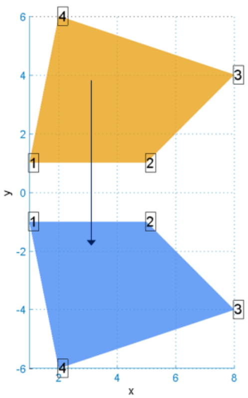
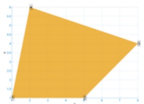
Transforming 2D shapes via matrix multiplication



$$\mathbf{V} = \begin{pmatrix} 1 & 5 & 8 & 2 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$



In this case a reflection in the x axis

$$\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 & 8 & 2 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 & 8 & 2 \\ -1 & -1 & -4 & -6 \end{pmatrix}$$

Inverse transformations

$$\mathbf{M}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$ Identity

$$\det(\mathbf{M}) \equiv |\mathbf{M}| = ad - bc$$ Determinant

Link between the transformation matrix and **basis vectors**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

The columns of the transformation matrix are where the **basis vectors** (1,0) and (0,1) go under the transformation

Reflection in the x axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflection in the x or y axes

Reflection in the x axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflection in the y axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Reflection in the lines y = x or y = -x

Reflection in the line y = x

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Reflection in the line y = -x

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Asymmetric stretch

1 1 Stretch by *a* in x and *b* in y directions, from (0,0) b a

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Note a negative stretch is a stretch in the opposite direction

1 1 Stretch by scale factors -1 in x and 2 in y directions, from (0,0) 2 1

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Enlargement

1 1 Enlargement Scale Factor *k* from (0,0) k k

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Rotation about (0,0) by multiples of 90°

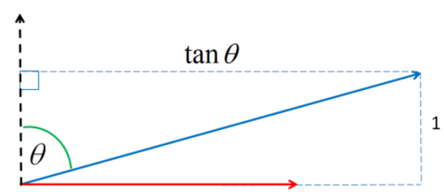
Rotation clockwise of 90° about (0,0)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Rotation anticlockwise of 90° about (0,0)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Shear by angle θ



$$\mathbf{S} = \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$$

The *shear factor* is often specified

$$k = \tan\theta$$

Proof of the inverse of a 2 x 2 matrix

The defining property of an *inverse matrix* is that multiplication of a matrix by its inverse yields the *identity*. We can use this fact to determine the four elements of the inverse matrix in terms of the elements *a,b,c,d* of the original matrix.

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

i.e. we don't yet know the relationship between *a,b,c,d* so lets use some different letters!

$|\mathbf{M}| = ad - bc$ is called the *determinant* of the matrix **M**
If it is zero then there is *no inverse* to a 2 x 2 matrix.

$$\boxed{\mathbf{MM}^{-1} = \mathbf{I}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} a\alpha + b\gamma &= 1 \quad (1) \\ c\alpha + d\gamma &= 0 \quad (2) \\ a\beta + b\delta &= 0 \quad (3) \\ c\beta + d\delta &= 1 \quad (4) \end{aligned}}$$

To find the Greek letters in terms of *a,b,c,d* we need to solve (1)....(4) *simultaneously*. Although there are no specific numbers (i.e. it is all algebra) we can use the same methods as we use to find the *intersection of straight lines of the form y = mx + c*, since every equation is *linear* in the unknown (Greek) letter.

$$a(2) - c(1): \quad ad\gamma - bc\gamma = -c$$

$$\therefore \boxed{\gamma = \frac{-c}{ad-bc}}$$

This is a nifty piece of 'fraction action!' Replace the 1 by (ad-bc)/ad-bc). It is still one, but now will enable factorization.

$$\text{In (1)}: \quad \alpha = \frac{1-b\gamma}{a} = \frac{1 - b\left(\frac{-c}{ad-bc}\right)}{a} = \frac{\frac{ad-bc}{ad-bc} - b\left(\frac{-c}{ad-bc}\right)}{a}$$

$$\alpha = \frac{1}{ad-bc}\,\frac{ad-bc+bc}{a}$$

$$\therefore \boxed{\alpha = \frac{d}{ad-bc}}$$

$$a(4) - c(3): \quad ad\delta - bc\delta = a$$

$$\therefore \boxed{\delta = \frac{a}{ad-bc}}$$

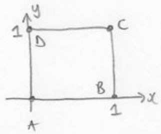
$$\text{In (3)}: \quad \beta = \frac{-b\delta}{a} = \frac{-b\left(\frac{a}{ad-bc}\right)}{a}$$

$$\therefore \boxed{\alpha = \frac{-b}{ad-bc}}$$

$$\therefore \boxed{\mathbf{M}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

"Proof" that if a shape is transformed by pre-multiplication by matrix $\underline{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its area will be preserved if $\det(\underline{T}) \equiv |\underline{T}|$ $= ad - bc = 1$.

