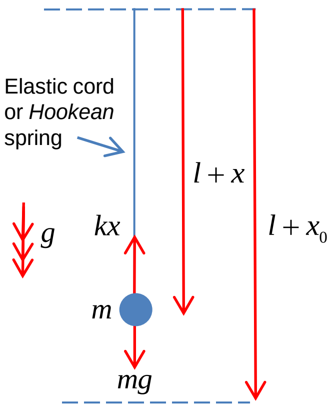
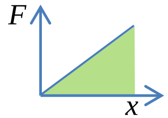
Hookean springs

If a spring or elastic cord obeys **Hooke's Law**, then the restoring force experienced is in direct proportion to the amount it is stretched x beyond its natural length l





$$\boxed{F = kx = \lambda \frac{x}{l}}$$

k is the spring constant
λ is the elastic modulus

$$\boxed{\lambda = kl}$$

When a spring is stretched, the **work done** to achieve this is

$$W = \int_0^x F dx = \int_0^x kx dx = \tfrac{1}{2}kx^2$$

i.e. the area of the green triangle!

The **potential energy** in a stretched spring is therefore

$$\boxed{\tfrac{1}{2}kx^2}$$

If the spring hangs in equilibrium $mg = kx_0 \Rightarrow x_0 = \dfrac{mg}{k}$

The equilibrium displacement from the hanging point of the mass is therefore $l + \dfrac{mg}{k}$

The total energy of the mass-spring system above is:

$$E = \tfrac{1}{2}m\dot{x}^2 + mg\left(l + \frac{mg}{k} - l - x\right) + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\dot{x}^2 + mg\left(\frac{mg}{k} - x\right) + \tfrac{1}{2}kx^2$$

$$E = \tfrac{1}{2}m\dot{x}^2 + mg\left(\frac{mg}{k} - x\right) + \tfrac{1}{2}kx^2$$

$$\dot{E} = m\dot{x}\ddot{x} - mg\dot{x} + kx\dot{x}$$

differentiate with respect to time

$$\dot{E} = m\dot{x}\left(\ddot{x} - g + \frac{kx}{m}\right)$$

Assume system is lossless

$$\therefore \dot{E} = 0$$

$$\Rightarrow \ddot{x} - g + \frac{kx}{m} = 0$$

$$x = z + \frac{mg}{k}$$

Define a new displacement, from the equilibrium position

$$\ddot{x} = \ddot{z}$$

$$\ddot{x} - g + \frac{kx}{m} = 0$$

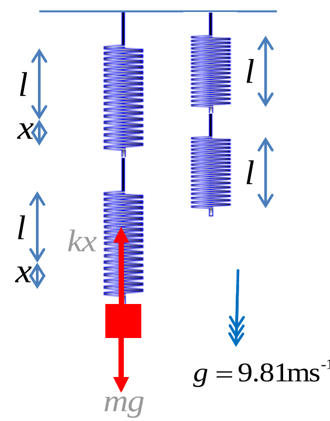
$$\Rightarrow \ddot{z} - g + \frac{k}{m}\left(z + \frac{mg}{k}\right) = 0$$

$$\therefore \boxed{\ddot{z} = -\frac{k}{m}z}$$

$$\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta$$

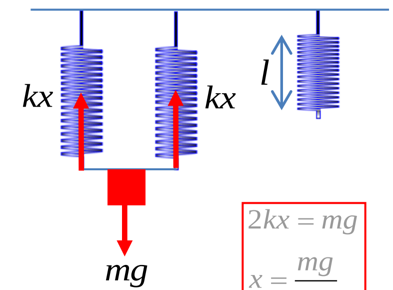
Compare to Simple Harmonic Motion (SHM)

Series springs – each stretches the *same*



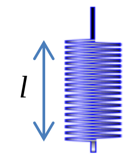
$$\boxed{kx = mg \quad \therefore x = \frac{mg}{k}}$$

Parallel springs – load is *shared*



$$\boxed{2kx = mg \qquad x = \frac{mg}{2k}}$$

Stretch is *half* the amount in this case



Unstretched spring

Robert Hooke
1635-1703
Born in Freshwater
Isle of Wight

Hence spring oscillations will have period

$$\left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Note in this case, **Newton's Second Law** can also be used to derive the equation of motion in a fairly straightforward fashion. In this case it is the most efficient method!

