

## Taylor & Maclaurin expansions

A *Taylor series* is a *polynomial expansion* of a function  $f(x)$  in the vicinity of a particular point  $(x_0, f(x_0))$

The *Maclaurin expansion* is a special case when  $x_0 = 0$

Taylor and Maclaurin expansions are useful as *approximations* to functions. They can enable numeric evaluations of functions such as sine, cosine etc and also integrals that cannot be analytically evaluated.

### Taylor series

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$

$$f^{(n)}(x_0) = \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}$$

### Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

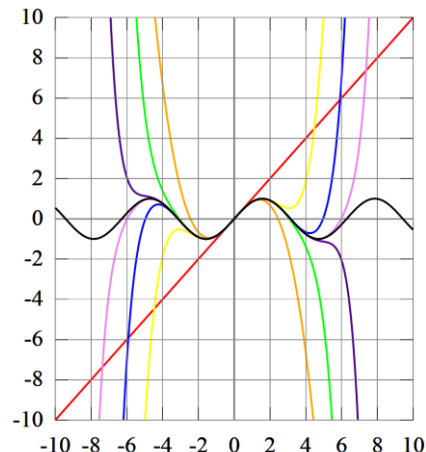
$$f^{(n)}(0) = \left. \frac{d^n f(x)}{dx^n} \right|_{x=0}$$



**Brook Taylor**  
(1685-1731)



**Colin Maclaurin**  
(1698-1746)



Maclaurin series for  $f(x) = \sin x$   
Increased number of terms  
improves the approximation  
further away from  $x = 0$

### Maclaurin expansion for trigonometric functions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

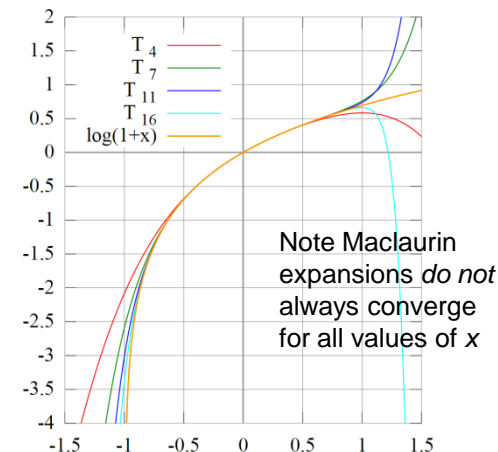
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \quad |x| < \frac{1}{2}\pi$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots \quad |x| < \frac{1}{2}\pi$$

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots \quad |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots \quad |x| < \pi$$



The Maclaurin series for  $f(x) = \ln(1+x)$   
*diverges* outside the range  $[-1, 1]$  regardless of the number of terms in the series. Indeed, adding more terms makes the series approximation worse outside  $[-1, 1]$

### Maclaurin expansion for exponential, logarithmic and hyperbolic functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad -1 < x \leq 1$$

$$\frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \quad |x| < 1$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots \quad |x| < \frac{1}{2}\pi$$

## Maclaurin series of functions defined as integrals

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2} dt$$

Generalized Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1+t^2)^{-1} = 1 + (-1)(t^2) + (-1)(-2)\frac{(t^2)^2}{2!} + (-1)(-2)(-3)\frac{(t^2)^3}{3!} + \dots \quad |t^2| < 1 \Rightarrow |t| < 1$$

$$(1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + \dots \quad |t| < 1$$

$$\int_0^x \frac{1}{1+t^2} dt = \left[ t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \dots \right]_0^x \quad |x| < 1$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad |x| < 1$$

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

$$(1-t^2)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})(-t^2) + (-\frac{1}{2})(-\frac{3}{2})\frac{(-t^2)^2}{2!} + (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\frac{(-t^2)^3}{3!} + \dots \quad |t^2| < 1 \Rightarrow |t| < 1$$

$$(1-t^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}t^2 + \frac{3}{2 \times 4}t^4 + \frac{3 \times 5}{2 \times 4 \times 6}t^6 + \dots \quad |t| < 1$$

