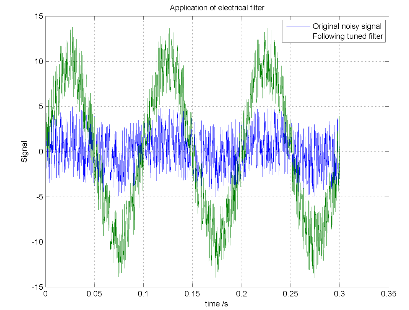
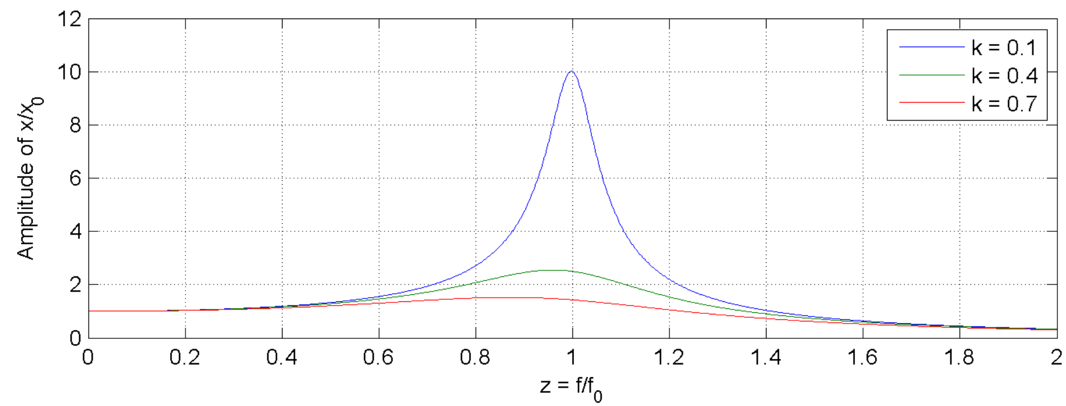
Simple Harmonic Motion (SHM) and Electrical Oscillations

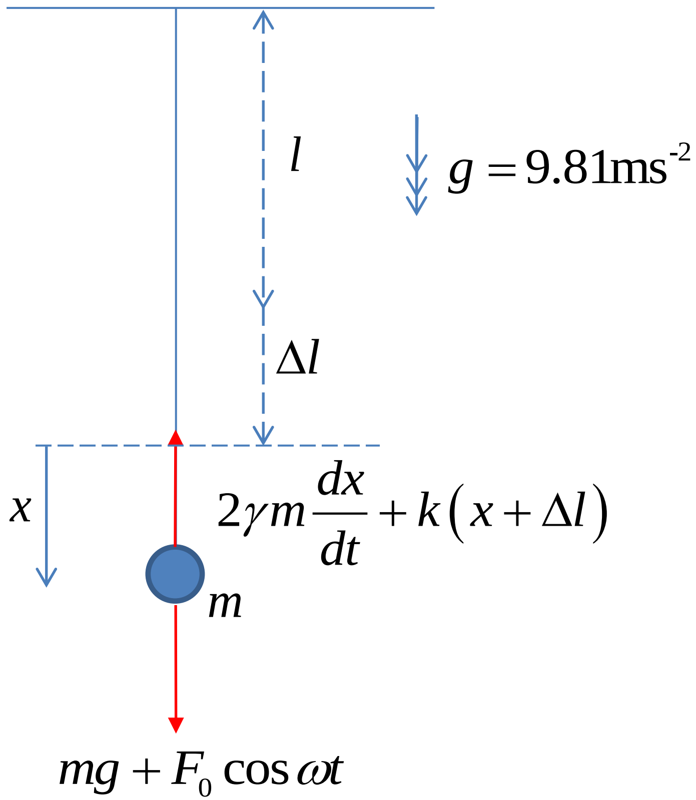


Dr Andrew French



$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 e^{i\omega t}$$

SHM equation from driven mechanical oscillations



$\boxed{\omega = 2\pi f}$ ← oscillation frequency

Consider a particle of mass *m* suspended from a light elastic string from a fixed surface. The string has natural length *l*.

Assume a *Hookean* law of elasticity i.e. restoring force is proportional to extension. The elastic constant in this case is *k*.

Also assume mass is subject to air resistance which is proportional to velocity and mass *m*.

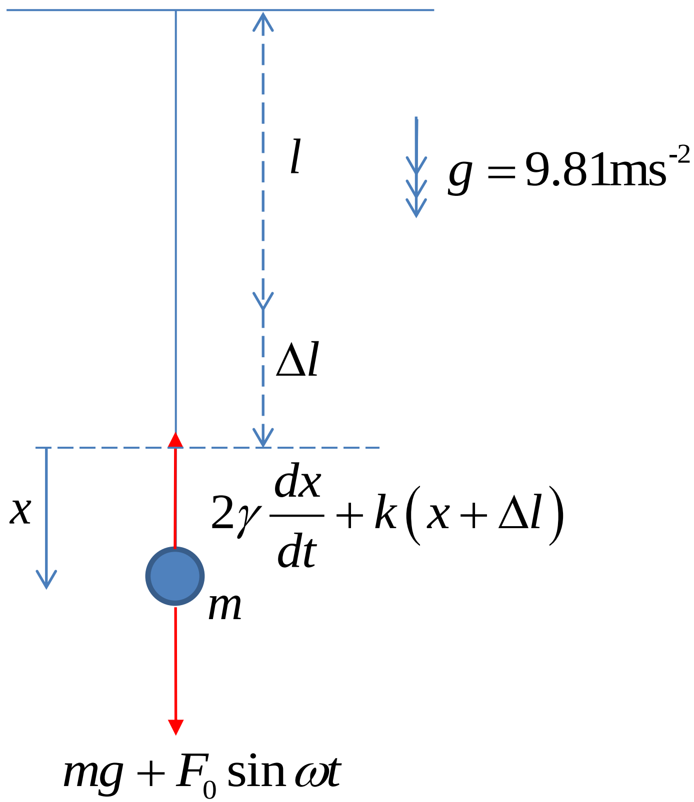
The mass is also pulled 'driven' via an oscillatory force of magnitude F_0 and frequency $f = \omega/2\pi$

In the absence of any driving force, the mass rests at string extension Δl. It is assumed at time $t = 0$ that extension from this equilibrium point, (x), is zero and the mass is at instantaneous rest.

By Newton's Second Law:

$$\boxed{m\frac{d^2x}{dt^2} = mg + F_0 \cos \omega t - 2\gamma m \frac{dx}{dt} - k(x + \Delta l)}$$

SHM equation from driven mechanical oscillations



By Newton's Second Law:

$$m\frac{d^2x}{dt^2} = mg + F_0 \sin \omega t - 2\gamma m \frac{dx}{dt} - k\left(x + \Delta l\right)$$

At equilibrium $x = 0$, $F_0 = 0$

$$0 = mg - k\Delta l$$

$$\therefore \quad \Delta l = \frac{mg}{k}$$

Hence:

$$m\frac{d^2x}{dt^2} + 2\gamma m \frac{dx}{dt} + kx = F_0 \sin \omega t$$

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega t}$$

This equation has a very similar form to the generic equation of **Simple Harmonic Motion (SHM)**

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = A_0 \sin \omega t}$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \sin\omega t$$ can have *oscillatory solutions* if $\gamma < \omega_0$

The general solution is: $$x(t) = Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2 - \gamma^2} - \Phi\right) + B\sin\left(\omega t - \phi\right)$$

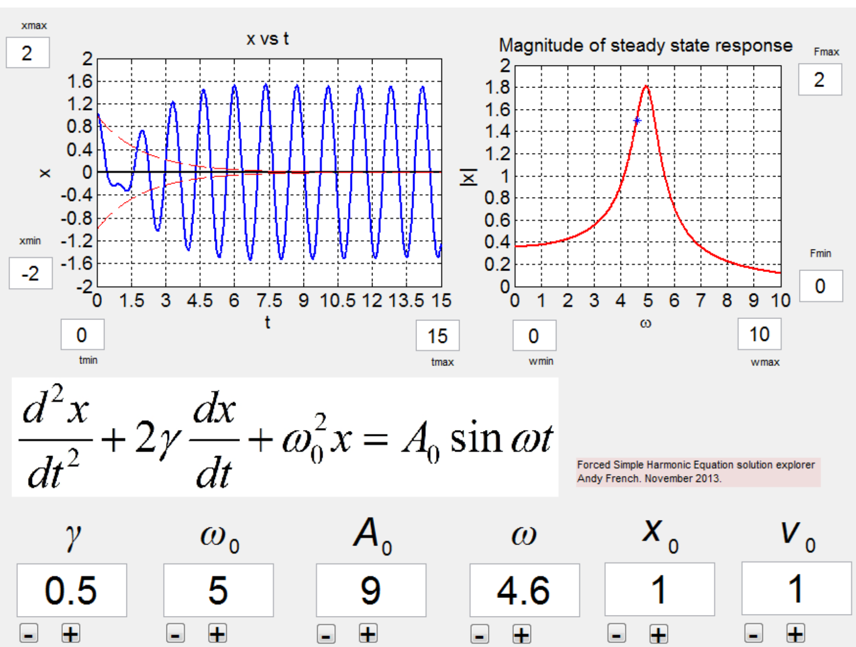
'Transient solution' (which exponentially decays)

Steady state oscillation at same frequency as driving force

Transient amplitude and phase

$$\Phi = \tan^{-1}\left(\frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{\left(x_0 + B\sin\phi\right)\sqrt{\omega_0^2 - \gamma^2}}\right)$$

$$A = \frac{x_0 + B\sin\phi}{\cos\Phi}$$



The *steady state amplitude* exhibits **resonance phenomenon** i.e. it has a *maximum* at a particular 'resonance frequency'