$$m\ddot{x} = mg - kx$$

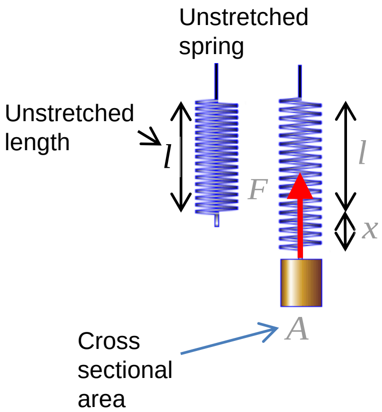
$$x = z + \frac{mg}{k}$$

z is the displacement from equilibrium

$$\therefore m\ddot{z} = mg - k\left(z + \frac{mg}{k}\right)$$

$$\boxed{\ddot{z} = -\frac{k}{m}z}$$

Stress and strain



$$\text{stress} = \frac{\text{force}}{\text{area}}$$

$$\sigma = \frac{F}{A}$$

$$\text{strain} = \frac{\text{extension}}{\text{original length}}$$

$$\varepsilon = \frac{x}{l}$$

Young's Modulus

Stress

$$Y = \frac{\sigma}{\varepsilon}$$

Strain

$$\therefore \sigma = Y\varepsilon$$

Elastic strain energy per unit volume

$$U = \frac{E}{Al} = \frac{\frac{1}{2}kx^2}{Al}$$

$$U = \tfrac{1}{2}\frac{kx}{A} \times \frac{x}{l}$$

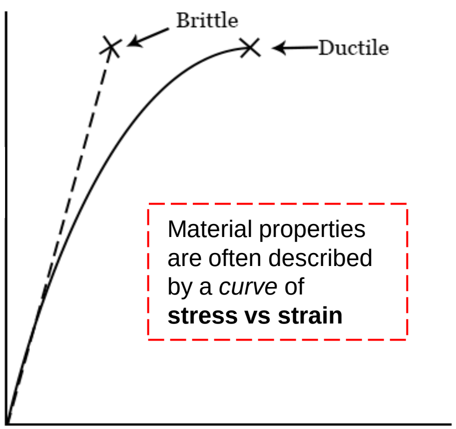
$$U = \tfrac{1}{2} \times \frac{\text{force}}{\text{area}} \times \frac{\text{extension}}{\text{original length}}$$

$$U = \tfrac{1}{2}\text{stress} \times \text{strain}$$

$$\boxed{U = \tfrac{1}{2}Y\varepsilon^2} \longleftarrow \sigma = Y\varepsilon$$

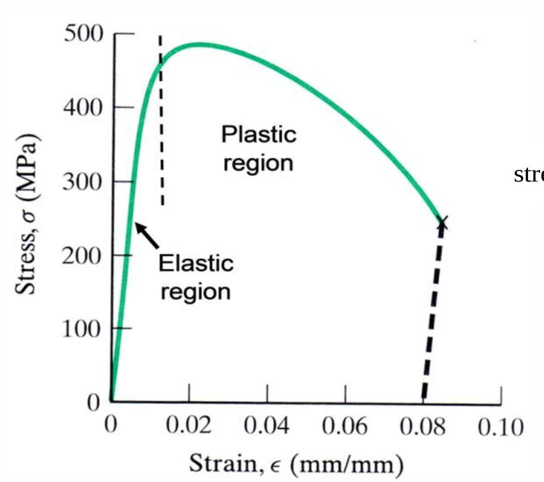
The energy U stored by a system undergoing deformation, per unit volume. When the load is removed, strain energy is released as the system returns to its original shape.

$$\sigma = \text{stress} = \frac{\text{force}}{\text{area}}$$



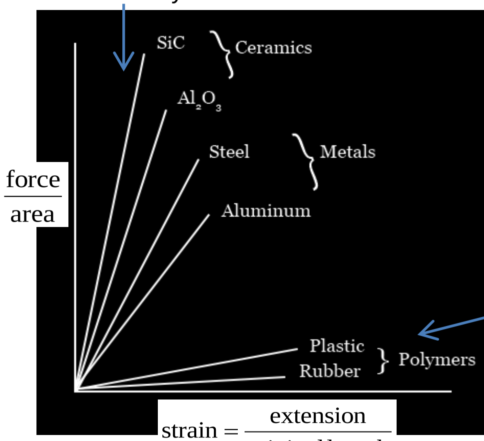
Material properties are often described by a *curve* of **stress vs strain**

$$\varepsilon = \text{strain} = \frac{\text{extension}}{\text{original length}}$$



