$
$\frac{dy}{dx} = \cos 2x$	$y = \sin x \cos x$	$\int \sin x \cos x dx = -\frac{1}{2} \cos^2 x + c$
$\frac{dy}{dx} = \sin 2x$	$y = \sin^2 x$	$\int \sin^2 x dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + c$
$\frac{dy}{dx} = -2 \sin x \cos x$	$y = \cos^2 x$	$\int \cos^2 x dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + c$
$\frac{dy}{dx} = 2 \tan x \sec^2 x$	$y = \tan^2 x$	$\int \tan^2 x dx = \tan x - x + c$
$\frac{dy}{dx} = 3 \sin^2 x \cos x$	$y = \sin^3 x$	$\int \sin^3 x dx = \frac{1}{12} (\cos 3x - 9 \cos x) + c$
$\frac{dy}{dx} = -3 \cos^2 x \sin x$	$y = \cos^3 x$	$\int \cos^3 x dx = \frac{1}{12} (9 \sin x + \sin 3x) + c$
$\frac{dy}{dx} = 3 \tan^2 x \sec^2 x$	$y = \tan^3 x$	$\int \tan^3 x dx = \frac{1}{2} \sec^2 x + \ln \cos x + c$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x) \quad \cos^3 x = \frac{1}{4}(3 \cos x + \cos 3x)$$

Derivatives and integrals of hyperbolic functions

$\frac{dy}{dx}$	$y = f(x)$	$\int f(x)dx$
$\frac{dy}{dx} = a \sinh ax$	$y = \cosh ax$	$\int \sinh ax dx = \frac{1}{a} \sinh ax + c$
$\frac{dy}{dx} = a \cosh ax$	$y = \sinh ax$	$\int \sinh ax dx = \frac{1}{a} \cosh ax + c$
$\frac{dy}{dx} = a \operatorname{sech}^2 ax$	$y = \tanh ax$	$\int \tanh ax dx = \frac{1}{a} \ln(\cosh ax) + c$
$\frac{dy}{dx} = -a \operatorname{sech} ax \times \tanh ax$	$y = \operatorname{sech} ax$	$\int \operatorname{sech} ax dx = \frac{2}{a} \tan^{-1}(e^{ax}) + c$
$\frac{dy}{dx} = -a \operatorname{cosech} ax \times \coth ax$	$y = \operatorname{cosech} ax$	$\int \operatorname{cosech} ax dx = \frac{1}{a} \ln\left(\tanh \frac{1}{2} ax\right) + c$
$\frac{dy}{dx} = -a \operatorname{cosech}^2 ax$	$y = \coth ax$	$\int \coth ax dx = \frac{1}{a} \ln(\sinh ax) + c$
$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 x^2 - 1}}$	$y = \cosh^{-1} ax$	$\int \cosh^{-1} ax dx = x \cosh^{-1} ax - \frac{1}{a} \sqrt{ax-1} \sqrt{ax+1} + c$
$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 x^2 + 1}}$	$y = \sinh^{-1} ax$	$\int \sinh^{-1} ax dx = x \sinh^{-1} ax - \frac{1}{a} \sqrt{a^2 x^2 + 1} + c$
$\frac{dy}{dx} = \frac{a}{1 - a^2 x^2}$	$y = \tanh^{-1} ax$	$\int \tanh^{-1} ax dx = x \tanh^{-1} ax + \frac{1}{2a} \ln(1 - a^2 x^2) + c$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{cosech}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\operatorname{coth}^{-1} x = \tanh^{-1} \frac{1}{x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\operatorname{coth}^2 x - \operatorname{cosech}^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$

$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^3 x = \frac{1}{4}(3 \cosh x + \cosh 3x)$$

$$\sinh^3 x = \frac{1}{4}(-3 \sinh x + \sinh 3x)$$

Standard integrals

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a \in \mathbb{Z}^+ \quad \text{i.e. positive integers}$$

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad a > 0$$

$$\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a}, \quad a > 0$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \begin{cases} 1 \times 3 \times 5 \times \dots (n-1) (2a)^{-\frac{n+1}{2}} \sqrt{\frac{\pi}{2}} & n > 0, n \text{ even} \\ 2 \times 4 \times 6 \times \dots (n-1) (2a)^{-\frac{n+1}{2}} & n > 1, n \text{ odd} \end{cases}$$

$$\int_{-\infty}^{\infty} e^{2bx-ax^2} dx = \sqrt{\frac{\pi}{a}} e^{\frac{1}{a}b^2}, \quad a > 0$$

$$\int_0^1 x^p (1-x)^q dx = \frac{p!q!}{(p+q+1)!}, \quad p, q \in \mathbb{Z}^+$$

$$\int_0^{\infty} \cos(ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}}, \quad a > 0$$

$$\int_0^{\infty} \sin(ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}}, \quad a > 0$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \pi$$