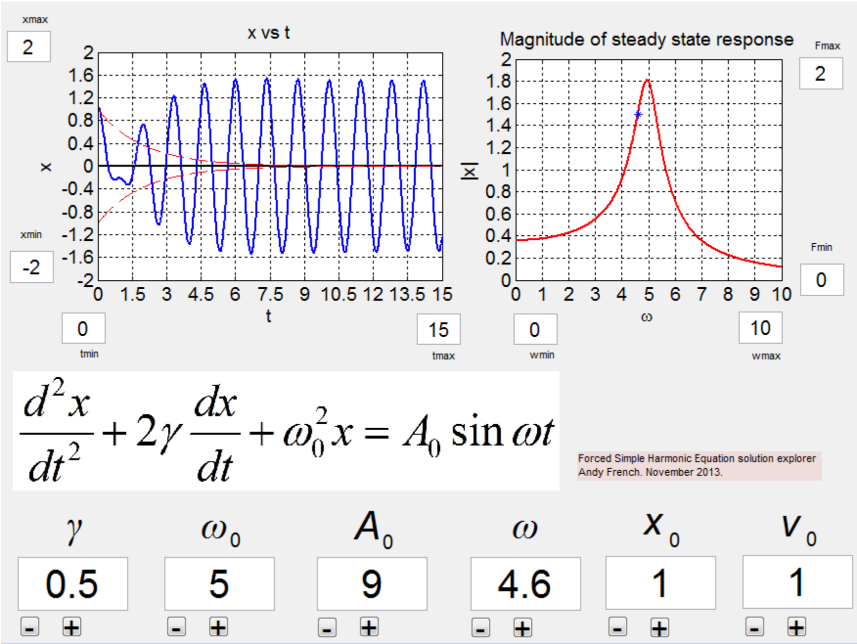
$$B = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

$B_{\max}$ when $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \sin \omega t$$ can have *oscillatory solutions* if $\gamma < \omega_0$

$$x(t) = B\sin\left(\omega t - \phi\right)$$ Steady state oscillation at same frequency as driving force



$$B = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}$$

$$B_{\max} \text{ when } \frac{d}{d\omega}\left\{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right\} = 0$$

$$2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 8\gamma^2\omega = 0$$

$$\omega\left(2\gamma^2 - \omega_0^2 + \omega^2\right) = 0$$

$$\omega^2 = \omega_0^2 - 2\gamma^2$$

$$\omega = \sqrt{\omega_0^2 - 2\gamma^2}$$

The *steady state amplitude* exhibits **resonance phenomenon** i.e. it has a *maximum* at a particular ‘resonance frequency’

$$\boxed{B_{\max} \text{ when } \quad \omega = \sqrt{\omega_0^2 - 2\gamma^2}}$$

$$\omega_{\max} = \sqrt{\omega_0^2 - 2\gamma^2}$$

$$\therefore \left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2$$

$$= \left(\omega_0^2 - \omega_0^2 - 2\gamma^2\right)^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)$$

$$= 4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^4$$

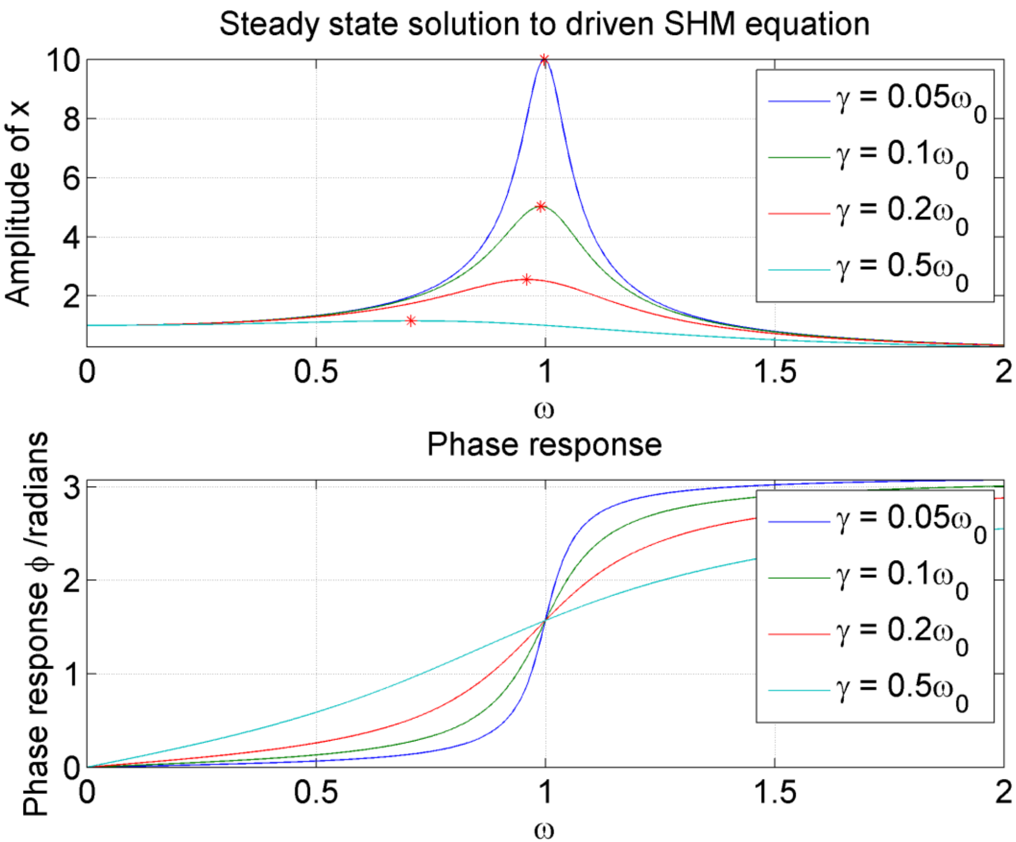
$$= 4\gamma^2\omega_0^2 - 4\gamma^4$$

$$\therefore B_{\max} = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}$$

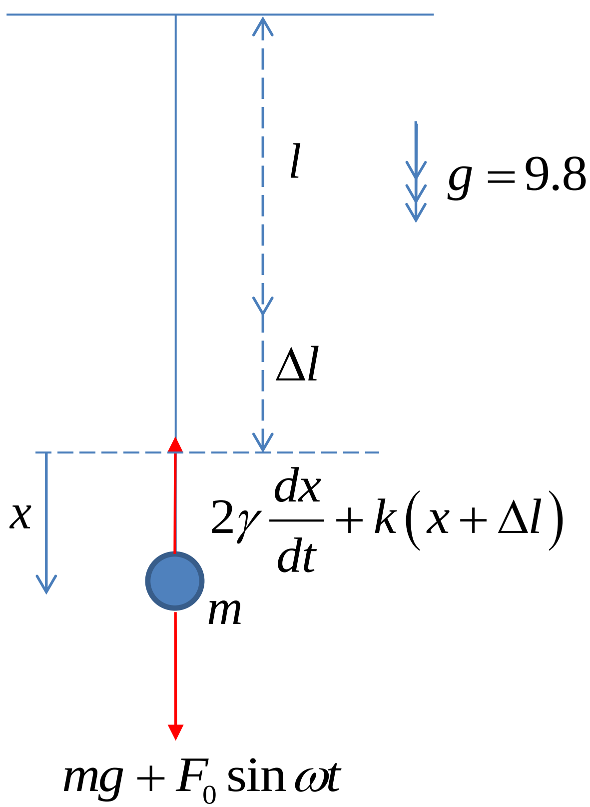
$$B_{\max} = \frac{A_0}{\sqrt{4\gamma^2\omega_0^2 - 4\gamma^4}}$$

$$B_{\max} = \frac{A_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

$$\boxed{\begin{aligned} &\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \sin\omega t \\ &x = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}} \sin\left(\omega t - \phi\right) \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$



SHM parameters for the driven mechanical oscillator



Mechanical oscillator $$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \frac{k}{m} x = \frac{F_0}{m} \sin \omega t$$

SHM $$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = A_0 \sin \omega t$$

$$\omega_0 = 2\pi f_0$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$ Natural oscillation frequency

$$A_0 = \frac{F_0}{m}$$

$$\omega_{\mathrm{res}} = \sqrt{\omega_0^2 - 2\gamma^2}$$ Resonance frequency

Solving the SHM equation (steady state) using **complex variables**

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 e^{i\omega t}$$

$$\boxed{x = Be^{i(\omega t - \phi)}}$$

$$\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right) Be^{i(\omega t - \phi)} = A_0 e^{i\omega t}$$

$$\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right) Be^{-i\phi} = A_0$$