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \left[ t + \frac{1}{2} \frac{1}{3}t^3 + \frac{3}{2 \times 4} \frac{1}{5}t^5 + \frac{3 \times 5}{2 \times 4 \times 6} \frac{1}{7}t^7 + \dots \right]_0^x \quad |x| < 1$$

$$\sin^{-1} x = x + \frac{1}{2} \frac{1}{3}x^3 + \frac{1 \times 3}{2 \times 4} \frac{1}{5}x^5 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \frac{1}{7}x^7 + \dots \quad |x| < 1$$

## Summary of Maclaurin expansions of inverse trigonometric functions

Note:

$$\cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x$$

$$\cot^{-1} x = \frac{1}{2}\pi - \tan^{-1} x$$

$$\sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2 \times 4} \frac{x^5}{5} + \frac{1}{2 \times 4 \times 6} \frac{x^7}{7} + \dots \quad |x| < 1$$

$$\tan^{-1} x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots & |x| \leq 1 \\ \frac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots & x > 1 \\ -\frac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots & x < -1 \end{cases}$$

## Approximations to integrals that can't be evaluated analytically

$$f(x) = \int_0^x e^{-\frac{1}{2}t^2} dt \quad \leftarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-\frac{1}{2}t^2} = 1 + \left(-\frac{1}{2}t^2\right) + \frac{1}{2}\left(-\frac{1}{2}t^2\right)^2 + \frac{1}{6}\left(-\frac{1}{2}t^2\right)^3 + \dots$$

$$e^{-\frac{1}{2}t^2} = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{48}t^6 + \dots$$

$$\int_0^x e^{-\frac{1}{2}t^2} dt = x - \frac{1}{2} \frac{1}{3}x^3 + \frac{1}{8} \frac{1}{5}x^5 - \frac{1}{48} \frac{1}{7}x^7 + \dots$$

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$z = \frac{x - \mu}{\sigma}$$

For more general random variable  $x$ , mean  $\mu$  and standard deviation  $\sigma$

This result has an application in *probability theory*. The *Normal Distribution* has a probability density function  $p(z)$  i.e. where the probability of 'normalized' random variable  $z$  being between  $z$  and  $z+dz$  is  $p(z)dz$

The probability of  $z$  being less than  $z^*$  is therefore:

$$P(z < z^*) = \int_{-\infty}^{z^*} p(z) dz$$

$$P(z < z^*) = \begin{cases} \frac{1}{2} + \int_0^{z^*} p(z) dz & z^* > 0 \\ \frac{1}{2} - \int_0^{|z^*|} p(z) dz & z^* < 0 \end{cases}$$

Since  $\int_{-\infty}^{\infty} p(z) dz = 1$  and  $p(z)$  is an even function

$$\int_0^{|z^*|} p(z) dz = |z^*| - \frac{1}{2} \frac{1}{3}|z^*|^3 + \frac{1}{8} \frac{1}{5}|z^*|^5 - \frac{1}{48} \frac{1}{7}|z^*|^7 + \dots$$

This is how tables of the *cumulative Normal distribution* are worked out numerically

$$R_n(x) = \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \quad \text{Define this function}$$

## Proof of the Taylor expansion

$$R_n(x) = \left[ \frac{(x-t)^n}{n!} f^{(n)}(t) \right]_{x_0}^x - \int_{x_0}^x f^{(n)}(t) \times \frac{n(x-t)^{n-1}(-1)}{n(n-1)!} dt \quad \text{Integration by parts}$$

$$R_n(x) = -\frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_{n-1}$$

$$R_n(x) = -\frac{(x-x_0)^n}{n!} f^{(n)}(x_0) - \frac{(x-x_0)^{n-1}}{(n-1)!} f^{(n)}(x_0) - \dots + f(x)$$

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n$$

Expect  $\lim_{n \rightarrow \infty} R_n = 0$