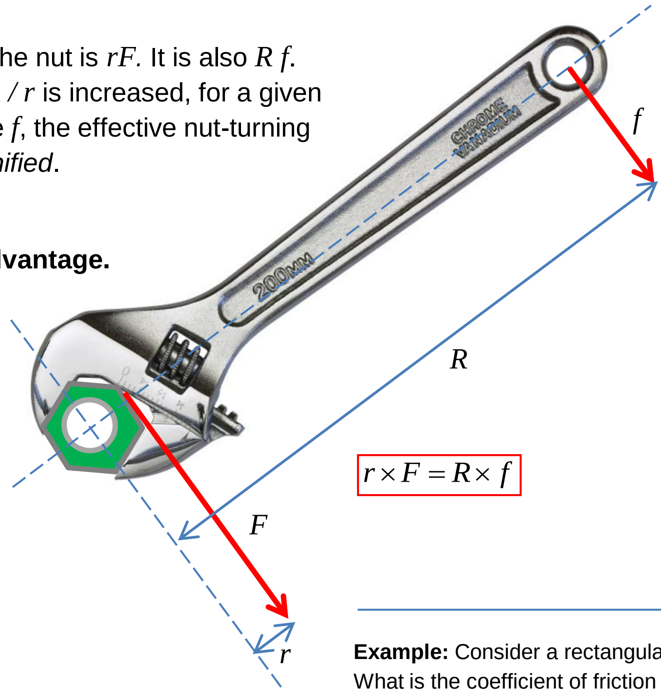
## Moments & Tools

A tool like a screwdriver or a wrench can deliver the required turning moment to tighten a screw or nut with less force.

The torque on the nut is $rF$. It is also $Rf$. So if the ratio $R/r$ is increased, for a given amount of force $f$, the effective nut-turning force $F$ is *magnified*.

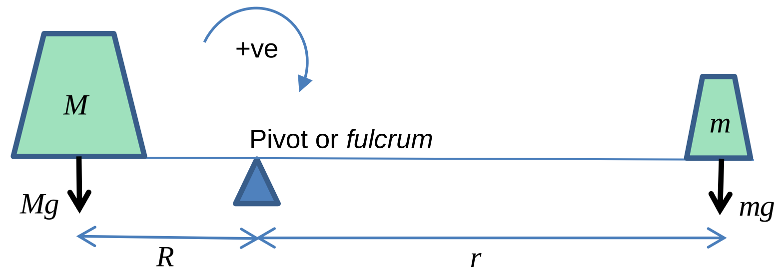
This is called a **mechanical advantage.**




$$\boxed{r \times F = R \times f}$$

**If a body is not rotating, or rotating with a constant angular velocity, then the sum of moments must equate to zero.** This is very useful in calculating forces in equilibrium problems involving rigid bodies, since it doesn't matter in this case which point 'we take moments about.' (They must always sum to zero).

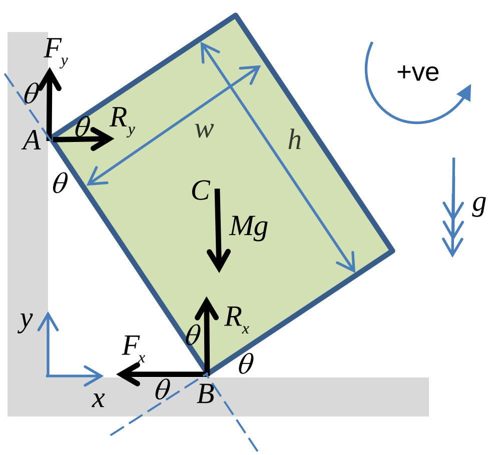
Only when there is net torque is it a good idea to choose the **centre of mass** of an object, and use symmetry axis to simplify the moment of inertia tensor.*




Using the convention that clockwise moments are positive the above pivot system is in rotational equilibrium if:

$$-MgR + mgr = 0 \quad \therefore \quad \boxed{mr = MR}$$

**Example:** Consider a rectangular block on the *verge of sliding* in the $x$ and –ve $y$ directions when it is inclined at angle $\theta$. What is the coefficient of friction $\mu$ between the block and the floor? Assume wall is smooth and floor is rough.




$F_y = 0$ Smooth wall

$F_x = \mu R_x$ About to slide on rough floor

$0 = R_y - F_x$ Newton II in $x$ direction

$0 = R_x - Mg$ Newton II in $y$ direction

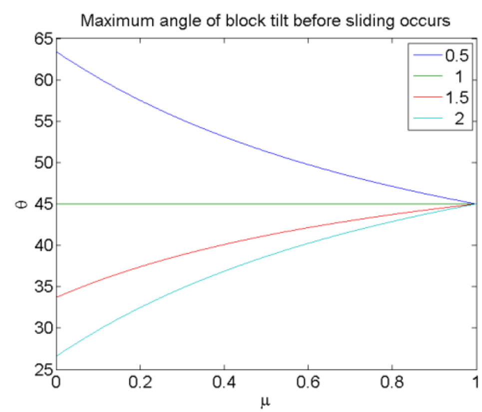
$$0 = -R_x \cos\theta \times \tfrac{1}{2}w + R_x \sin\theta \times \tfrac{1}{2}h + ...$$
$$-F_x \cos\theta \times \tfrac{1}{2}h + F_x \sin\theta \times \tfrac{1}{2}w + ...$$
$$-R_y \cos\theta \times \tfrac{1}{2}h + R_y \sin\theta \times \tfrac{1}{2}w$$

Moments about centre of mass $C$

$$Mg\left(-w\cos\theta + h\sin\theta\right) +$$
$$+\mu Mg\left(-h\cos\theta + w\sin\theta\right) + ...$$
$$+\mu Mg\left(-h\cos\theta + w\sin\theta\right) = 0$$

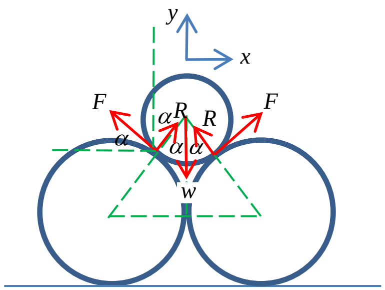
Substituting into the moments equation

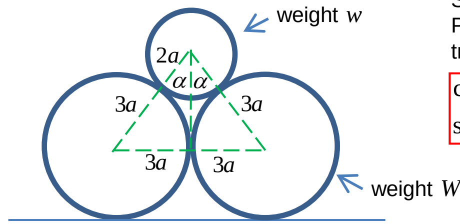
$$-w\cos\theta + h\sin\theta + ...$$
$$+\mu\left(-h\cos\theta + w\sin\theta\right) = 0$$
$$\mu = \frac{\cos\theta - \frac{h}{w}\sin\theta}{\sin\theta - \frac{h}{w}\cos\theta} = \frac{1 - \frac{h}{w}\tan\theta}{\tan\theta - \frac{h}{w}}$$
$$\therefore \left(\mu + \tfrac{h}{w}\right)\tan\theta = 1 + \tfrac{h}{w}\mu$$
$$\boxed{\tan\theta = \frac{1 + \frac{h}{w}\mu}{\mu + \frac{h}{w}}}$$




* See following pages for a definition of this quantity!