High Young's Modulus i.e. very **stiff**

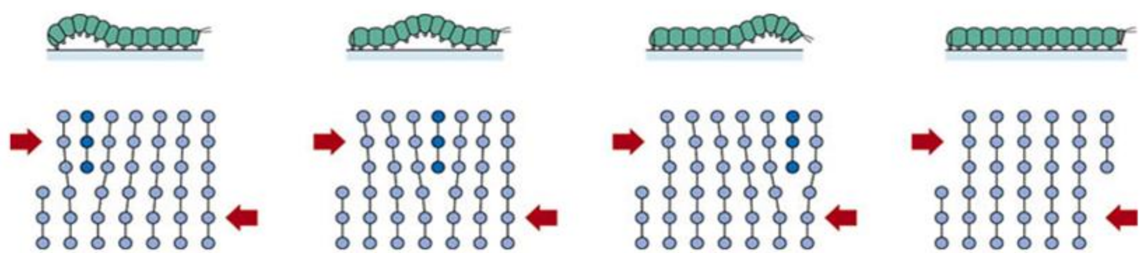
$$\text{stress} = \frac{\text{force}}{\text{area}}$$



$$\text{strain} = \frac{\text{extension}}{\text{original length}}$$

Low Young's Modulus i.e. **elastic** or '**stretchy**'

For small strains we often have a linear, *elastic* region. i.e. force is proportional to extension or stress/strain is a constant. The constant is called the **Young's Modulus**

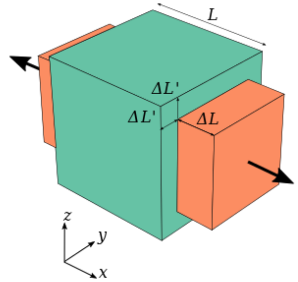


A *slip* in a material can be described by the movement of a **dislocation** in the atomic structure.

Poisson's ratio

$$\nu = -\frac{\text{transverse strain change}}{\text{axial strain change}} \approx \frac{\Delta L'}{\Delta L}$$

i.e. most materials will shrink *transversely* as they are stretched. This is a *positive* Poisson ratio.



Young's Modulus

$$Y = \frac{\sigma}{\varepsilon}$$

Stress

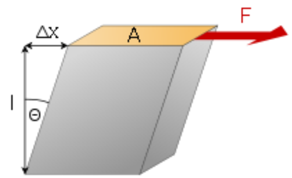
Strain

For *isotropic* materials

$$Y = 2G(1+\nu)$$

Shear modulus

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{Fl}{A\Delta x}$$



Note fluids and gases *flow* rather than shear.
So a fluid or gas will have a shear modulus of zero.

Speed of sound in elastic solids

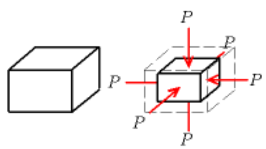
$$c_p = \sqrt{\frac{K + \frac{4}{3}G}{\rho}} = \sqrt{\frac{Y(1-\nu)}{\rho(1+\nu)(1-2\nu)}}$$ Pressure (P) waves

$$c_s = \sqrt{\frac{G}{\rho}}$$ Shear (S) waves

Note speed of sound in air is about 330ms^{-1} and 1480ms^{-1} in water

Bulk modulus

$$K = -V\frac{dP}{dV} = \rho\frac{dP}{d\rho}$$



This is the *compressibility* of a material i.e. the ratio of a change in pressure P applied to the consequential fractional change in volume V

For *isotropic* materials (i.e. movement in any direction is the *same*, there is no particular direction where the material is weaker or stronger or more stretchy..)

$$K = \frac{Y}{3(1-2\nu)}$$

Typical elastic moduli for materials
(assumed to be isotropic)

$K = \frac{Y}{3(1-2\nu)}$ used where data unavailable

Material	Young's modulus Y /GPa	Poisson ratio ν	Shear modulus G /GPa	Bulk modulus K /GPa	Density ρ /kgm^{-3}	Speed* of sound c_P /ms^{-1}
Rubber	0.01	0.5	0.0006	1	801	1,120
Steel	200	0.3	79.3	160	7800	5,840
Copper	117	0.33	44.7	123	8960	4,510
Plastic	0.5-3	0.3-0.5	0.1	2.9	930	1,810
Concrete	30	0.1-0.2	21	14.3	2400	4,200
Diamond	1050-1210	0.07	478	443	3510	17,540
Wood	11	0.2-0.7	13	36.7	600-900	8,490
Glass	50-90	0.18-0.3	26.2	35-55	2500	5,560

*Assume pressure (P) waves