$$B = \frac{A_0}{\left|-\omega^2 + 2i\gamma\omega + \omega_0^2\right|}$$

$$\boxed{B = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}}$$

$$\phi = \arg\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right)$$

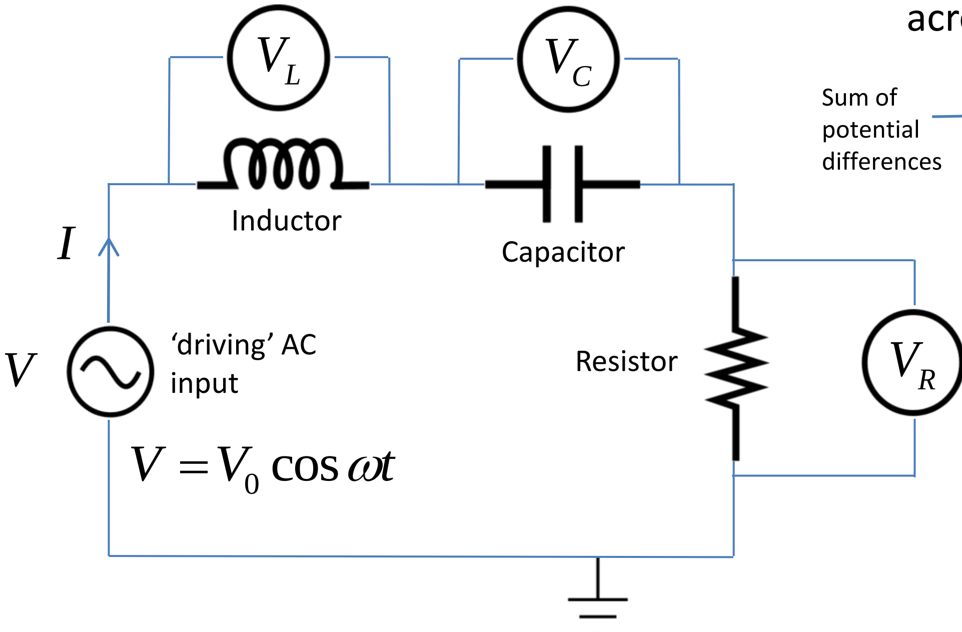
$$\boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)}$$

$$\boxed{\begin{aligned} x &= Be^{i(\omega t - \phi)} \\ x &= B\cos\left(\omega t - \phi\right) + iB\sin\left(\omega t - \phi\right) \end{aligned}}$$

$$\boxed{\begin{aligned} &\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \cos\omega t \\ &x = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}\cos\left(\omega t - \phi\right) \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} &\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \sin\omega t \\ &x = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}\sin\left(\omega t - \phi\right) \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$

Steady state solution to the LCR circuit



Let current *I* flow through the circuit. The net EMF *V* - V_L must equal the sum of the potential drops across each electrical component.

Sum of potential differences → $V_R + V_C = V - V_L$ ← EMF – 'back EMF' due to induction in the coil

$$V_R = IR$$

$$V_C = \frac{1}{C}\int I dt \longleftarrow Q = CV$$

$$V_L = L\frac{dI}{dt} \qquad I = \frac{dQ}{dt} \therefore CV = \int I dt$$

$$IR + \frac{1}{C}\int I dt = V_0 \cos \omega t - L\frac{dI}{dt}$$

$$R\frac{dI}{dt} + \frac{I}{C} + L\frac{d^2 I}{dt^2} = -\omega V_0 \sin \omega t$$

Steady state solution to SHM equation

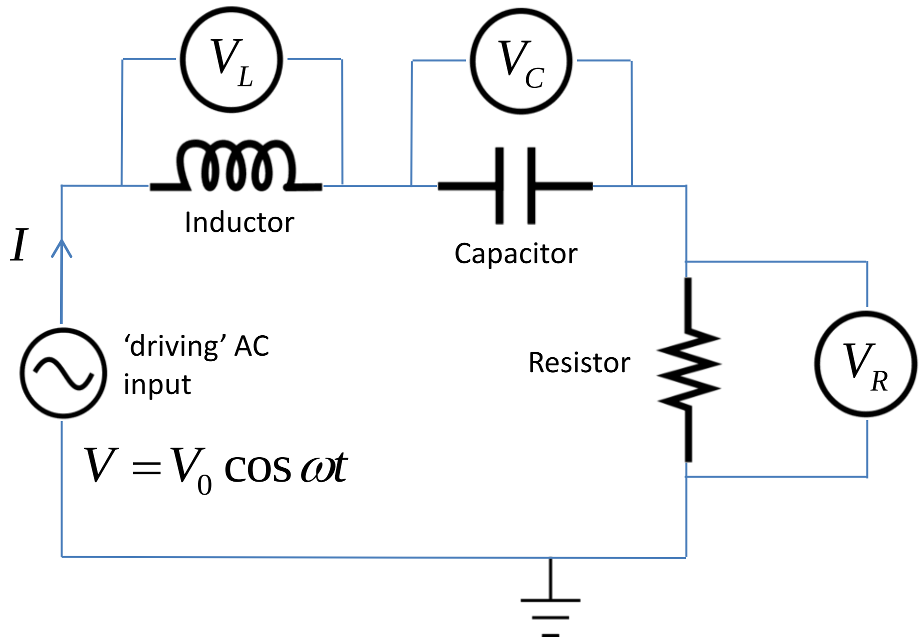
$$\boxed{\begin{aligned} &\frac{d^2 x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0 \sin \omega t \\ &x = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}\sin\left(\omega t - \phi\right) \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$

→

$$\boxed{\frac{d^2 I}{dt^2} + \tfrac{R}{L}\frac{dI}{dt} + \tfrac{1}{LC} I = \frac{-\omega V_0}{L}\sin \omega t}$$

$$\gamma = \frac{R}{2L}\ , \quad \omega_0^2 = \frac{1}{LC}\ , \quad A_0 = \frac{-\omega V_0}{L}$$

Steady state solution to the LCR circuit .... cont



$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC} = \frac{-\omega V_0}{L} \sin \omega t$$

$$\gamma = \frac{R}{2L} \ , \quad \omega_0^2 = \frac{1}{LC} \ , \quad A_0 = \frac{-\omega V_0}{L}$$

$$\boxed{I = \frac{-\omega V_0 / L}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{RC}{LC}\right)^2 \omega^2}} \sin(\omega t - \phi) \qquad \phi = \tan^{-1}\left(\frac{RC\omega}{1 - LC\omega^2}\right)}$$

Steady state solution to SHM equation

$$\boxed{\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = A_0 \sin \omega t \qquad x = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2 \omega^2}} \sin(\omega t - \phi) \qquad \phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)}$$

$$I_{max} \text{ when } \quad \omega = \sqrt{\omega_0^2 - 2\gamma^2}$$

$$f_{max} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{RC}{LC}\right)^2}$$

$$f_{max} = f_0 \sqrt{1 - \frac{(RC)^2}{LC}}$$

$$\boxed{f_{max} = f_0 \sqrt{1 - 4\pi^2 (f_0 \tau)^2}}$$

$$\omega = 2\pi f$$

$$f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$\tau = RC$$

$$LC = \frac{1}{4\pi^2 f_0^2}$$

$$4\pi^2 (f_0 \tau)^2 < 1$$

$$\boxed{f_0 \tau < \tfrac{1}{2\pi}}$$

Using dimensionless variables ...

$$\frac{d^2 I}{dt^2}+\frac{R}{L}\frac{dI}{dt}+\frac{I}{LC}=\frac{-\omega V_0}{L}\sin\omega t$$

$$\gamma=\frac{R}{2L}\ ,\quad \omega_0^2=\frac{1}{LC}\ ,\quad A_0=\frac{-\omega V_0}{L}$$

$$\frac{d^2 x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x=A_0\sin\omega t$$

$$x_0=\frac{A_0}{\omega_0^2}\qquad z=\frac{\omega}{\omega_0}\qquad k=\frac{2\gamma}{\omega_0}$$

$$x=\frac{x_0}{\sqrt{\left(1-z^2\right)^2+k^2 z^2}}\sin\left(\omega t-\phi\right)$$

$$\phi=\tan^{-1}\left(\frac{kz}{1-z^2}\right)$$

$$f_0=\tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$z=\frac{f}{f_0}$$

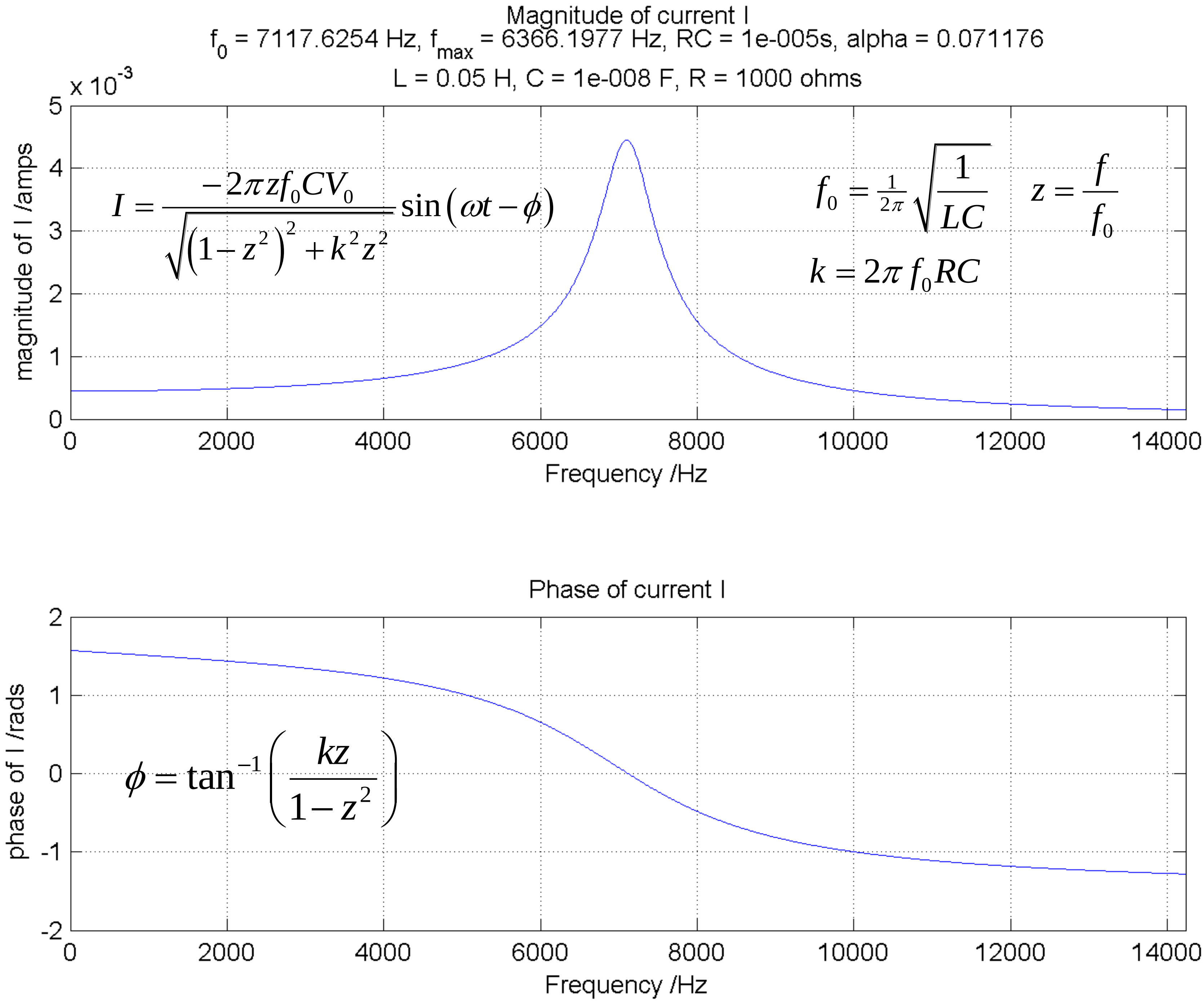
$$I_0=\frac{-\omega V_0}{L\omega_0^2}=-\frac{2\pi fV_0 LC}{L}=-2\pi fCV_0\qquad=-2\pi zf_0CV_0$$

