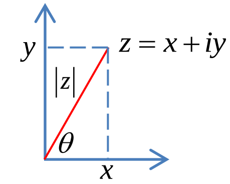
Argand Diagram loci

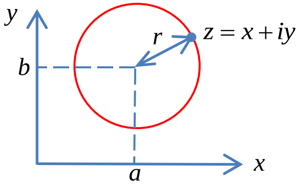
Regions of the complex plane (the Argand Diagram) can be defined via algebraic expressions involving **complex numbers**.

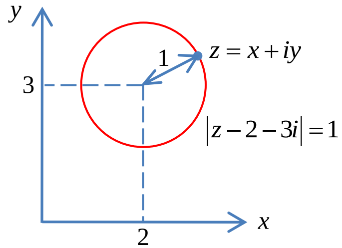
$$\boxed{\begin{aligned} &z = x+iy \\ &|z| = \sqrt{x^2+y^2} \\ &\arg z = \tan^{-1}\tfrac{y}{x} \\ \\ &e^{i\theta} = \cos\theta + i\sin\theta \\ &z = |z|e^{i\theta} \\ &\arg z = \theta \end{aligned}}$$



Circle, centre (a,b), **radius** r

$$\boxed{|z-a-ib| = r}$$





$$|z-a-ib|^2 = r^2$$
$$|x+iy-a-ib|^2 = r^2$$
$$|(x-a)+i(y-b)|^2 = r^2$$
$$\boxed{(x-a)^2+(y-b)^2 = r^2}$$

It is easy to show that this form results in the *Cartesian equation of the circle*

Another similar form of modulus-based equation also yields a **circular locus**

$$\boxed{|z-a-ib| = k|z-c-id|}$$

$$|z-a-ib|^2 = k^2|z-c-id|^2$$
$$(x-a)^2+(y-b)^2 = k^2(x-c)^2+k^2(y-d)^2$$
$$x^2-2ax+a^2+y^2-2by+b^2 = k^2\left(x^2-2cx+c^2+y^2-2dy+d^2\right)$$
$$x^2\left(1-k^2\right)+y^2\left(1-k^2\right)-2x\left(a+k^2c\right)-2y\left(b+k^2d\right)+a^2+b^2-k^2\left(c^2+d^2\right)=0$$
$$x^2+y^2-2x\frac{a+k^2c}{1-k^2}-2y\frac{b+k^2d}{1-k^2}+\frac{a^2+b^2-k^2\left(c^2+d^2\right)}{1-k^2}=0$$
$$\left(x-\frac{a+k^2c}{1-k^2}\right)^2+\left(y-\frac{b+k^2d}{1-k^2}\right)^2=\left(\frac{b+k^2d}{1-k^2}\right)^2+\left(\frac{a+k^2c}{1-k^2}\right)^2-\frac{a^2+b^2-k^2\left(c^2+d^2\right)}{1-k^2}$$

$$\boxed{\textbf{Circle}\text{: centre }\left(\frac{a+k^2c}{1-k^2},\frac{b+k^2d}{1-k^2}\right) \quad \text{radius } \sqrt{\left(\frac{b+k^2d}{1-k^2}\right)^2+\left(\frac{a+k^2c}{1-k^2}\right)^2-\frac{a^2+b^2-k^2\left(c^2+d^2\right)}{1-k^2}}}$$

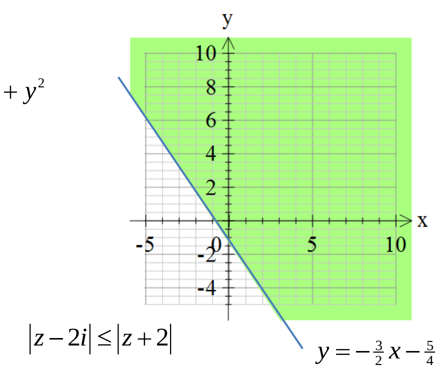
The exception is when $k=1$, which defines a **straight line locus**

$$\boxed{|z-a-ib| = |z-c-id|}$$
$$|z-a-ib|^2 = |z-c-id|^2$$
$$(x-a)^2+(y-b)^2 = (x-c)^2+(y-d)^2$$
$$x^2-2ax+a^2+y^2-2by+b^2 = x^2-2cx+c^2+y^2-2dy+d^2$$
$$x(2c-2a)+a^2+b^2-c^2-d^2 = y(2b-2d)$$
$$y = \frac{c-a}{b-d}x+\frac{a^2+b^2-c^2-d^2}{2b-2d}$$

$$\boxed{\text{Straight line with gradient } \frac{c-a}{b-d} \text{ and } y \text{ intercept } \frac{a^2+b^2-c^2-d^2}{2b-2d}}$$