## Example: Three stacked cylinders




Spot the Pythagorean triple!

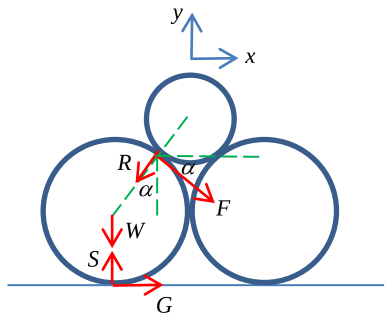
$\boxed{\cos\alpha = \tfrac{4}{5} \quad \sin\alpha = \tfrac{3}{5}}$

Assume static **equilibrium**.

- Sum of **moments** about any point is **zero**
- Vector sum of all **forces** is **zero**

Newton II for upper cylinder ($y$ direction)

$$0 = 2R\cos\alpha - w + 2F\sin\alpha$$
$$w = 2R\tfrac{4}{5} + 2F\tfrac{3}{5}$$
$$w = \tfrac{8}{5}R + \tfrac{6}{5}F$$
$$\therefore 5w = 8R + 6F$$




Clockwise moments about centre of lower cylinder

$$0 = F \times 3a - G \times 3a$$
$$\therefore \boxed{F = G}$$

Newton II for lower cylinder ($x$ direction)

$$0 = G + F\cos\alpha - R\sin\alpha$$
$$R\tfrac{3}{5} = G + F\tfrac{4}{5}$$
$$F = G \therefore R\tfrac{3}{5} = F\tfrac{9}{5}$$
$$\therefore R = 3F$$

Hence: $5w = 8R + 6F$

$$5w = 24F + 6F$$
$$\tfrac{5}{30}w = F$$
$$\therefore \boxed{F = \tfrac{1}{6}w} \qquad \boxed{G = \tfrac{1}{6}w}$$
$$\therefore \boxed{R = \tfrac{1}{2}w}$$

Newton II for lower cylinder ($y$ direction)

$$0 = S - R\cos\alpha - F\sin\alpha - W$$
$$\therefore S = W + \tfrac{4}{5}R + \tfrac{3}{5}F$$
$$\therefore S = W + \tfrac{4}{5}\left(\tfrac{1}{2}w\right) + \tfrac{3}{5}\left(\tfrac{1}{6}w\right)$$
$$\therefore S = W + \left(\tfrac{4}{10} + \tfrac{1}{10}\right)w$$
$$\therefore \boxed{S = W + \tfrac{1}{2}w}$$

No slip at ground-sphere interface

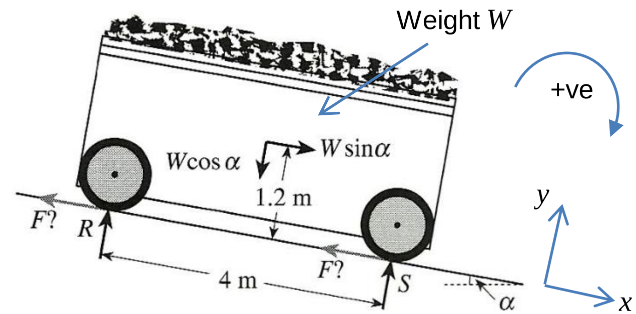
$$G \le \mu_A S$$
$$\tfrac{1}{6}w \le \mu_A\left(W + \tfrac{1}{2}w\right)$$
$$\therefore \boxed{\mu_A \ge \frac{w}{6W + 3w}}$$

No slip at sphere-sphere interface

$$F \le \mu_B R$$
$$\tfrac{1}{6}w \le \mu_A \tfrac{1}{2}w$$
$$\therefore \boxed{\mu_A \ge \tfrac{1}{3}}$$

## Example: Quarry truck problem

(Quadling, Mechanics 2, pp96)




A quarry truck has wheels 4 metres apart. When fully loaded with stone, the centre of mass is 1.2 metres from the tracks and midway between the wheels. A brake can be used to lock one of the pairs of wheels when the truck is on a slope. Is it better for the brake to be on the lower or the upper wheels? If the coefficient of friction between the wheels and the rails is 0.4, what is the steepest slope on which the truck can stand when fully loaded?

Moments about lower wheel: $4R + 1.2W\sin\alpha - 2W\cos\alpha = 0$

$$\therefore \boxed{R = W\left(0.5\cos\alpha - 0.3\sin\alpha\right)}$$

Moments about upper wheel: $-4S + 1.2W\sin\alpha + 2W\cos\alpha = 0$

$$\therefore \boxed{S = W\left(0.5\cos\alpha + 0.3\sin\alpha\right)}$$

Therefore $S > R$ for any $\alpha$ chosen. Since break force $F$ is proportional to $S$ or $R$, get a larger $F$ if you break with the **lower wheels.**

For no sliding down the slope $\quad F < 0.4S$

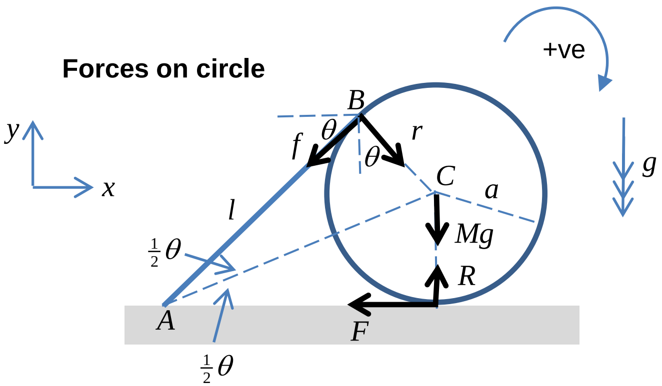
Newton II in $x$ direction: $\quad 0 = W\sin\alpha - F \quad \therefore F = W\sin\alpha$

$$F < 0.4S$$
$$W\sin\alpha < 0.4W\left(0.5\cos\alpha + 0.3\sin\alpha\right)$$
$$\sin\alpha < 0.2\cos\alpha + 0.12\sin\alpha$$
$$0.88\tan\alpha < 0.2$$
$$\alpha < \tan^{-1}\tfrac{5}{22}$$
$$\boxed{\alpha < 12.8^{\circ}}$$

## Example: Stick on a circle problem (Morin, Introduction to Classical Mechanics pp38)

What are the coefficients of friction between (i) the stick and a circle and (ii) between the circle and the floor necessary to maintain equilibrium?