$$k=\frac{R}{L\omega_0}=\frac{R\sqrt{LC}}{L}=R\sqrt{\frac{C}{L}}=RC\sqrt{\frac{1}{LC}}=2\pi f_0RC$$

$$I=\frac{-2\pi zf_0CV_0}{\sqrt{\left(1-z^2\right)^2+k^2z^2}}\sin\left(\omega t-\phi\right)$$

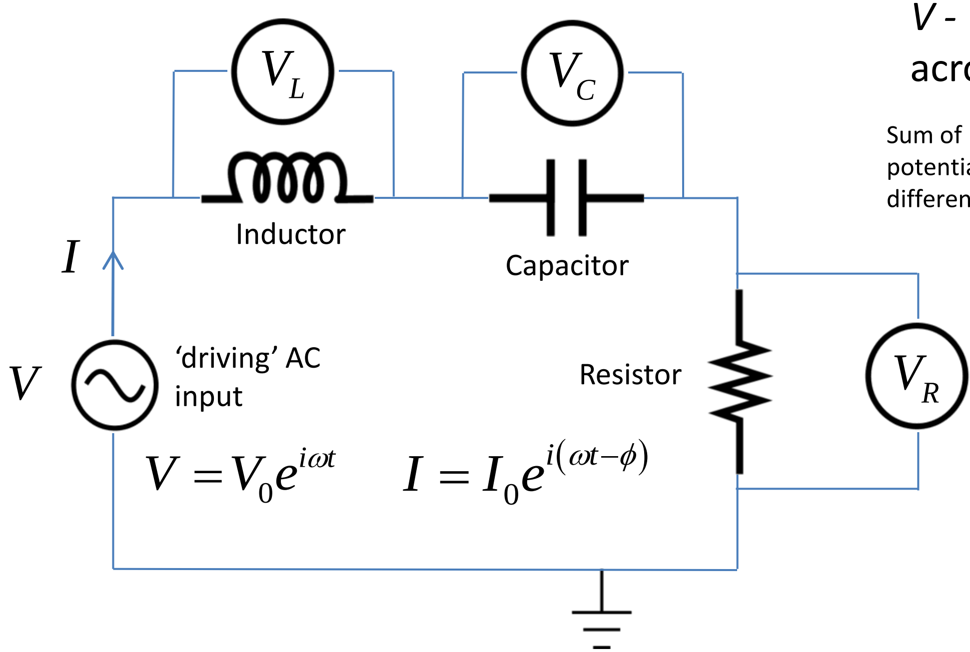
$$\phi=\tan^{-1}\left(\frac{kz}{1-z^2}\right)$$

Note f_0CV_0
is the average current
when the maximum amount
of charge stored in the capacitor
is discharged over one complete
period at frequency f_0



Magnitude of current I
f0 = 7117.6254 Hz, fmax = 6366.1977 Hz, RC = 1e-005s, alpha = 0.071176
L = 0.05 H, C = 1e-008 F, R = 1000 ohms
x 10-3
magnitude of I /amps
Frequency /Hz
$I = \frac{-2\pi z f_0 C V_0}{\sqrt{(1-z^2)^2 + k^2 z^2}} \sin(\omega t - \phi)$
$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$
$z = \frac{f}{f_0}$
$k = 2\pi f_0 RC$
Phase of current I
phase of I /rads
Frequency /Hz
$\phi = \tan^{-1}\left(\frac{kz}{1-z^2}\right)$

Complex *impedance*



$$V = V_0 e^{i\omega t} \qquad I = I_0 e^{i(\omega t - \phi)}$$

Let current *I* flow through the circuit. The net EMF $V - V_L$ must equal the sum of the potential drops across each electrical component.

Sum of potential differences → $V_R + V_C = V - V_L$ ← EMF – 'back EMF' due to induction in the coil

$$V_R = IR$$

$$V_C = \frac{1}{C}\int I dt \quad \leftarrow \quad Q = CV$$

$$I = \frac{dQ}{dt} \therefore CV = \int I dt$$

$$V_L = L\frac{dI}{dt}$$

$$IR + \frac{1}{C}\int I dt = V_0 e^{i\omega t} - L\frac{dI}{dt}$$

$$IR + \frac{1}{C}\int I dt + L\frac{dI}{dt} = V_0 e^{i\omega t}$$

$$I_0 \, e^{i(\omega t - \phi)}\left(R + \frac{1}{i\omega C} + i\omega L\right) = V_0 e^{i\omega t}$$

$$\boxed{Z = R + \frac{1}{i\omega C} + i\omega L}$$

$$\therefore V = I\left|Z\right| e^{i\arg(Z)}$$

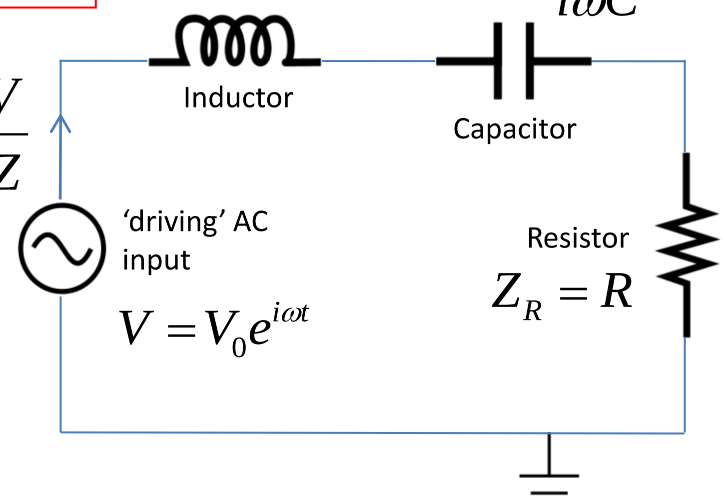
$$\phi = \arg(Z)$$

$$\therefore \boxed{V = IZ}$$

Ohm's Law, generalized for AC

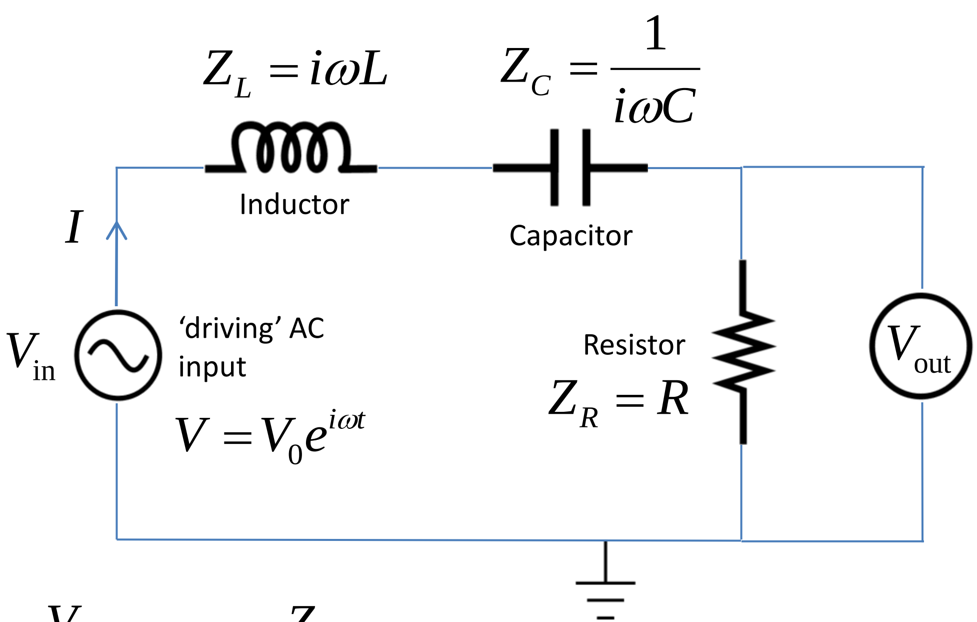
$$\boxed{Z = Z_L + Z_C + Z_R} \qquad Z_L = i\omega L \qquad Z_C = \frac{1}{i\omega C}$$

$$I = \frac{V}{Z}$$



$$V = V_0 e^{i\omega t} \qquad Z_R = R$$

Impedances for L,C,R components



$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{2\pi f \tau}{2\pi f \tau + i\left(\frac{4\pi^2 f^2}{4\pi^2 f_0^2} - 1\right)}$$