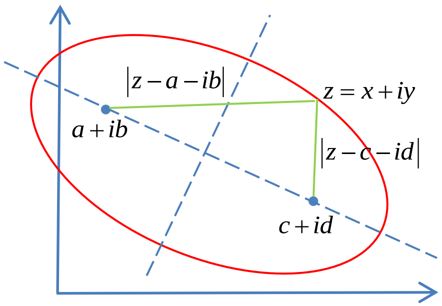
Example: Find the locus defined by $|z-2i| \le |z+3|$

$$|z-2i|^2 \le |z+3|^2$$
$$x^2+(y-2)^2 \le (x+3)^2+y^2$$
$$x^2+y^2-4y+4 \le x^2+6x+9+y^2$$
$$0 \le 4y+6x+5$$
$$y \le \frac{-6x-5}{4}$$
$$\boxed{y \le -\tfrac{3}{2}x-\tfrac{5}{4}}$$



Elliptical loci

$$\boxed{|z-a-ib|+|z-c-id|=k} \qquad k>0$$



i.e. *foci* of the ellipse are the coordinates (a,b) and (c,d)

A defining property of an ellipse is the direct distance from one focus, to a point on the ellipse, to the focus is always a *constant*.

Elliptical geometry

$$\boxed{\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1}$$

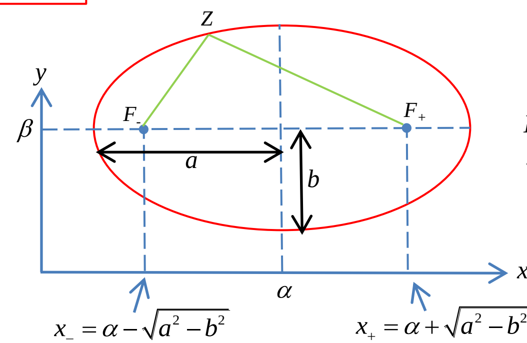
Cartesian equation i.e. a two-way stretch of a unit circle

$a \geq b$

$$\boxed{\begin{array}{l} x_{\pm}=\alpha \pm \sqrt{a^2-b^2} \\ y_{\pm}=\beta \end{array}}$$

Coordinates of *foci*

Eccentricity $\boxed{\varepsilon=\sqrt{1-\frac{b^2}{a^2}}}$ A circle has zero eccentricity since $a=b$



$$F_-Z+ZF_+=2a$$

This is a *defining* property of an ellipse
See below!

Example: Consider the locus defined by $\left|z+\sqrt{a^2-b^2}\right|+\left|z-\sqrt{a^2-b^2}\right|=2a$

Note any ellipse with long axis of symmetry (the "semi-major axis") can be described in this way if we shift the origin of the Argand diagram to the geometric centre of the ellipse.

If the solution is an *ellipse* centred on $(0,0)$ then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\left|z+\sqrt{a^2-b^2}\right|+\left|z-\sqrt{a^2-b^2}\right|=2a$$

$$\sqrt{\left(x+\sqrt{a^2-b^2}\right)^2+y^2}+\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}=2a$$

$$\left(x+\sqrt{a^2-b^2}\right)^2+y^2=\left(2a-\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}\right)^2$$

$$\left(x+\sqrt{a^2-b^2}\right)^2+y^2=4a^2-4a\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}+\left(x-\sqrt{a^2-b^2}\right)^2+y^2$$

$$\left(x+\sqrt{a^2-b^2}\right)^2-\left(x-\sqrt{a^2-b^2}\right)^2-4a^2=-4a\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}$$

$$(2x)\left(2\sqrt{a^2-b^2}\right)-4a^2=-4a\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}$$

$$\left(x\sqrt{a^2-b^2}-a^2\right)^2=a^2\left(\left(x-\sqrt{a^2-b^2}\right)^2+y^2\right)$$

$$\left(x\sqrt{a^2-b^2}-a^2\right)^2=a^2\left(\left(x-\sqrt{a^2-b^2}\right)^2+y^2\right)$$

$$x^2\left(a^2-b^2\right)+a^4-2xa^2\sqrt{a^2-b^2}=a^2\left\{x^2-2x\sqrt{a^2-b^2}+a^2-b^2+y^2\right\}$$

$$a^2x^2-b^2x^2+a^4-2xa^2\sqrt{a^2-b^2}=a^2x^2-2xa^2\sqrt{a^2-b^2}+a^4-a^2b^2+a^2y^2$$