Consider a square

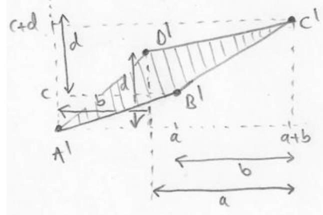


Area = 1 unit^2

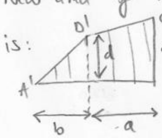
with vertices $\underline{V} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\quad\quad\quad\quad\quad\quad$ A B C D

$$\underline{T}\,\underline{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{array}{cccc} \begin{pmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{pmatrix} \\ A' \quad B' \quad C' \quad D' \end{array}$$

For illustrative purposes, let $a, b, c, d > 0$

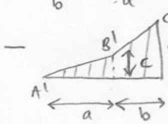


New area of A'B'C'D' is:



(I)

−



(II)

Now $(\mathrm{I}) = \frac{1}{2}bd + \frac{1}{2}(c+2d)a$

$(\mathrm{II}) = \frac{1}{2}ac + \frac{1}{2}(d+2c)b$

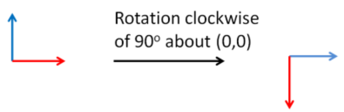
$\therefore \quad (\mathrm{I}) - (\mathrm{II}) = \cancel{\tfrac{1}{2}bd} + \cancel{\tfrac{1}{2}ac} + da - \cancel{\tfrac{1}{2}ac} - \cancel{\tfrac{1}{2}db} - cb$

$= \boxed{ad - bc}$

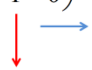
So if $ad - bc = 1 \Rightarrow$ area of transformed square is preserved

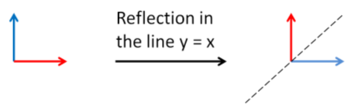
This is not a complete proof, as we have only demonstrated this result for the unit square. However, via a more sophisticated means of finding areas in general, we will find the result holds.

Multiple transformations (1)

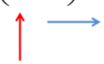


$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

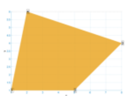




$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

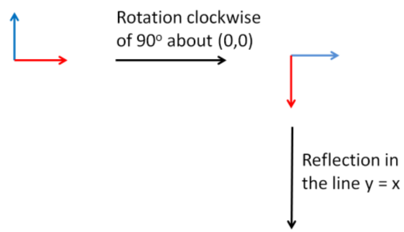


$$\mathbf{V} = \begin{pmatrix} 1 & 5 & 8 & 2 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$

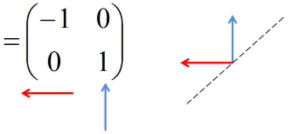


FRV = RFV ?

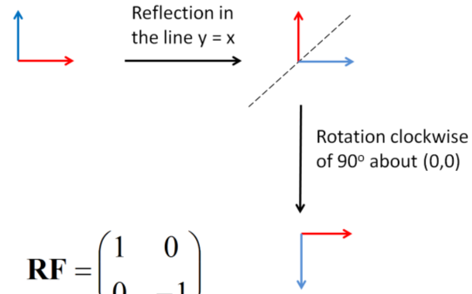
Multiple transformations (2)



$$\mathbf{FR} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Multiple transformations (3)



$$\mathbf{RF} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

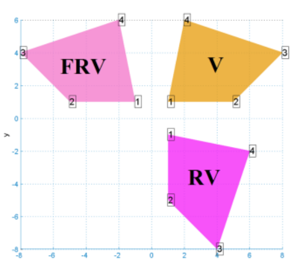
Multiple transformations (4)

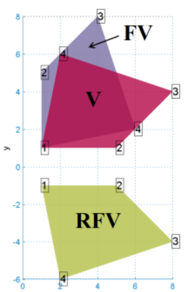
$$\mathbf{FR} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{RF} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So the *order* of the transformations, in general, matters.
Matrix multiplication is NON COMMUTATIVE

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$$





$$\mathbf{FR} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{RF} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$