$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{2\pi x f_0 \tau}{2\pi x f_0 \tau + i\left(x^2 - 1\right)}$$

$$\boxed{\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{2\pi x \alpha}{2\pi x \alpha + i\left(x^2 - 1\right)}}$$

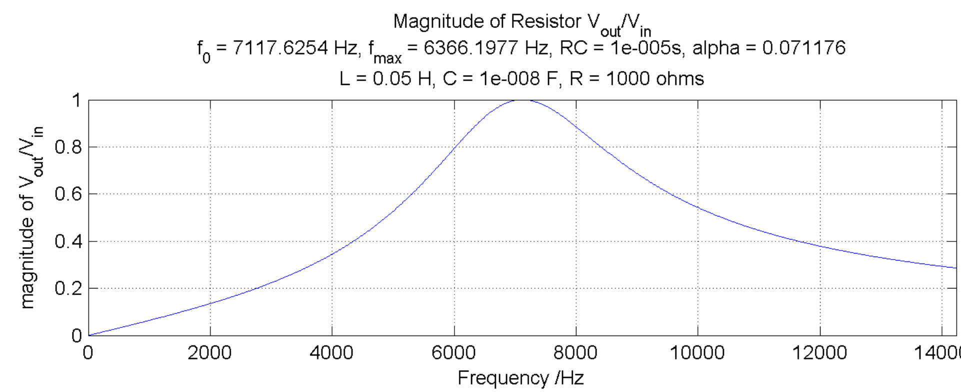
$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{Z_R}{Z_L + Z_C + Z_R}$$

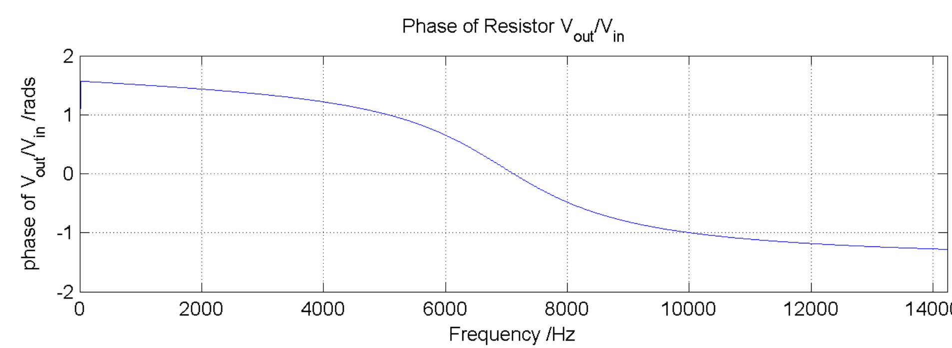
$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{R}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

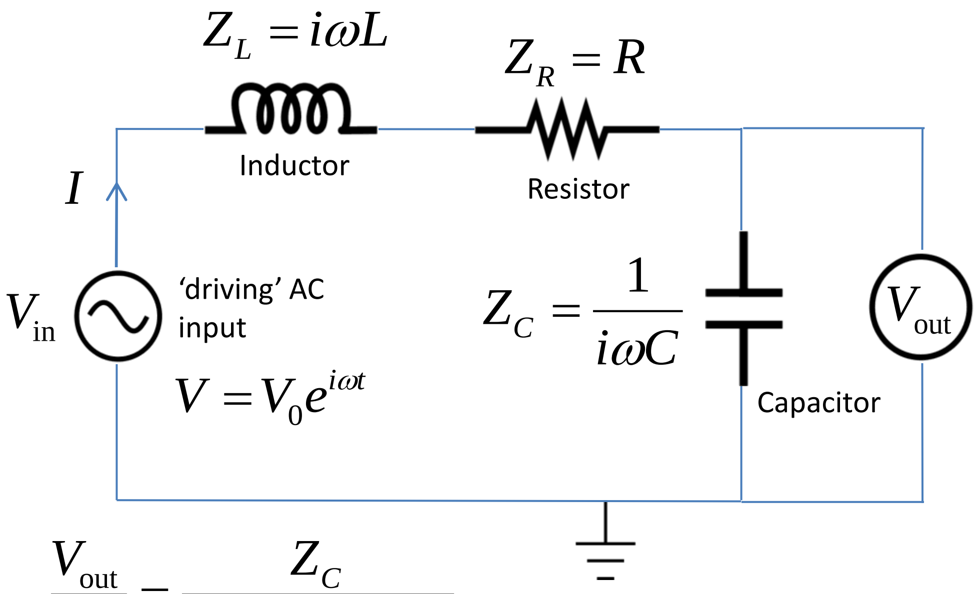
$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{RC\omega}{RC\omega + i\left(\omega^2 LC - 1\right)}$$

$$x = \frac{f}{f_0} \quad f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$\tau = RC \quad \alpha = f_0 \tau$$







$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_C}{Z_L + Z_C + Z_R}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\dfrac{1}{i\omega C}}{R + i\left(\omega L - \dfrac{1}{\omega C}\right)}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{iRC\omega - \left(\omega^2 LC - 1\right)}$$

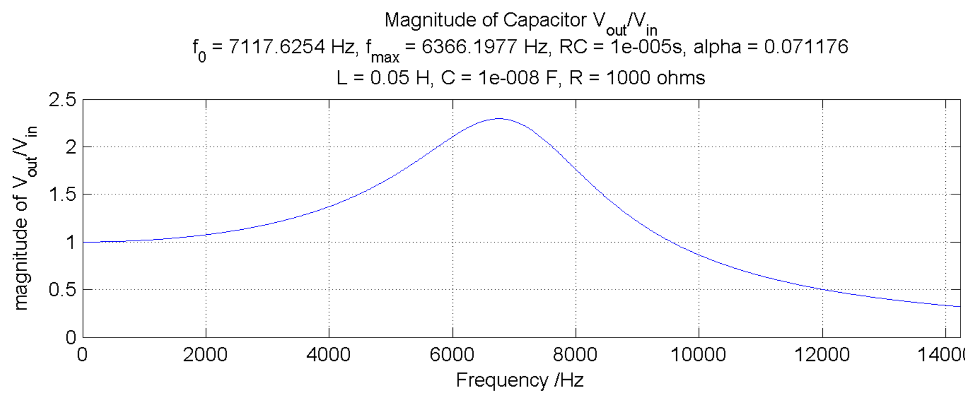
$$x = \frac{f}{f_0} \quad f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

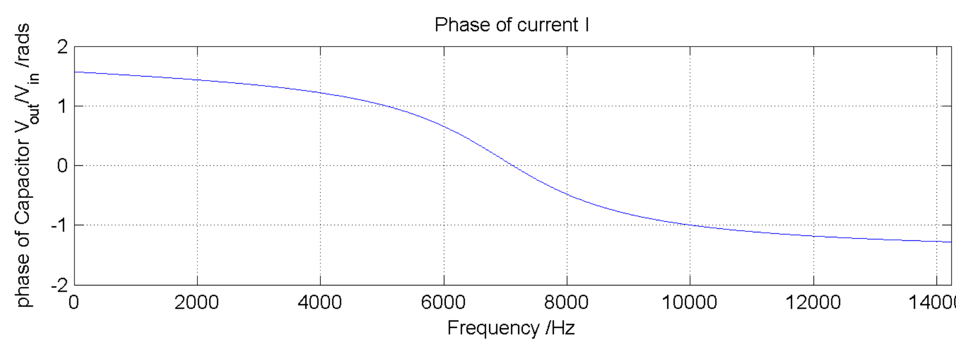
$$\tau = RC \quad \alpha = f_0 \tau$$

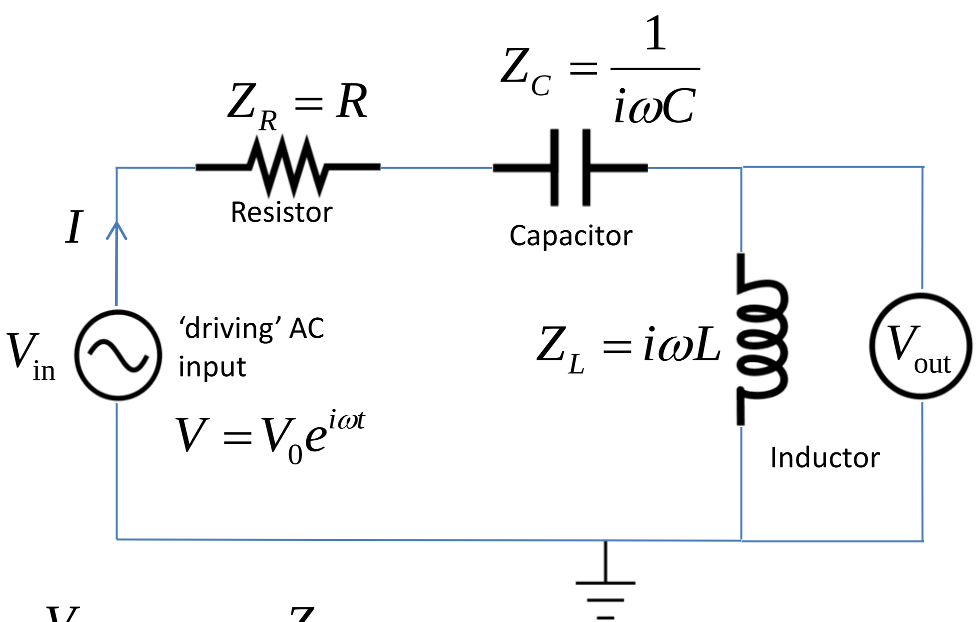
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{2\pi f \tau + i\left(\dfrac{4\pi^2 f^2}{4\pi^2 f_0^2} - 1\right)}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{2\pi x f_0 \tau + i\left(x^2 - 1\right)}$$

$$\boxed{\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{2\pi x \alpha + i\left(x^2 - 1\right)}}$$







$$\frac{V_{out}}{V_{in}} = \frac{i\dfrac{4\pi^2 f^2}{4\pi^2 f_0^2}}{2\pi f\tau + i\left(\dfrac{4\pi^2 f^2}{4\pi^2 f_0^2} - 1\right)}$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{ix}{2\pi\alpha x + i\left(x^2 - 1\right)}}$$