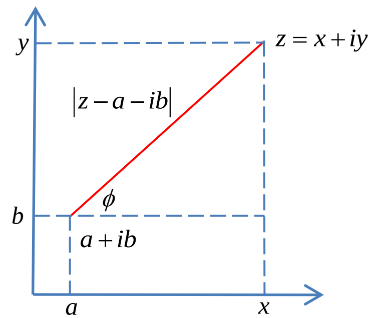
$$a^2b^2=b^2x^2+a^2y^2$$

$$1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

So the locus is indeed an ellipse centred on $(0,0)$

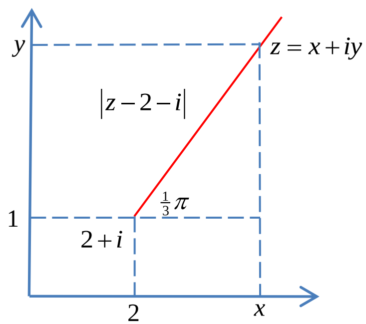
Straight line segments

$\boxed{\arg(z-a-ib)=\phi}$



Example: Sketch the locus of the equation

$\arg(z-2-i)=\frac{1}{3}\pi$



Intersection of line segment and circle example

Find z such that

$|z-1-i|=1$

$\arg(z+i)=\frac{\pi}{4}$

Cartesian equations of intersecting loci are:

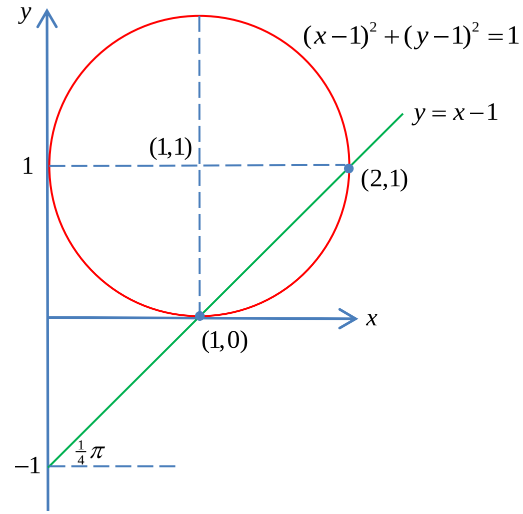
$|z-1-i|=1$

$\boxed{(x-1)^2+(y-1)^2=1}$

$\arg(z+i)=\frac{\pi}{4}$

$y=x\tan\frac{\pi}{4}-1$

$\boxed{y=x-1}$



Substituting for y

$(x-1)^2+(y-1)^2=1$

$(x-1)^2+(x-2)^2=1$

$x^2-2x+1+x^2-4x+4=1$

$2x^2-6x+4=0$

$x^2-3x+2=0$

$(x-1)(x-2)=0$

$\therefore x=1,2$

$\therefore y=0,1$

Hence solutions to $|z-1-i|=1$, $\arg(z+i)=\frac{\pi}{4}$ are $\boxed{z=1}$, $\boxed{z=2+i}$

Circle tangents example

What are the range of values of $\arg(z-4+2i)$ such that $|z-2-3i|=2$?

From diagram $\phi_{\min}\leq\arg(z-4+2i)\leq\phi_{\max}$

i.e. range of possible intersections of line segments from $(4,-2)$ with the circle are bounded by the tangents to the circle.

Pythagoras: $OA=\sqrt{2^2+5^2}=\sqrt{29}$

$\sqrt{29}\sin\theta=2$

$\therefore\theta=\sin^{-1}\left(\frac{2}{\sqrt{29}}\right)$

$180^\circ-2\theta-\phi_{\min}=\tan^{-1}\frac{5}{2}$

$\therefore\phi_{\min}=180^\circ-2\sin^{-1}\left(\frac{2}{\sqrt{29}}\right)-\tan^{-1}\frac{5}{2}\approx\boxed{68.2^\circ}$

$\phi_{\max}=\phi_{\min}+2\theta$

$\therefore\phi_{\max}=180^\circ-\tan^{-1}\frac{5}{2}\approx\boxed{111.8^\circ}$

$|z-2-3i|=2$

$\therefore(x-2)^2+(x-3)^2=4$