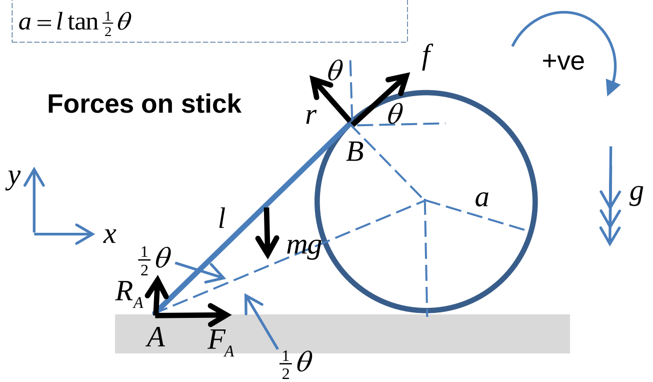
**Forces on circle**




Since stick is on a circle tangent

$a = l\tan\frac{1}{2}\theta$

**Forces on stick**




**Stick: Moments about $A$**

$$0 = \tfrac{1}{2} l\cos\theta \times mg - rl \qquad \therefore mg = \frac{2r}{\cos\theta}$$

$$R_A = \frac{2r}{\cos\theta} - \frac{r\sin^2\theta}{1+\cos\theta} - r\cos\theta$$

$$R_A = r\left\{\frac{2}{\cos\theta} - \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{1+\cos\theta}\right\}$$

$$R_A = r\left\{\frac{2}{\cos\theta} - 1\right\}$$

$$R_A = \frac{r(2-\cos\theta)}{\cos\theta}$$

**Circle: Newton II**

$$x:\ 0 = -F + r\sin\theta - f\cos\theta$$

$$y:\ 0 = R - Mg - r\cos\theta - f\sin\theta$$

**Circle: Moments about centre $C$**

$$f = F \qquad \therefore f = \frac{r\sin\theta}{1+\cos\theta} = r\tan\tfrac{1}{2}\theta$$

$f \le \mu_B r$ **For no sliding at point $B$**

$$\therefore \mu_B \ge \frac{f}{r}$$

$$\boxed{\mu_B \ge \frac{\sin\theta}{1+\cos\theta} = \tan\tfrac{1}{2}\theta}$$

**Stick: Newton II**

$$x:\ 0 = F_A + f\cos\theta - r\sin\theta$$

$$y:\ 0 = R_A - mg + f\sin\theta + r\cos\theta$$

$$\therefore R_A = mg - \frac{r\sin^2\theta}{1+\cos\theta} - r\cos\theta$$

$$F_A = r\sin\theta - \frac{r\sin\theta\cos\theta}{1+\cos\theta}$$

$$F_A = r\sin\theta\left(1 - \frac{\cos\theta}{1+\cos\theta}\right)$$

$$\therefore F_A = \frac{r\sin\theta}{1+\cos\theta} = f$$

$F_A \le \mu_A R_A$ **For no sliding at point $A$**

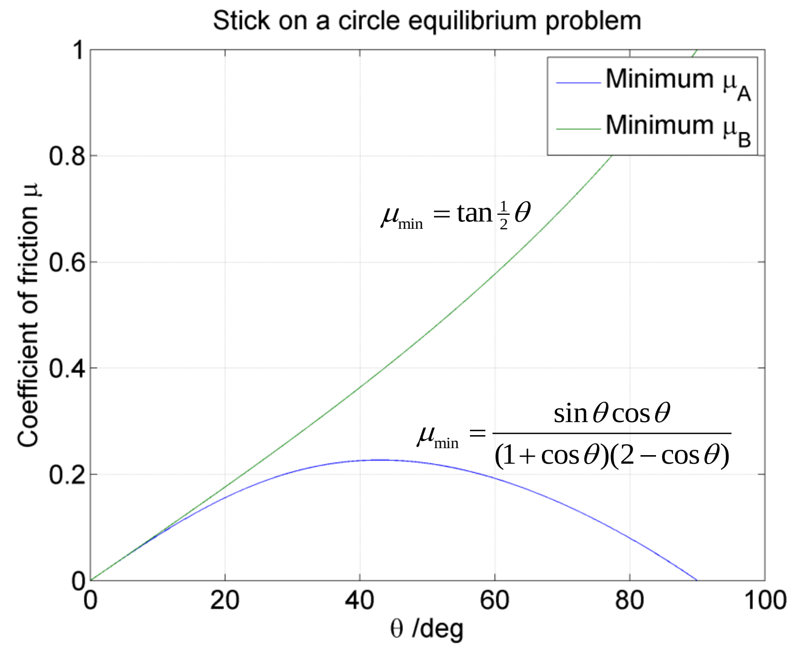
$$\therefore \mu_A \ge \frac{F_A}{R_A}$$

$$\boxed{\mu_A \ge \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}}$$

**Note** if there is no sliding at the stick and floor interface (A) *and* at the stick and circle interface (B), this is a *sufficient condition* for the system to be in equilibrium.

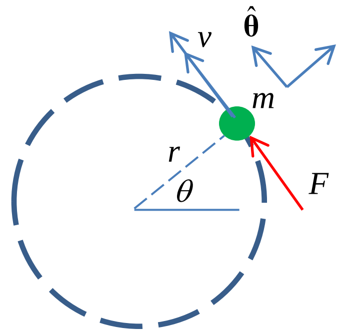
An additional no-slip condition at the floor-cylinder junction does not need to be obeyed, so therefore cannot be generally true.

i.e. $F \le \mu_D R$ may *not* be generally true



**Torque, moment of inertia and angular acceleration**

Consider a mass $m$ undergoing circular motion. The radius is assumed to be constant.




Force $F$ is applied *tangential* to the motion

$\omega = \dot{\theta} = \dfrac{d\theta}{dt}$ Angular velocity /radians per second

$v = r\omega$ Tangential velocity

$v = r\dot{\theta}$

$\therefore \dfrac{dv}{dt} = r\dot{\omega} = r\ddot{\theta}$

Newton II

$m\dfrac{dv}{dt} = F$

$\therefore mr\dot{\omega} = F$

The **moment** (or **torque**) of the force about the centre of the circle is

$$\tau = rF$$

Hence: $\boxed{\tau = mr^2\dot{\omega}}$

This is a useful general result for rotational motion - essentially a **rotational equivalent of Newton II**

**"torque = moment of inertia x angular acceleration"**

$\tau = rF$ $\boxed{I = mr^2}$ $\dot{\omega} = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2} = \ddot{\theta}$

$$\boxed{\tau = I\dot{\omega}}$$

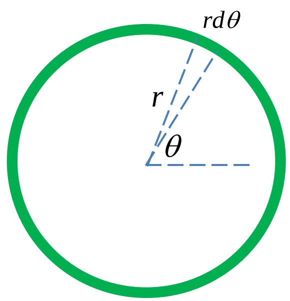
The **kinetic energy** is

$E = \tfrac{1}{2}mv^2$

$E = \tfrac{1}{2}mr^2\omega^2$

$\boxed{E = \tfrac{1}{2}I\omega^2}$

We can calculate the **moment of inertia** $I$ for **extended objects** (i.e. not just particles) and hence apply the formulae above more generally for **rigid body rotation**