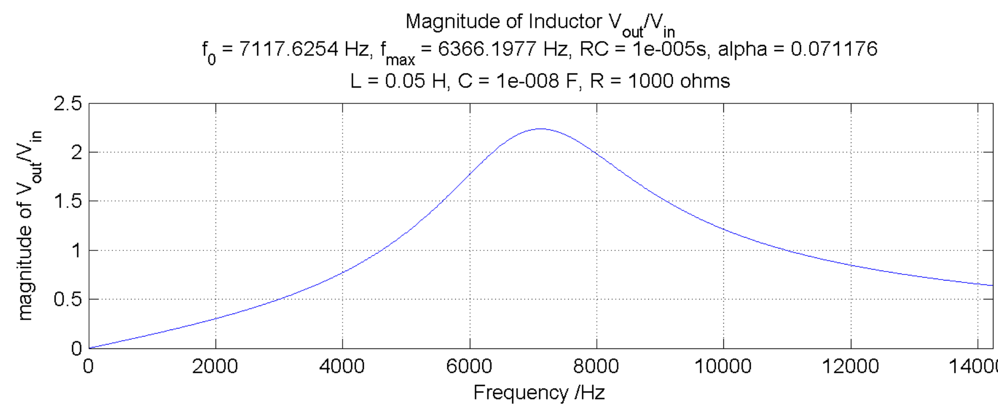
$$\frac{V_{out}}{V_{in}} = \frac{Z_L}{Z_L + Z_C + Z_R}$$

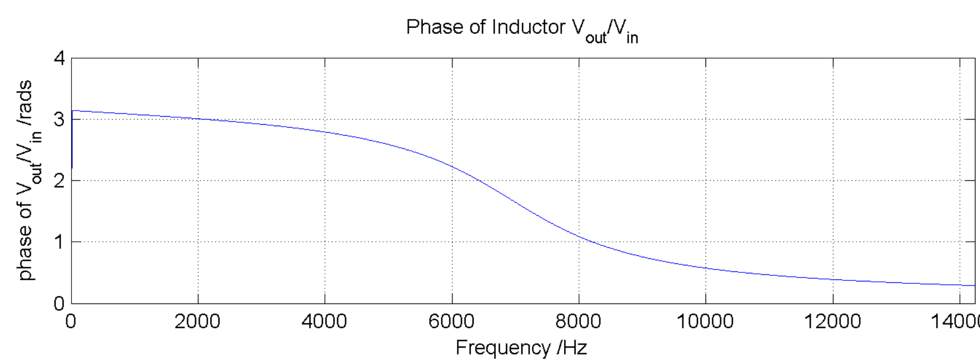
$$\frac{V_{out}}{V_{in}} = \frac{i\omega L}{R + i\left(\omega L - \dfrac{1}{\omega C}\right)}$$

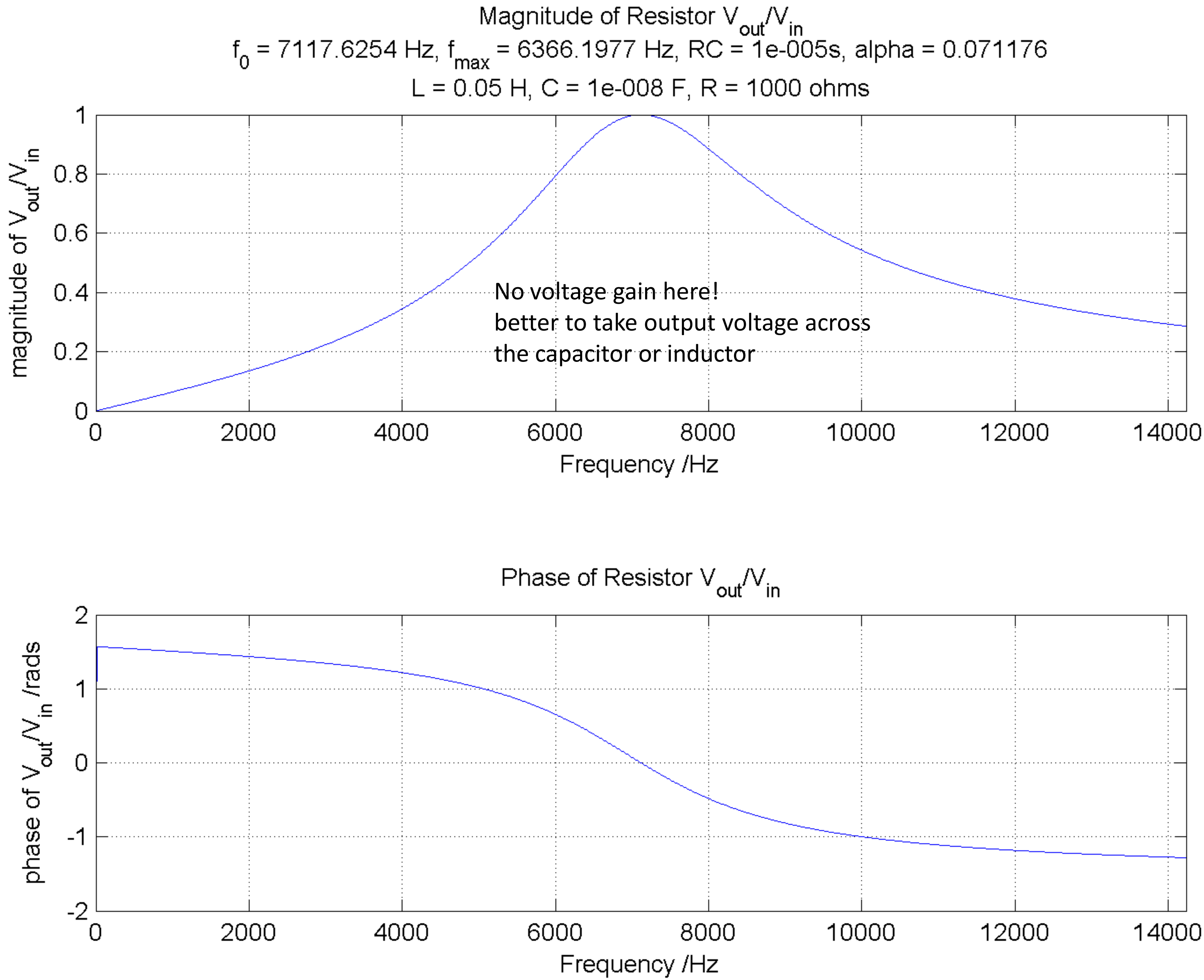
$$\frac{V_{out}}{V_{in}} = \frac{i\omega^2 LC}{RC\omega + i\left(\omega^2 LC - 1\right)}$$

