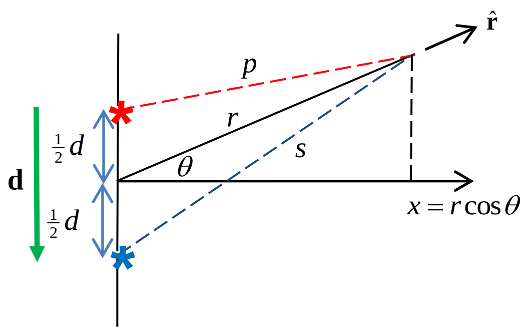
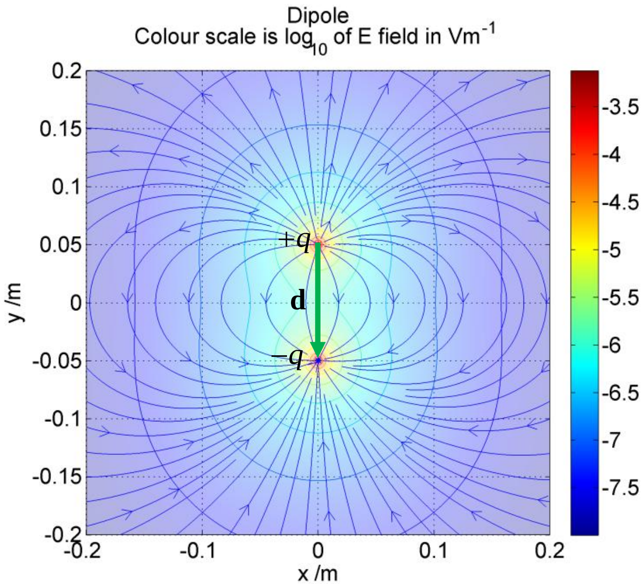
Electric field of a dipole

$$\boxed{\begin{aligned} V(r) &\approx \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2} \\ \mathbf{E}(r) &\approx \frac{3q(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - q\mathbf{d}}{4\pi\varepsilon_0 r^3} \end{aligned}}$$

d is the vector separation of opposing charges of magnitude q





A **dipole** is a pair of opposing charges separated by a vector $\mathbf{d}$. In many scenarios in electromagnetism it is useful to calculate the electric field due to a set of dipoles rather than individual charges.

In this note we will derive the far field results for the electric potential V and electric field $\mathbf{E}$. Far field means the radial distance from the centre of the dipole is much larger than the charge separation d.

Cosine Rule:

$$p^2 = r^2 + \tfrac{1}{4}d^2 - rd\cos(90^\circ - \theta)$$

$$\therefore p^2 = r^2 + \tfrac{1}{4}d^2 - rd\sin\theta$$

$$s^2 = r^2 + \tfrac{1}{4}d^2 - rd\cos(90^\circ + \theta)$$

$$\therefore s^2 = r^2 + \tfrac{1}{4}d^2 + rd\sin\theta$$

Hence:

$$p = r\sqrt{1 + \tfrac{1}{4}\frac{d^2}{r^2} - \frac{d\sin\theta}{r}}$$

$$s = r\sqrt{1 + \tfrac{1}{4}\frac{d^2}{r^2} + \frac{d\sin\theta}{r}}$$

Electric potential is

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{q}{p} - \frac{1}{4\pi\varepsilon_0}\frac{q}{s}$$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{p} - \frac{1}{s}\right)$$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left(\frac{s-p}{ps}\right)$$

$$ps = r^2\sqrt{\left(1 + \tfrac{1}{4}\frac{d^2}{r^2} - \frac{d\sin\theta}{r}\right)\left(1 + \tfrac{1}{4}\frac{d^2}{r^2} + \frac{d\sin\theta}{r}\right)}$$

$$ps = r^2\sqrt{\left(1 + \tfrac{1}{4}\frac{d^2}{r^2}\right)^2 - \frac{d^2\sin^2\theta}{r^2}}$$

Assume $r \gg d \quad \therefore ps \approx r^2$

Binomial expansion:

$$p \approx r + \tfrac{1}{8}\frac{d^2}{r} - \tfrac{1}{2}d\sin\theta$$

$$s \approx r + \tfrac{1}{8}\frac{d^2}{r} + \tfrac{1}{2}d\sin\theta$$

$$\therefore s - p \approx d\sin\theta$$

Hence:

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{d\sin\theta}{r^2}$$

$$\mathbf{d}\cdot\hat{\mathbf{r}} = d\sin\theta$$

$$\therefore \boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2}}$$

$$\mathbf{E} = -\nabla V$$

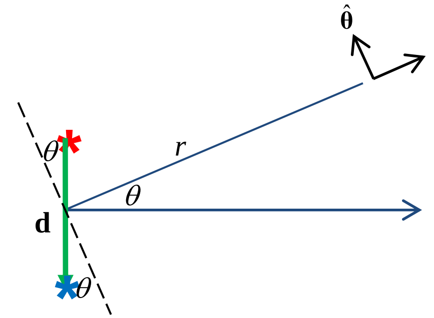
$$\mathbf{E} = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}$$

$$V = \frac{q}{4\pi\varepsilon_0}\frac{d\sin\theta}{r^2}$$

$$\frac{\partial V}{\partial r} = \frac{-2q}{4\pi\varepsilon_0}\frac{d\sin\theta}{r^3}$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^3}$$

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 r^3}\left(2\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta\right)$$



$$\mathbf{d} = -\hat{\boldsymbol{\theta}}d\cos\theta - \hat{\mathbf{r}}d\sin\theta$$

$$\therefore \hat{\boldsymbol{\theta}}d\cos\theta = -\hat{\mathbf{r}}d\sin\theta - \mathbf{d}$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left(2\hat{\mathbf{r}}d\sin\theta - \hat{\boldsymbol{\theta}}d\cos\theta\right)$$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left(3\hat{\mathbf{r}}d\sin\theta - \mathbf{d}\right)$$

$$\mathbf{d}\cdot\hat{\mathbf{r}} = d\sin\theta$$

$$\boxed{\mathbf{E} = \frac{3q(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - q\mathbf{d}}{4\pi\varepsilon_0 r^3}}$$