$\boxed{I = \sum_i m_i r_i^2}$

mass element

(squared) distance of mass from axis of rotation

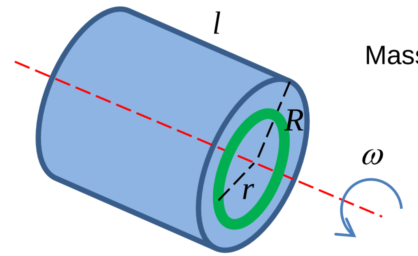
Moment of inertia of a thin hoop* of mass $M$ and radius $r$

$$I = \int_{\theta=0}^{2\pi} \frac{M}{2\pi r} \times rd\theta \times r^2$$

$$I = \frac{Mr^2}{2\pi}\int_{\theta=0}^{2\pi} d\theta$$

Unsurprisingly, the same algebraic form as for a particle $\boxed{I = Mr^2}$

**Moment of inertia of a solid cylinder**




Determine centre of mass by summing 'extruded hoops'

$$I = \int_{r=0}^{R} \rho \times 2\pi r dr \times l \times r^2$$

tube volume

Density $\rho = \dfrac{M}{\pi r^2 l}$

$$I = 2\pi l \frac{M}{\pi R^2 l}\int_{r=0}^{R} r^3 dr$$

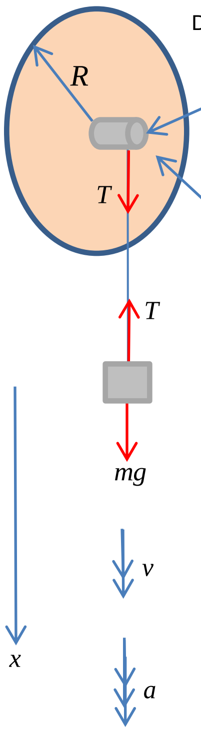
$$I = \frac{2M}{R^2}\tfrac{1}{4}R^4$$

$$\boxed{I = \tfrac{1}{2}MR^2}$$

Note this will be the same formula for the moment of inertia of a **disc** about a rotation axis perpendicular to the plane of the disc, and through the disc centre.

*Which must have the same moment of inertia as an 'extruded hoop' i.e. a cylindrical tube of mass $M$, since it comprises of identical 'hooplets' with the same moment of inertia.

## Spin up spool experiment




Disc, mass $M$ and radius $R$

Cylinder, mounted in disc. Mass $M_c$, radius $r$

Note disc has a *hole* for the reel to fit

Mass $m$, connected to spool via light inextensible string

**Newton II for mass** $m$

$$ma = mg - T$$

Now assume the disc can vary in radius, but the same density and thickness of material is used. i.e. a **constant mass per unit area** $\rho$

$$\therefore M = \rho\pi R^2 - \rho\pi r^2$$ to account for hole

**Torque = moment of inertia x angular acceleration for spool + disc**

$$Tr = \left(\tfrac{1}{2}M_c r^2 + \tfrac{1}{2}\rho\pi R^2 R^2 - \tfrac{1}{2}\rho\pi r^2 r^2\right)\dot{\omega}$$

to account for hole

Now the 'spooling velocity' must equal $v$ since the string is inextensible

$$v = r\omega \;\; \therefore a = r\dot{\omega}$$

Hence:

$$\dot{\omega} = \frac{a}{r}$$

$$\therefore T = \left(\tfrac{1}{2}M_c r^2 + \tfrac{1}{2}\rho\pi R^4 - \tfrac{1}{2}\rho\pi r^4\right)\frac{a}{r^2}$$

$$\therefore ma = mg - \left(\tfrac{1}{2}M_c r^2 + \tfrac{1}{2}\rho\pi R^4 - \tfrac{1}{2}\rho\pi r^4\right)\frac{a}{r^2}$$

$$\therefore a\left(m + \tfrac{1}{2}M_c + \tfrac{1}{2}\frac{\rho\pi R^4}{r^2} - \tfrac{1}{2}\rho\pi r^2\right) = mg$$

$$\therefore \boxed{a = \frac{g}{1 + \frac{1}{2}\frac{M_c}{m} - \frac{1}{2}\frac{\rho\pi}{m}r^2 + \frac{\rho\pi}{m}\frac{R^4}{r^2}}}$$

Let the mass drop a fixed distance $x$ and let the drop time be $t$

For a given disc and spool, the acceleration is constant. Hence:

$$\boxed{x = \tfrac{1}{2}at^2}$$

$$\therefore \frac{2x}{t^2} = \frac{g}{1 + \frac{1}{2}\frac{M_c}{m} - \frac{1}{2}\frac{\rho\pi}{m}r^2 + \frac{\rho\pi}{m}\frac{R^4}{r^2}}$$

$$\therefore \boxed{t = \sqrt{\frac{2x}{g}}\left(1 + \tfrac{1}{2}\frac{M_c}{m} - \tfrac{1}{2}\frac{\rho\pi r^2}{m} + \frac{\rho\pi R^4}{mr^2}\right)^{\frac{1}{2}}}$$

Therefore a graph of:

$$1 + \tfrac{1}{2}\frac{M_c}{m} - \tfrac{1}{2}\frac{\rho\pi r^2}{m} + \frac{\rho\pi R^4}{mr^2} \quad \text{vs} \quad \frac{t^2}{2x}$$

should be a **straight line**, with gradient $g$

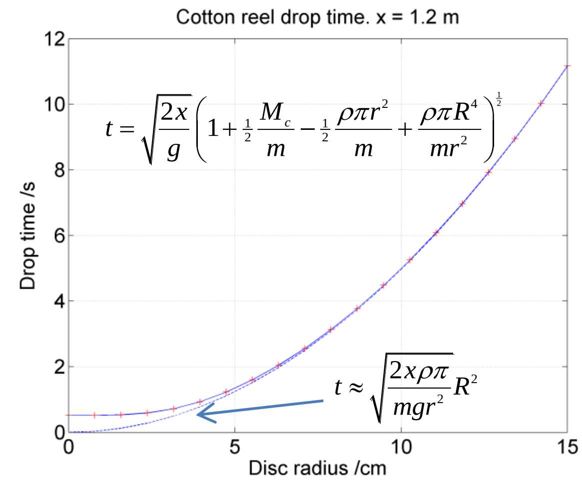
One might expect $\frac{\rho\pi R^4}{mr^2} \gg 1 + \frac{1}{2}\frac{M_c}{m} - \frac{1}{2}\frac{\rho\pi r^2}{m}$