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{1}{2} \pi$$

$$\int_0^{\infty} \frac{1}{(1+x)x^a} dx = \frac{\pi}{\sin a\pi}, \quad (0 < a < 1)$$

$$\int \frac{1}{a+bx} dx = \frac{1}{b} \ln(a+bx) + c$$

$$\int \frac{1}{(a+bx)^2} dx = \frac{-1}{b(a+bx)} + c$$

$$\int \frac{1}{x(a+bx)} dx = -\frac{1}{a} \ln \left(\frac{a+bx}{x} \right) + c$$

$$\int \frac{1}{a^2+b^2x^2} dx = \frac{1}{ab} \tan^{-1} \left(\frac{bx}{a} \right) + c$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$$

$$\int \frac{x}{(x^2 \pm a^2)^n} dx = \frac{-1}{2(n-1)(x^2 \pm a^2)^{n-1}} + c$$

$$\int \frac{1}{x(x^n+a)} dx = \frac{1}{an} \ln \left| \frac{x^n}{x^n+a} \right| + c$$

$$\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln |x^2 \pm a^2| + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln |x + \sqrt{x^2 \pm a^2}| + c$$

$$\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{1}{a}x \right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{1}{a}x \right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + c$$

$$\int x \ln ax dx = \frac{1}{2} x^2 \left(\ln ax - \frac{1}{2} \right) + c$$

$$\int \frac{x^n}{(1+tx)^{n+2}} dx = \frac{1}{n+1} \left(\frac{x}{1+tx} \right)^{n+1} + c$$

Integration by substitution

Integral includes	Try substitution
$(ax+b)^n$	$u = ax+b$
$\sqrt[n]{ax+b}$	$u^n = ax+b$
$a-bx^2$	$a \sin^2 u = bx^2$
$a+bx^2$	$a \tan^2 u = bx^2$
bx^2-a	$a \sec^2 u = bx^2$
e^x	$u = e^x$
$\ln(ax+b)$	$e^u = ax+b$

e.g. $I = \int \frac{1}{1+x^2} dx$

$$x = \tan u \quad \therefore u = \tan^{-1} x$$

$$\frac{dx}{du} = \sec^2 u \quad \therefore dx = \sec^2 u du$$

$$1+x^2 = 1+\tan^2 u = \sec^2 u$$

$$I = \int \frac{1}{1+x^2} dx = \int \frac{\sec^2 u}{\sec^2 u} du$$

$$I = \int du = u + c$$

$$I = \tan^{-1} x + c$$

Special functions

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{Error function}$$

$$C(x) = \int_0^x \cos\left(\frac{1}{2}\pi t^2\right) dt \quad \text{Fresnel integrals}$$

$$S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$$

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x > 0 \quad \text{Exponential integral}$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0 \quad \text{Gamma function}$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n! \quad \text{Note Stirling's Approximation} \quad n! \approx \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}$$

$$\mathfrak{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \quad \text{Fourier transform}$$

$$f(t) = \int_{-\infty}^{\infty} \mathfrak{F}(\omega) e^{2\pi i \omega t} d\omega \quad \text{Inverse Fourier transform}$$

$$L(s) = \int_0^{\infty} f(t) e^{-st} dt \quad \text{Laplace transform}$$

Fourier series

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right\}$$

$$a_n = \frac{2}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t) \cos\left(\frac{2n\pi t}{T}\right) dt$$

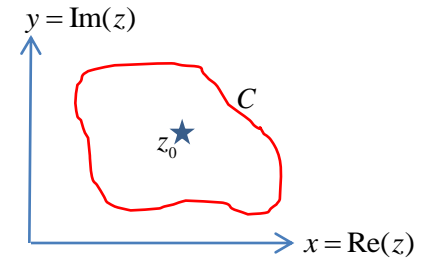
$$b_n = \frac{2}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t) \sin\left(\frac{2n\pi t}{T}\right) dt$$