$$x = \frac{f}{f_0} \quad f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

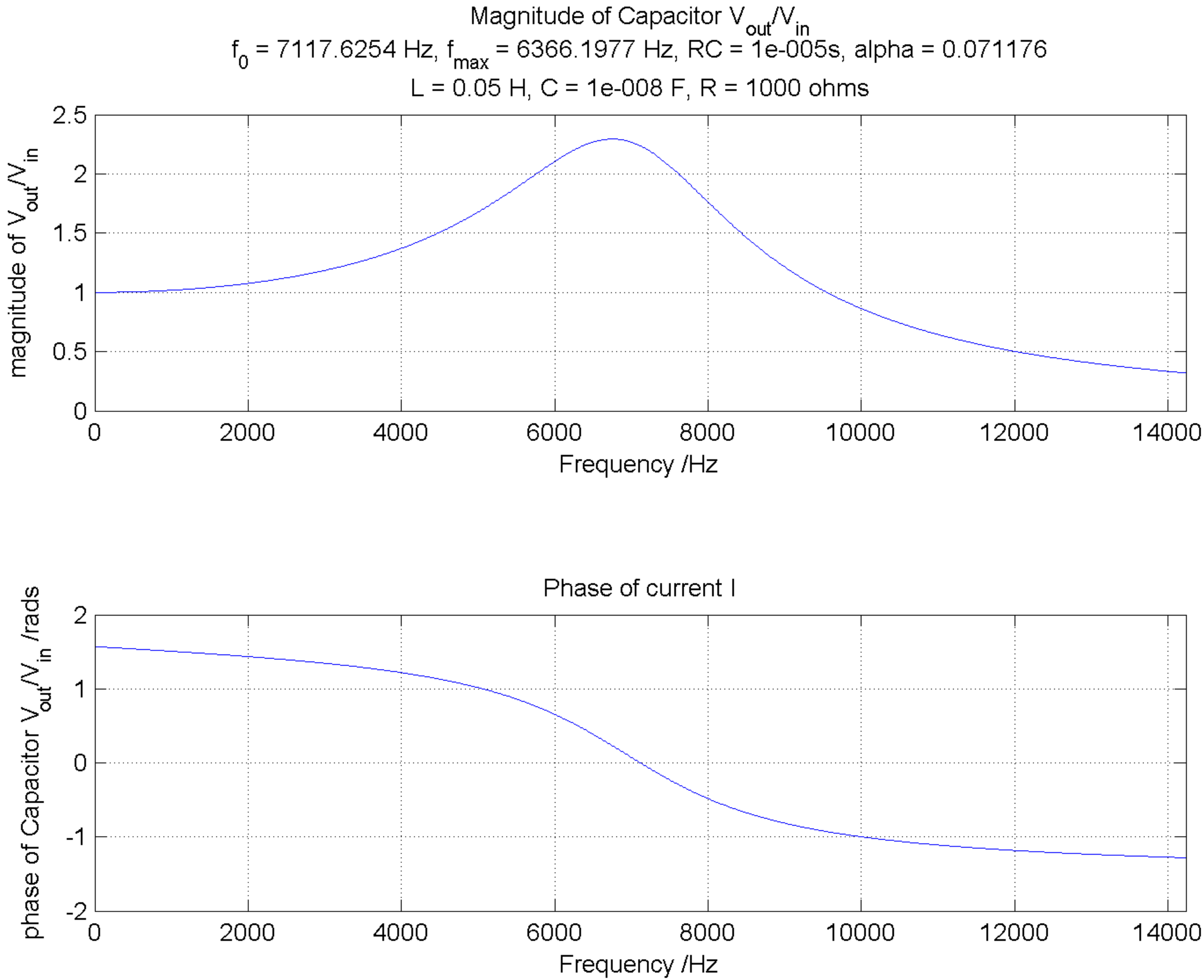
$$\tau = RC \quad \alpha = f_0\tau$$

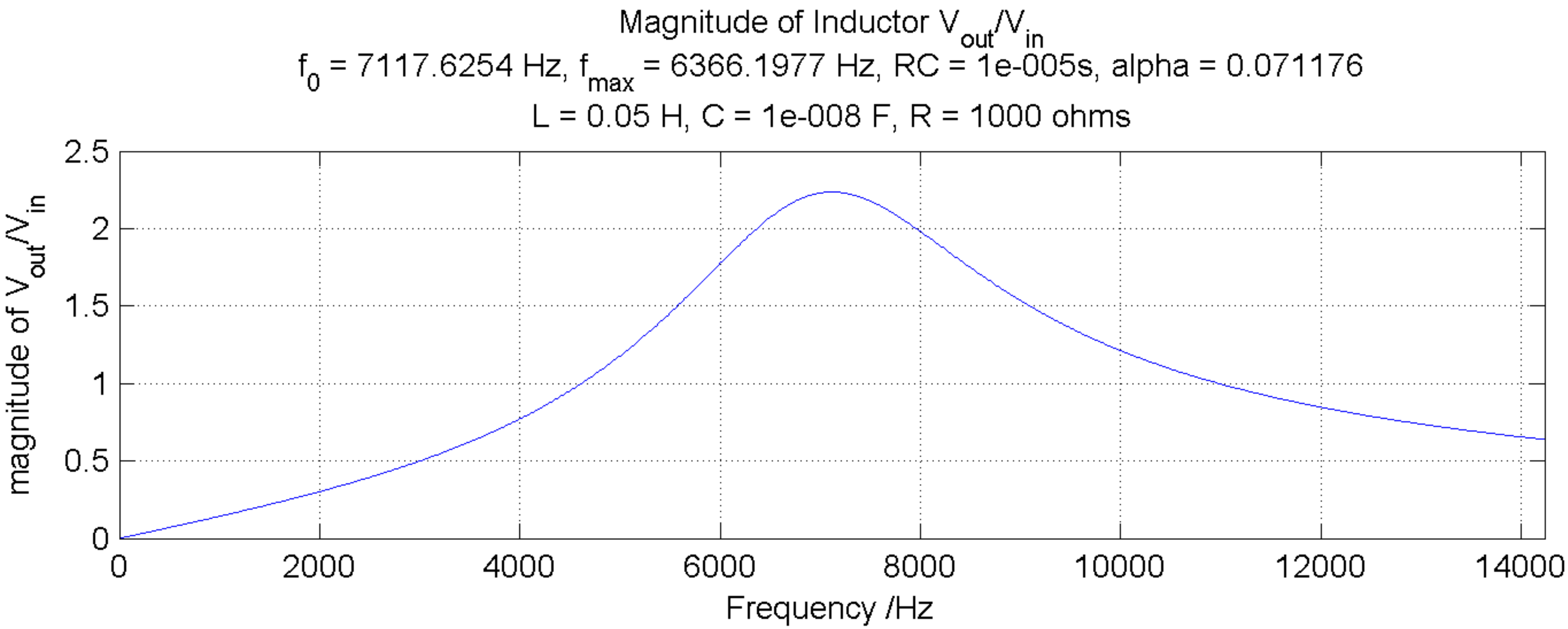




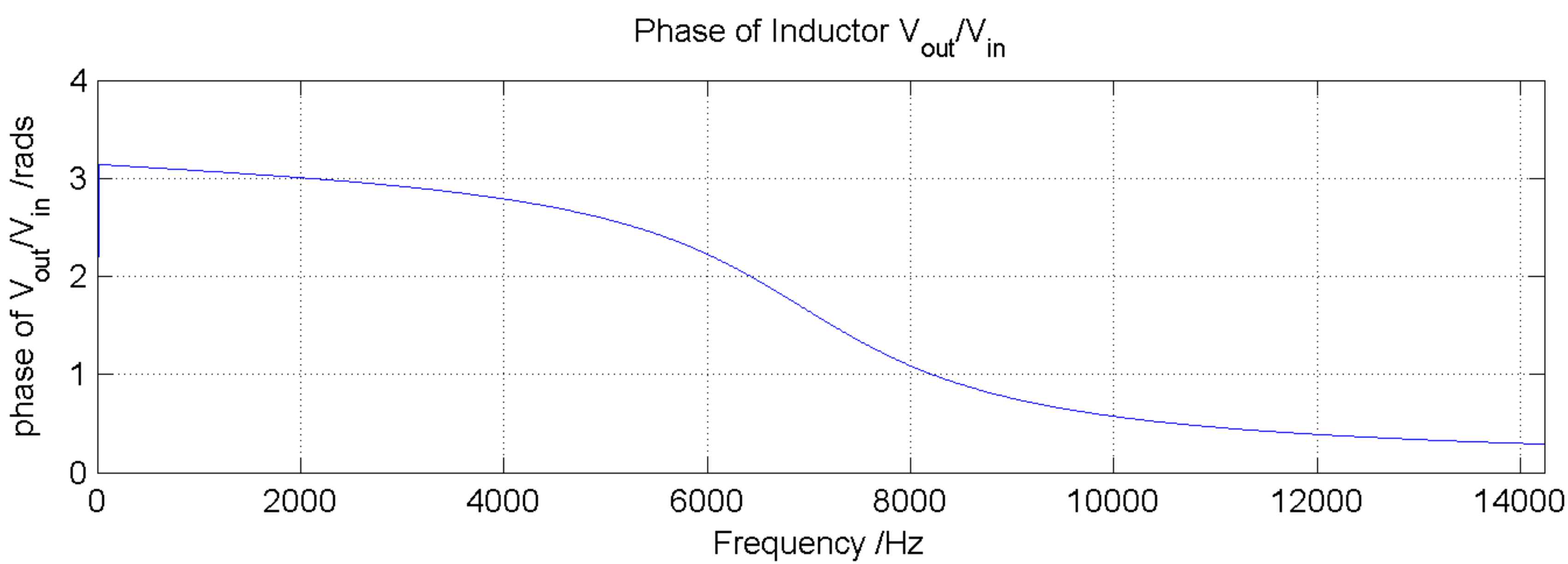


Magnitude of Resistor Vout/Vin
f0 = 7117.6254 Hz, fmax = 6366.1977 Hz, RC = 1e-005s, alpha = 0.071176
L = 0.05 H, C = 1e-008 F, R = 1000 ohms
magnitude of Vout/Vin
Frequency /Hz
No voltage gain here!
better to take output voltage across
the capacitor or inductor
Phase of Resistor Vout/Vin
phase of Vout/Vin /rads
Frequency /Hz


Magnitude of Capacitor V_{out}/V_{in}
f_0 = 7117.6254 Hz, f_{max} = 6366.1977 Hz, RC = 1e-005s, alpha = 0.071176
L = 0.05 H, C = 1e-008 F, R = 1000 ohms
magnitude of V_{out}/V_{in}
Frequency /Hz
Phase of current I
phase of Capacitor V_{out}/V_{in} /rads
Frequency /Hz