$$\boxed{\therefore t \approx \sqrt{\frac{2x\rho\pi}{mgr^2}}R^2}$$

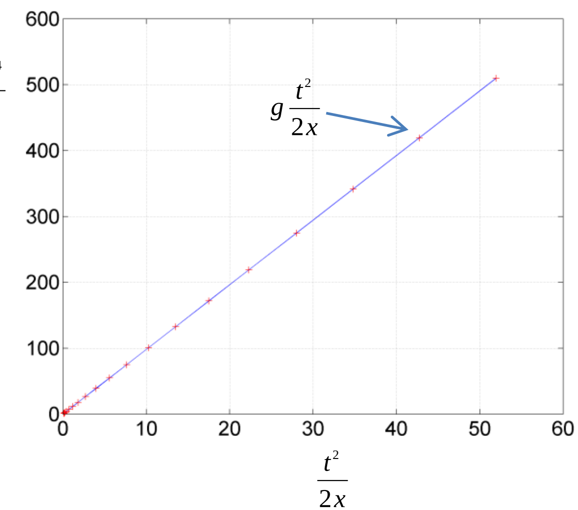
$$k = 1 + \tfrac{1}{2}\frac{M_c}{m} - \tfrac{1}{2}\frac{\rho\pi r^2}{m}$$ 'Correction'

**MATLAB simulation**

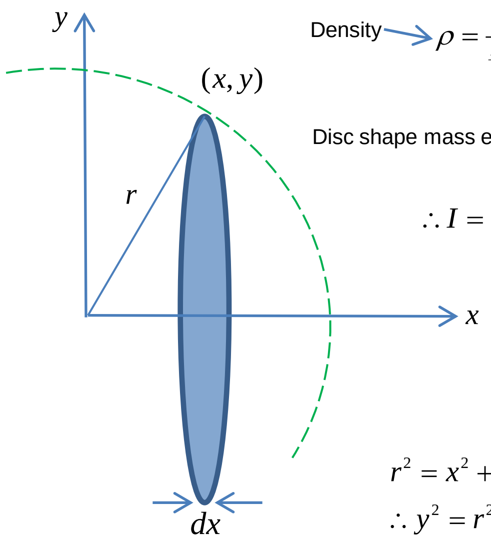
$$M_c = 0.0145\text{kg}$$
$$m = 0.05\text{kg}$$
$$r = 0.013825\text{m}$$
$$\rho = \frac{0.1627\text{kg} - M_c}{\pi\left(0.0125^2 - r^2\right)}$$
$$\rho \approx 3.06\text{kgm}^{-2}$$
$$g = 9.81\text{ms}^{-2}$$
$$x = 1.20\text{m}$$




$$1 + \tfrac{1}{2}\frac{M_c}{m} - \tfrac{1}{2}\frac{\rho\pi r^2}{m} + \frac{\rho\pi R^4}{mr^2} \approx \frac{\rho\pi R^4}{mr^2}$$



## Moment of inertia of a solid sphere




Density $\rightarrow \rho = \dfrac{M}{\frac{4}{3}\pi r^3}$ ← mass, ← radius

Disc shape mass element $dm = \rho \pi y^2 dx$

$$\therefore I = 2\int_{x=0}^{r} \tfrac{1}{2}\rho \pi y^2 dx \times y^2$$

i.e. two hemispheres

Moment of inertia of a disc of mass $m$ and radius $y$ is

$$I = \tfrac{1}{2} m y^2$$

$r^2 = x^2 + y^2$ Pythagoras' Theorem

$$\therefore y^2 = r^2 - x^2$$

$$\therefore y^4 = r^4 - 2x^2r^2 + x^4$$

$$I = \rho\pi \int_{x=0}^{r} y^4 dx$$

$$I = \tfrac{3}{4}\frac{M}{r^3}\int_{x=0}^{r} y^4 dx$$

$$I = \tfrac{3}{4}\frac{M}{r^3}\int_{x=0}^{r} \left(r^4 - 2x^2r^2 + x^4\right) dx$$

$$I = \tfrac{3}{4}\frac{M}{r^3}\left[r^4 x - \tfrac{2}{3}x^3 r^2 + \tfrac{1}{5}x^5\right]_0^r$$

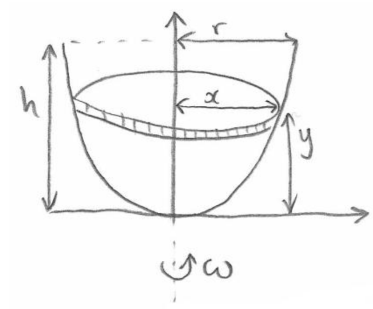
$$I = \tfrac{3}{4}Mr^2\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)$$

$$\boxed{I = \tfrac{2}{5}Mr^2}$$

## Moment of inertia of a parabolic cap

$$\boxed{y = \frac{h}{r^2}x^2}$$

$V = \int_0^h \pi x^2 dy$ Volume




$$x^2 = \frac{yr^2}{h}$$

$$\therefore V = \frac{\pi r^2}{h}\int_0^h y\,dy$$

$$\therefore V = \frac{\pi r^2}{h}\left[\tfrac{1}{2}y^2\right]_0^h$$

$$\therefore \boxed{V = \tfrac{1}{2}\pi r^2 h}$$

$$\rho = \frac{M}{V}$$

$\boxed{\rho = \dfrac{2M}{\pi r^2 h}}$ Density

$$\therefore I = \int_{y=0}^{h} \tfrac{1}{2}\rho\pi x^2 dy \times x^2$$

i.e. sum of moments of inertia of cylindrical mass elements

$$I = \tfrac{1}{2}\rho\pi\int_{y=0}^{h}\left(x^2\right)^2 dy$$

$$I = \tfrac{1}{2}\frac{2M}{\pi r^2 h}\pi\int_{y=0}^{h}\left(\frac{yr^2}{h}\right)^2 dy$$

$$I = \frac{M}{r^2 h}\frac{r^4}{h^2}\int_{y=0}^{h} y^2 dy$$

$$I = \frac{Mr^2}{h^3}\tfrac{1}{3}h^3$$

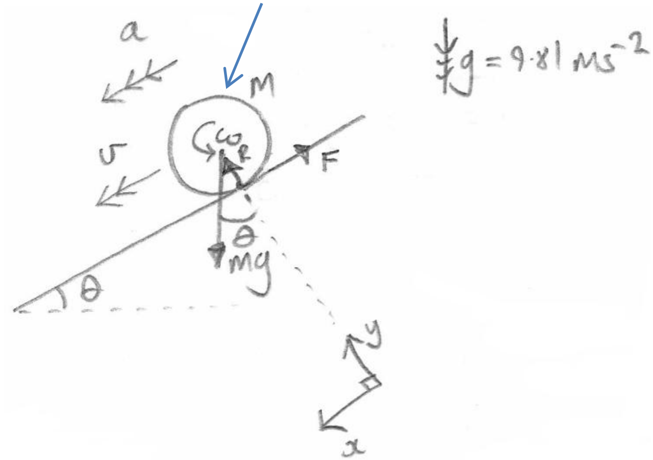
$$\boxed{I = \tfrac{1}{3}Mr^2}$$

## Rolling a cylinder (or a sphere) down a slope

If the cylinder (or sphere) is *not* slipping

$$v = r\omega \quad \therefore a = \frac{dv}{dt} = r\dot{\omega}$$

cylinder or sphere of mass $m$ and radius $r$




friction force

Newton II $\quad x: \; ma = mg\sin\theta - F$

$\qquad\qquad\; y: \; 0 = R - mg\cos\theta$

Normal contact force

**"torque = moment of inertia x angular acceleration"**

$$Fr = I\dot{\omega}$$

$$\therefore Fr = I\frac{a}{r}$$

$$\therefore F = \frac{Ia}{r^2}$$

Hence using Newton II:

$$a = g\sin\theta - \frac{Ia}{mr^2}$$

$$a\left(1 + \frac{I}{mr^2}\right) = g\sin\theta$$

$$\therefore \boxed{a = \frac{g\sin\theta}{1 + \frac{I}{mr^2}}}$$

For no slip:


$F \le \mu R \qquad$ coefficient of friction $\mu$

$$\frac{Ia}{r^2} \le \mu mg\cos\theta$$

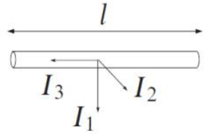
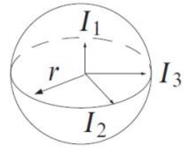
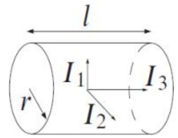
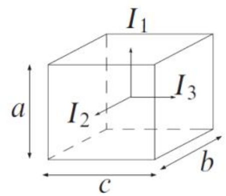
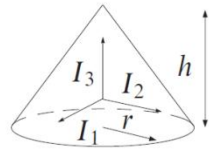
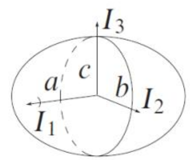
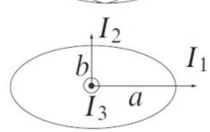
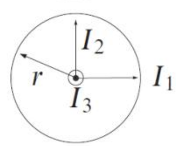
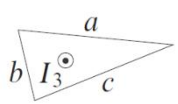
$$\frac{I}{r^2}\frac{g\sin\theta}{1 + \frac{I}{mr^2}} \le \mu mg\cos\theta$$

$$\mu \ge \frac{I}{mr^2}\frac{\tan\theta}{1 + \frac{I}{mr^2}}$$

$$\boxed{\mu \ge \frac{\tan\theta}{\frac{mr^2}{I} + 1}}$$

| | | | |
|---|---|---|---|
| **Cylinder:** | $I = \frac{1}{2}mr^2$ | $\therefore a = \frac{2}{3}g\sin\theta$ | $\mu \ge \frac{1}{3}\tan\theta$ |
| **Sphere:** | $I = \frac{2}{5}mr^2$ | $\therefore a = \frac{5}{7}g\sin\theta$ | $\mu \ge \frac{2}{7}\tan\theta$ |

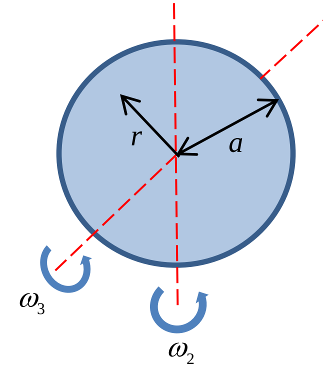
Note a particle sliding down the slope without friction would accelerate at $a = g\sin\theta$

| | | |
|---|---|---|
| Thin rod, length $l$ | $I_1 = I_2 = \dfrac{ml^2}{12}$ <br> $I_3 \simeq 0$ |   |
| Solid sphere, radius $r$ | $I_1 = I_2 = I_3 = \dfrac{2}{5}mr^2$ |   |
| Spherical shell, radius $r$ | $I_1 = I_2 = I_3 = \dfrac{2}{3}mr^2$ | |
| Solid cylinder, radius $r$, length $l$ | $I_1 = I_2 = \dfrac{m}{4}\left(r^2 + \dfrac{l^2}{3}\right)$ <br> $I_3 = \dfrac{1}{2}mr^2$ |   |
| Solid cuboid, sides $a,b,c$ | $I_1 = m(b^2+c^2)/12$ <br> $I_2 = m(c^2+a^2)/12$ <br> $I_3 = m(a^2+b^2)/12$ |   |
| Solid circular cone, base radius $r$, height $h$[b] | $I_1 = I_2 = \dfrac{3}{20}m\left(r^2 + \dfrac{h^2}{4}\right)$ <br> $I_3 = \dfrac{3}{10}mr^2$ |   |
| Solid ellipsoid, semi-axes $a,b,c$ | $I_1 = m(b^2+c^2)/5$ <br> $I_2 = m(c^2+a^2)/5$ <br> $I_3 = m(a^2+b^2)/5$ |   |
| Elliptical lamina, semi-axes $a,b$ | $I_1 = mb^2/4$ <br> $I_2 = ma^2/4$ <br> $I_3 = m(a^2+b^2)/4$ |   |
| Disk, radius $r$ | $I_1 = I_2 = mr^2/4$ <br> $I_3 = mr^2/2$ |   |
| Triangular plate[c] | $I_3 = \dfrac{m}{36}(a^2+b^2+c^2)$ |   |

**Moments of Inertia** about principal axes for various basic shapes
From Woan, *The Cambridge Handbook of Physics Formulas.*

Note the convention $I_1, I_2, I_3$ has the same meaning as $I_x, I_y, I_z$

**Example: Calculate the moment of inertia of a disc about its symmetry axes**




**Out of the plane**

$$\rho = \frac{M}{\pi a^2}$$

$$I_3 = \int r^2 dm$$

d$m$ is a hoop of width d$r$ and radius $r$, of density $\rho$

$$I_3 = \int_{r=0}^{a} r^2 \times \rho \times 2\pi r dr$$

$$I_3 = \frac{2M}{a^2}\int_{r=0}^{a} r^3 dr$$

$$I_3 = \frac{2M}{4a^2}a^4$$

$$\boxed{I_3 = \tfrac{1}{2}Ma^2}$$

**Lateral or longitudinal flip**

$$I_3 = \int\left(x^2+y^2\right)dm = \int x^2 dm + \int y^2 dm$$

$$\therefore I_3 = I_2 + I_1$$

$$I_1 = I_2$$

Since a disc and symmetric in $x,y$ plane

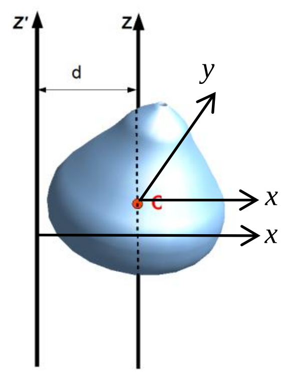
$$\therefore I_2 = \tfrac{1}{2}I_3$$

$$\boxed{I_2 = \tfrac{1}{4}Ma^2}$$

**Perpendicular axis theorem**
This works for *laminae only* i.e. with no '$z$' extent

$$I_1 = \int\left(y^2+z^2\right)dm = \int y^2 dm$$

$$I_2 = \int\left(x^2+z^2\right)dm = \int x^2 dm$$




**Parallel axis theorem**

$$I_z = \int\left(x^2+y^2\right)dm$$

$$I_{z'} = \int\left(x'^2+y^2\right)dm = \int\left(\left(x+d\right)^2+y^2\right)dm$$