Contour integration of complex variables

$$f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{Cauchy-Riemann equations}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



$$\oint_C f(z) dz = 2\pi i \sum_k R_k \quad \text{Cauchy (Residue) theorem}$$

$$R_k = a_{-1}^{(k)} \quad \text{Residue - i.e. the } -1^{\text{th}} \text{ term of the Laurent expansion}$$

$$f(z) = \sum_{n=-m}^{\infty} a_n^{(k)} (z - z_0^{(k)})^n \quad \text{Laurent expansion of } f(z) \text{ about poles } z_0 \text{ of order } m$$

$$R_k = \lim_{z \rightarrow z_0^{(k)}} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0^{(j)})^m f(z) \right) \right\} \quad \text{Computing the Residue}$$

e.g. $f(z) = \frac{1}{(z - z_0)^m}$
has a pole at $z = z_0$ of order m

$$\iint_A \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \oint_C (p dy - q dx)$$

Green's Theorem in a plane, for functions p, q which have continuous derivatives within area A bounded by contour C

Example:

$$\int_0^{\infty} \frac{2x \sin x}{1+x^2} dx = \frac{\pi}{e} \quad \text{Consider this complex contour integral. Simple poles (i.e. } m=1) \text{ at } z_0 = -i, i$$

$$I = \oint_C \frac{ze^{iz}}{1+z^2} dz = \oint_C \frac{ze^{iz}}{(z+i)(z-i)} dz$$

Residues are:

$$R_{\pm} = \lim_{z_0 \rightarrow \pm i} \left\{ (z - z_0) f(z) \right\}$$

$$R_{\pm} = \begin{cases} \frac{ie^{-1}}{i+i} & z_0 = i \\ \frac{-ie}{-i-i} & z_0 = -i \end{cases} = \begin{cases} \frac{1}{2}e^{-1} & z_0 = i \\ \frac{1}{2}e & z_0 = -i \end{cases}$$

Now consider a semi-circular contour of radius R in the upper half plane of the Argand diagram

$$I = \int_{-R}^R \frac{x(\cos x + i \sin x)}{1+x^2} dx + \int_{\theta=0}^{\pi} \frac{Re^{iR(\cos \theta + i \sin \theta)}}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

As R become infinite the second integral tends to zero. Hence, using the residue theorem and considering imaginary parts:

$$\int_{-\infty}^{\infty} \frac{ix \sin x}{1+x^2} dx = 2\pi i \times \frac{1}{2} e^{-1} \quad \text{Contour only encloses the pole at } +i$$

$$\int_0^{\infty} \frac{2x \sin x}{1+x^2} dx = \frac{\pi}{e}$$

Since the integrand is even

Vector calculus

Cartesian coordinate unit vectors

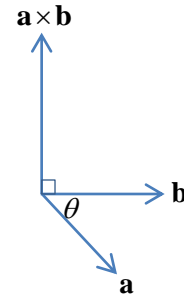
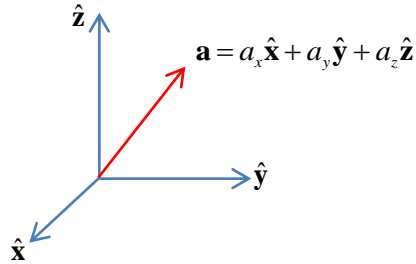
$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$$



$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \quad \text{Gradient}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{Laplacian}$$

$$dV = dxdydz \quad \text{Volume element}$$

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}} \quad \text{Generic vector}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{Divergence}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad \text{Curl}$$

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} \quad \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}} \quad \text{Displacement, velocity and acceleration}$$

$$\int_V (\nabla \cdot \mathbf{A}) dV = \oint_S \mathbf{A} \cdot d\mathbf{s} \quad \text{Gauss's Theorem ('Divergence theorem'). Surface } S \text{ bounds volume } V$$

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_L \mathbf{A} \cdot d\mathbf{l} \quad \text{Stokes' Theorem. Surface } S \text{ extends from loop } L.$$

$$\oint_S (f \nabla g) \cdot d\mathbf{s} = \int_V (f \nabla^2 g + \nabla f \cdot \nabla g) dV \quad \text{Green's Theorems}$$

$$\oint_S (f \nabla g - g \nabla f) \cdot d\mathbf{s} = \int_V (f \nabla^2 g - g \nabla^2 f) dV$$

Vector scalar (dot) and cross products

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

$$\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}$$

$$\mathbf{b} = b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}$$

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{x}}(a_y b_z - a_z b_y) + \hat{\mathbf{y}}(a_z b_x - a_x b_z) + \hat{\mathbf{z}}(a_x b_y - a_y b_x)$$

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Identities involving the vector derivative operator

$$\nabla(fg) = f \nabla g + g \nabla f$$

$$\nabla \cdot (f\mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

$$\nabla \times (f\mathbf{A}) = f \nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$$

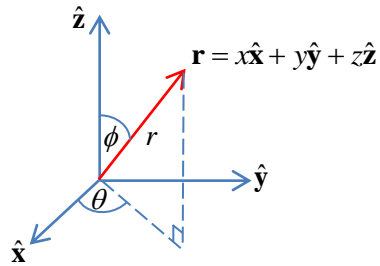
$$\nabla \cdot \nabla f = \nabla^2 f$$

$$\nabla \times \nabla f = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Vector calculus using cylindrical and spherical polar coordinates



$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

Conversion between Cartesian and polar coordinates

Polar coordinate unit vectors as Cartesian coordinates

$$\hat{\mathbf{r}} = \sin \phi \cos \theta \hat{\mathbf{x}} + \sin \phi \sin \theta \hat{\mathbf{y}} + \cos \phi \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = \cos \phi \cos \theta \hat{\mathbf{x}} + \cos \phi \sin \theta \hat{\mathbf{y}} - \sin \phi \hat{\mathbf{z}}$$

Spherical polar coordinates

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r \sin \phi} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

$$dV = r^2 \sin \phi dr d\theta d\phi$$

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial A_\theta}{\partial \theta} + \frac{1}{r \sin \phi} \frac{\partial (\sin \phi A_\phi)}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{\hat{\mathbf{r}}}{r \sin \phi} \left(\frac{\partial (\sin \phi A_\theta)}{\partial \phi} - \frac{\partial A_\phi}{\partial \theta} \right) + \frac{\hat{\boldsymbol{\theta}}}{r} \left(\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) + \frac{\hat{\boldsymbol{\phi}}}{r} \left(\frac{1}{\sin \phi} \frac{\partial A_r}{\partial \theta} - \frac{\partial (r A_\theta)}{\partial r} \right)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \sin \phi \dot{\theta} \hat{\boldsymbol{\theta}} + \dot{\phi} \hat{\boldsymbol{\phi}} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta} (\sin \phi \hat{\mathbf{r}} + \cos \phi \hat{\boldsymbol{\phi}}) \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi} \hat{\mathbf{r}} + \cos \phi \dot{\theta} \hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{r}} = \frac{d}{dt} (r \hat{\mathbf{r}}) = \dot{r} \hat{\mathbf{r}} + r \sin \phi \dot{\theta} \hat{\boldsymbol{\theta}} + r \dot{\phi} \hat{\boldsymbol{\phi}}$$

$$\ddot{\mathbf{r}} = \frac{d^2}{dt^2} (r \hat{\mathbf{r}}) = (\ddot{r} - r \dot{\phi}^2 - r \sin^2 \phi \dot{\theta}^2) \hat{\mathbf{r}} + (2r \cos \phi \dot{\theta} \dot{\phi} + 2 \sin \phi \dot{\phi} \dot{r} + r \sin \phi \ddot{\theta}) \hat{\boldsymbol{\theta}} + \dots$$

$$\dots + (2\dot{r} \dot{\phi} + r \ddot{\phi} - r \sin \phi \cos \phi \dot{\theta}^2) \hat{\boldsymbol{\phi}}$$

Cartesian coordinates

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \quad \text{Gradient}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{Laplacian}$$

$$dV = dx dy dz \quad \text{Volume element}$$

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}} \quad \text{Generic vector}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{Divergence}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad \text{Curl}$$

$$\mathbf{r} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}} \quad \dot{\mathbf{r}} = \dot{x} \hat{\mathbf{x}} + \dot{y} \hat{\mathbf{y}} + \dot{z} \hat{\mathbf{z}} \quad \ddot{\mathbf{r}} = \ddot{x} \hat{\mathbf{x}} + \ddot{y} \hat{\mathbf{y}} + \ddot{z} \hat{\mathbf{z}} \quad \text{Displacement, velocity and acceleration}$$

Cylindrical polar coordinates

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{r}} + \frac{\partial f}{\partial y} \hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial z^2}$$

$$dV = r dr d\theta dz$$

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_z \hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) + \hat{\boldsymbol{\theta}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta} \hat{\boldsymbol{\theta}} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta} \hat{\mathbf{r}}$$

$$\dot{\mathbf{r}} = \frac{d}{dt} (r \hat{\mathbf{r}}) = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}} + \dot{z} \hat{\mathbf{z}}$$

$$\ddot{\mathbf{r}} = \frac{d^2}{dt^2} (r \hat{\mathbf{r}}) = (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{r}} + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\boldsymbol{\theta}} + \ddot{z} \hat{\mathbf{z}}$$

Note alternative representations of time derivatives of polar angle

$$\omega = \dot{\theta} = \frac{d\theta}{dt}$$

$$\dot{\omega} = \ddot{\theta} = \frac{d^2\theta}{dt^2}$$