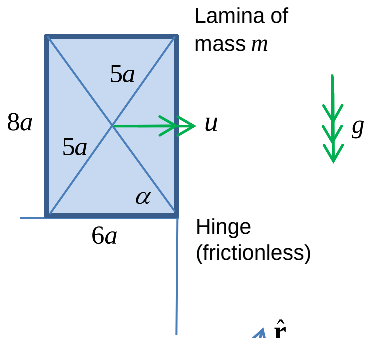
$$I_{z'} = \int\left(x^2+y^2\right)dm + 2d\int xdm + d^2\int dm$$

$$\int xdm = 0$$

Since $C$ is the centre of mass

$$\boxed{I_{z'} = I_z + Md^2}$$

**Rectangular lamina hinge rotation problem – what is the hinge force?**

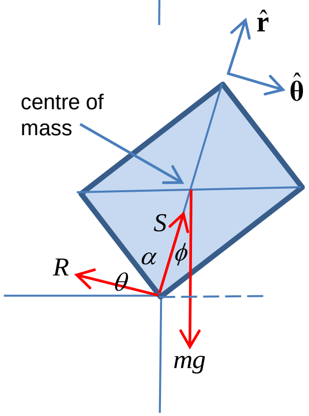



**Initial energy**

$$mg\times 4a+\tfrac{1}{2}mu^2$$

$$\cos\alpha=\tfrac{3}{5}$$

$$\sin\alpha=\tfrac{4}{5}$$




$$\phi=180^\circ-90^\circ-(180^\circ-\theta-\alpha)$$

$$\phi=\theta+\alpha-90^\circ$$

$$\cos\phi=\sin(\theta+\alpha)$$

$$\sin\phi=-\cos(\theta+\alpha)$$

**Newton II**

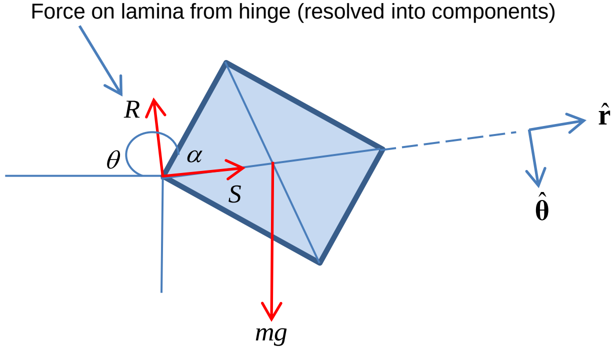
$$\hat{\mathbf{r}}:\ -m(5a)\dot{\theta}^2=S-mg\cos\phi$$

$$\hat{\boldsymbol{\theta}}:\ m(5a)\ddot{\theta}=mg\sin\phi-R$$

**Energy** $\quad mg\times 4a+\tfrac{1}{2}mu^2=mg\times 5a\cos\phi+\tfrac{1}{2}I\dot{\theta}^2$

$$\therefore \dot{\theta}^2=\frac{4mga+\frac{1}{2}mu^2-5mga\cos\phi}{\frac{1}{2}I}$$

Force on lamina from hinge (resolved into components)




**Moment of inertia of a rectangular lamina about hinge**

$$I=\tfrac{1}{12}m\left((6a)^2+(8a)^2\right)+m(5a)^2$$

Moment of inertia of rectangular plate about centre

Parallel axis theorem

$$\therefore \boxed{I=\tfrac{100}{3}ma^2}$$

$$\therefore \dot{\theta}^2=\frac{4mga+\frac{1}{2}mu^2-5mga\cos\phi}{\frac{50}{3}ma^2}$$

$$\therefore \dot{\theta}^2=\tfrac{3}{50}\frac{4ga+\frac{1}{2}u^2-5ga\cos\phi}{a^2}$$

**Torque about hinge = moment of inertia x angular acceleration**

$$5a\times mg\sin\phi=I\ddot{\theta}$$

$$\therefore \ddot{\theta}=\frac{5mga\sin\phi}{I}=\frac{5mga\sin\phi}{\frac{100}{3}ma^2}=\frac{3g\sin\phi}{20a}$$

**Newton II**

$$S=mg\cos\phi-5ma\dot{\theta}^2$$

$$R=mg\sin\phi-5ma\ddot{\theta}$$

$$S=mg\cos\phi-5ma\left(\tfrac{3}{50}\frac{4ga+\frac{1}{2}u^2-5ga\cos\phi}{a^2}\right)$$

$$\boxed{S=\tfrac{5}{2}mg\cos\phi-\tfrac{6}{5}mg-\tfrac{3}{20}\frac{mu^2}{a}}$$

$$R=mg\sin\phi-5ma\left(\frac{3g\sin\phi}{20a}\right)$$

$$\boxed{R=\tfrac{1}{4}mg\sin\phi}$$

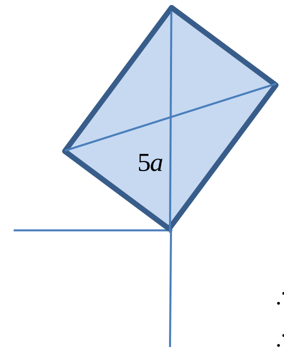
**Magnitude of hinge force**

$$\boxed{F=\sqrt{R^2+S^2}}$$

The initial kinetic energy must exceed the maximum gain in GPE in order for the lamina to continue to fall

$$\tfrac{1}{2}mu^2>mg\times 5a-mg\times 4a$$

$$\therefore \boxed{u>\sqrt{2ga}}$$




Maximum GPE is when centre of mass is directly over the hinge

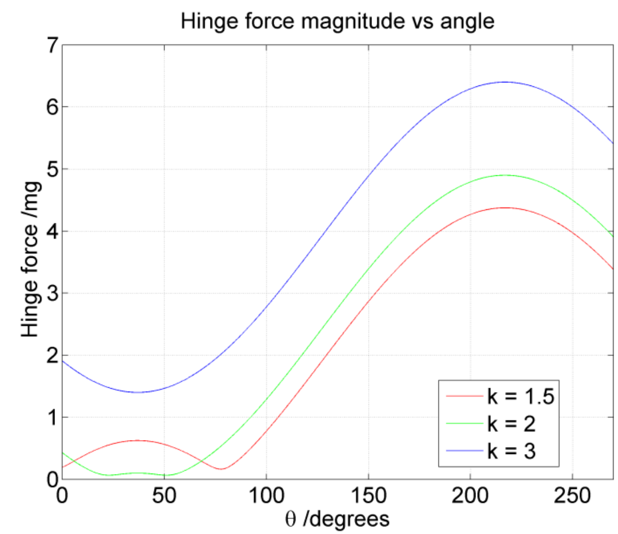
Let $\quad u=k\sqrt{2ga},\quad k>1$

$$\therefore \frac{u^2}{a}=2gk^2$$

$$\therefore S=\tfrac{5}{2}mg\cos\phi-\tfrac{6}{5}mg-\tfrac{3}{10}mgk^2$$

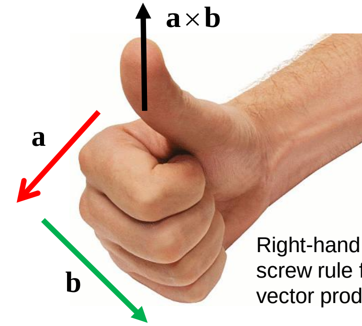
$$\therefore S=\tfrac{1}{10}mg\left(25\cos\phi-12-3k^2\right)$$

$$\boxed{\therefore F=mg\sqrt{\tfrac{1}{16}\cos^2(\theta+\alpha)+\tfrac{1}{100}\left(25\sin(\theta+\alpha)-12-3k^2\right)^2}}$$



The **moment of a force**, or **torque**, is the *vector cross product* of a displacement $\mathbf{r}$ and a force $\mathbf{f}$. $\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}}$ If $\mathbf{r}$ is the displacement from the centre of mass of a rigid body then a net applied torque will result in a rotation of the body about an axis through its centre of mass. Note unless torque is applied via two equal and opposite direction forces (this is called a **couple**) the centre of mass will also accelerate due to the net force upon the body.

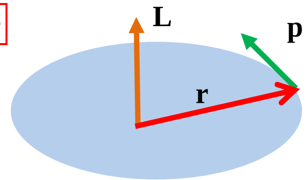
If $\mathbf{r}$ and $\mathbf{f}$ are perpendicular, then the magnitude of the torque (or **turning moment**) is **force x distance from the rotation axis.**



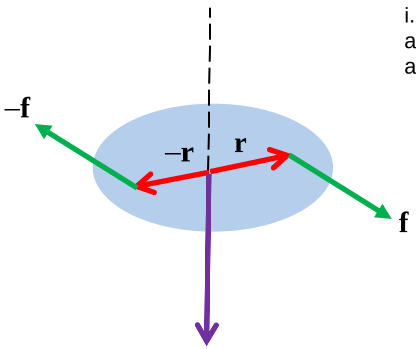

Right-hand screw rule for vector products

**Angular momentum** is defined as $\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$

i.e. the cross product between the momentum of a particle and a displacement vector from an axis of rotation




Complex rotational motion (e.g. 'tumbling') results when the direction of the angular momentum vector *changes*, as well as it's magnitude. e.g. in a **gyroscope.**




$\boldsymbol{\tau} = 2\mathbf{r} \times \mathbf{f}$

A **couple** applied to a rigid body creates torque $\boldsymbol{\tau} = 2\mathbf{r} \times \mathbf{f}$

This will not cause any translational movement of the centre of mass of the body but will cause it to rotate.

For a rigid body $\boxed{\mathbf{L} = \mathbf{I}\boldsymbol{\omega}}$

$\mathbf{I}$ is the **moment of inertia tensor** and $\boldsymbol{\omega}$ is the angular velocity vector which describes the *instantaneous rate of rotation* about an axis parallel to $\boldsymbol{\omega}$.

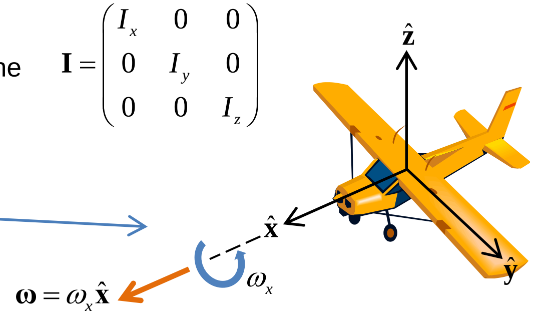
With respect to Cartesian $x,y,z$ coordinates:

$$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2)\, dm & -\int xy\, dm & -\int xz\, dm \\ -\int xy\, dm & \int (x^2 + z^2)\, dm & -\int yz\, dm \\ -\int xz\, dm & -\int yz\, dm & \int (x^2 + y^2)\, dm \end{pmatrix}$$

The **inertia tensor** can often be simplified if one defines Cartesian coordinates to match the **symmetry axes** of a rigid body

$$\mathbf{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

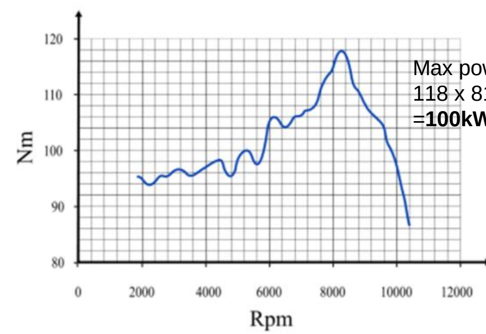
In this example, the aircraft is executing a *roll* (and not a *pitch* or *yaw*)




Most engines comprise of a rotating shaft which is then used to drive wheels etc. The **power** is given by $\boxed{P = \boldsymbol{\tau} \cdot \boldsymbol{\omega}}$

and the rotational kinetic energy

$$\boxed{E = \tfrac{1}{2}(\mathbf{I}\boldsymbol{\omega}) \cdot \boldsymbol{\omega} = \tfrac{1}{2} I_x \omega_x^2 + \tfrac{1}{2} I_y \omega_y^2 + \tfrac{1}{2} I_z \omega_z^2}$$




Max power is 118 x 8100 x 0.105 =**100kW**

Analogous to Newton's Second Law, **applied torque equates to the rate of change of angular momentum**

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

Newton's Second Law describes how the centre of mass of a body responds to a force, of the total mass is constant.

$$\boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt}}$$

**Torque** equates to a rate of change of **Angular Momentum.**

Note $\mathbf{I}$ must be constant here!

Typically we are interested in situations when $\mathbf{r}$, $\mathbf{f}$ and $\mathbf{t}$ are all mutually perpendicular. In the top left figure, if the body is a disk of radius $a$ and mass $m$

$$I = \tfrac{1}{2} M a^2 \quad \therefore 2rF = \tfrac{1}{2} M a^2 \frac{d\omega}{dt} \quad \therefore \boxed{\omega = \frac{4rF}{Ma^2} t + \omega_0}$$

1 rpm = $\pi$/30 = 0.105 rads$^{-1}$