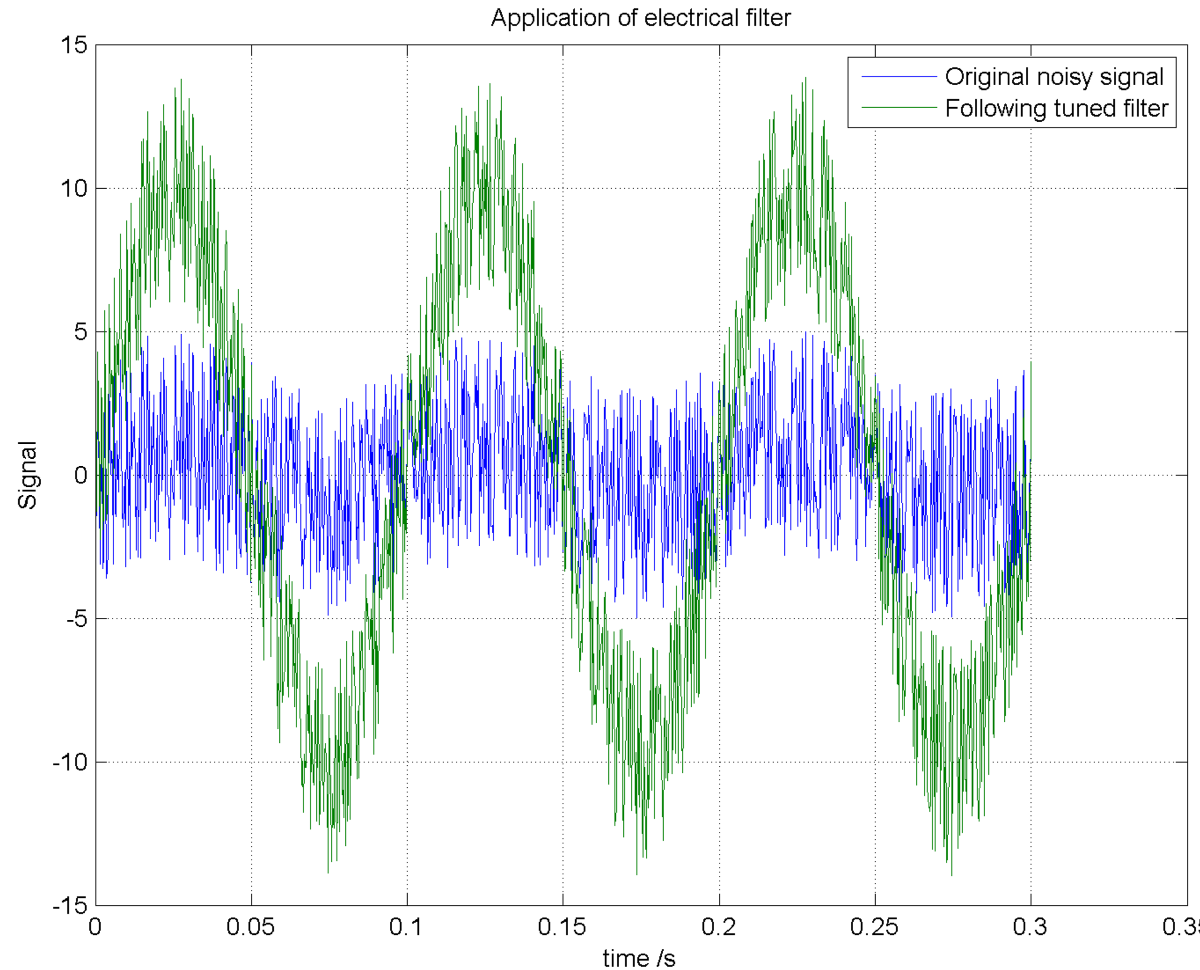
Magnitude of Inductor Vout/Vin
f0 = 7117.6254 Hz, fmax = 6366.1977 Hz, RC = 1e-005s, alpha = 0.071176
L = 0.05 H, C = 1e-008 F, R = 1000 ohms
magnitude of Vout/Vin
2.5
2
1.5
1
0.5
0
0
2000
4000
6000
8000
10000
12000
14000
Frequency /Hz

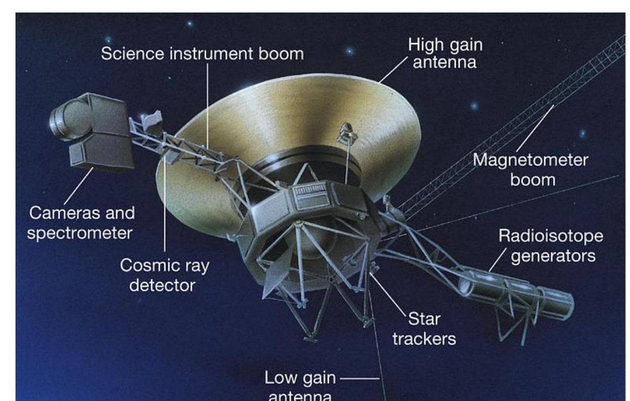


Phase of Inductor Vout/Vin
phase of Vout/Vin /rads
4
3
2
1
0
0
2000
4000
6000
8000
10000
12000
14000
Frequency /Hz

LCR circuits can be used as **electrical filters**. If a signal consists of a superposition of oscillations at different frequencies, an LCR circuit car be tuned to preferentially boost signal components whose frequencies are near tl resonance peak of the circuit.

This has *enormous* applicatior in communications or Radar technology, whereby a weak signal of a known frequency (e.g. a local radio station, or indeed the broadcasts from Voyager!) may be buried in electrical noise 'across the waveband'.





Designing a mains noise filter

In many applications we would like to filter out electrical signals associated with mains AC at 50Hz

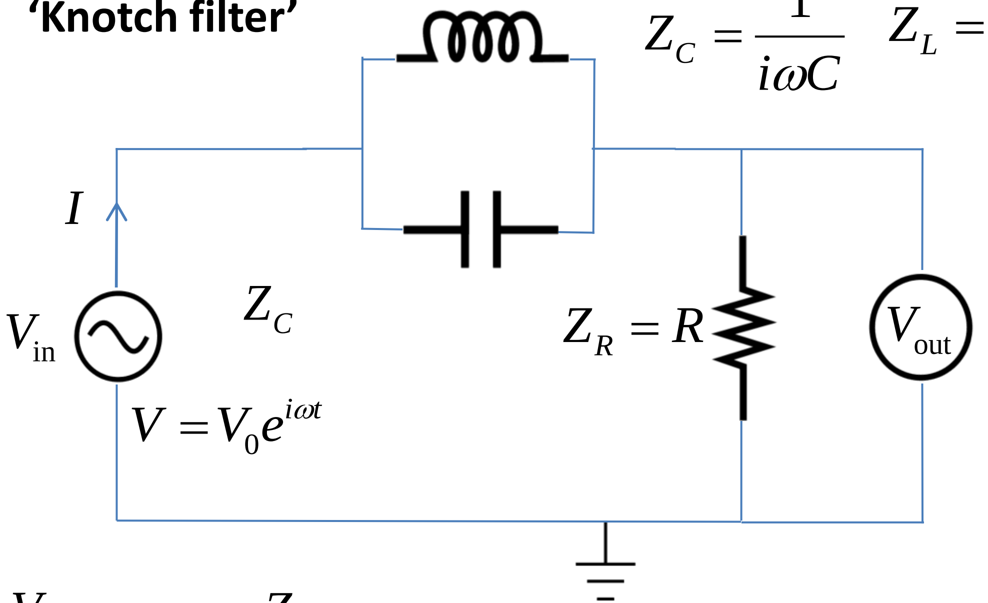
$$f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$\therefore LC = \frac{1}{4\pi^2 f_0^2}$$

If L = 0.05, therefore $\boxed{C = 2.03 \times 10^{-4}\ \mathrm{F}}$

‘Knotch filter’

$$Z_C = \frac{1}{i\omega C} \qquad Z_L = i\omega L$$



$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{i\omega RC - i\dfrac{RC}{\omega LC}}{1 + i\omega RC - i\dfrac{RC}{\omega LC}}$$

$$\boxed{\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{i2\pi\alpha\left(x - \dfrac{1}{x}\right)}{1 + i2\pi\alpha\left(x - \dfrac{1}{x}\right)}}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_R}{\dfrac{1}{\dfrac{1}{Z_C} + \dfrac{1}{Z_L}} + Z_R}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R}{\dfrac{1}{i\omega C + \dfrac{1}{i\omega L}} + R}$$

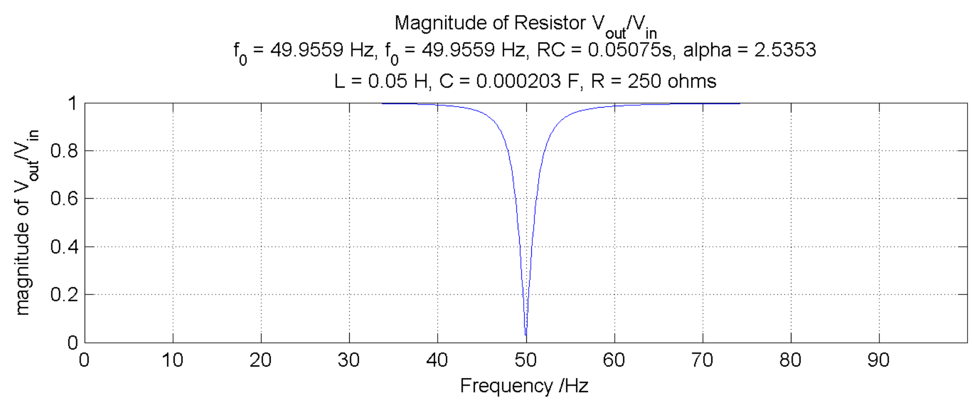
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{\dfrac{1}{i\omega RC - i\dfrac{RC}{\omega LC}} + 1}$$

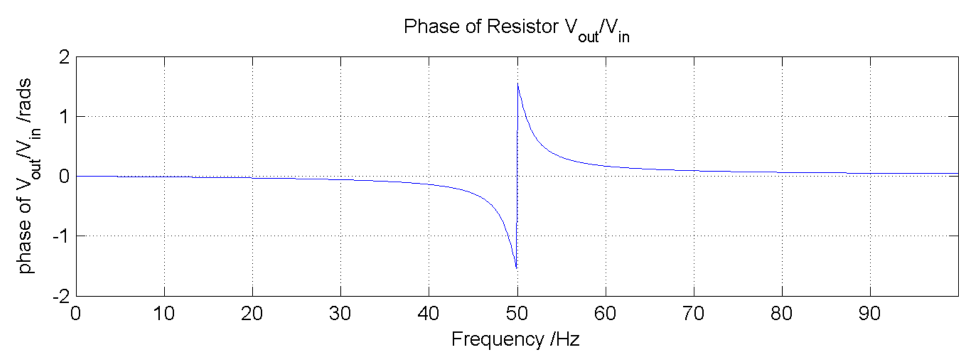
$$x = \frac{f}{f_0}$$

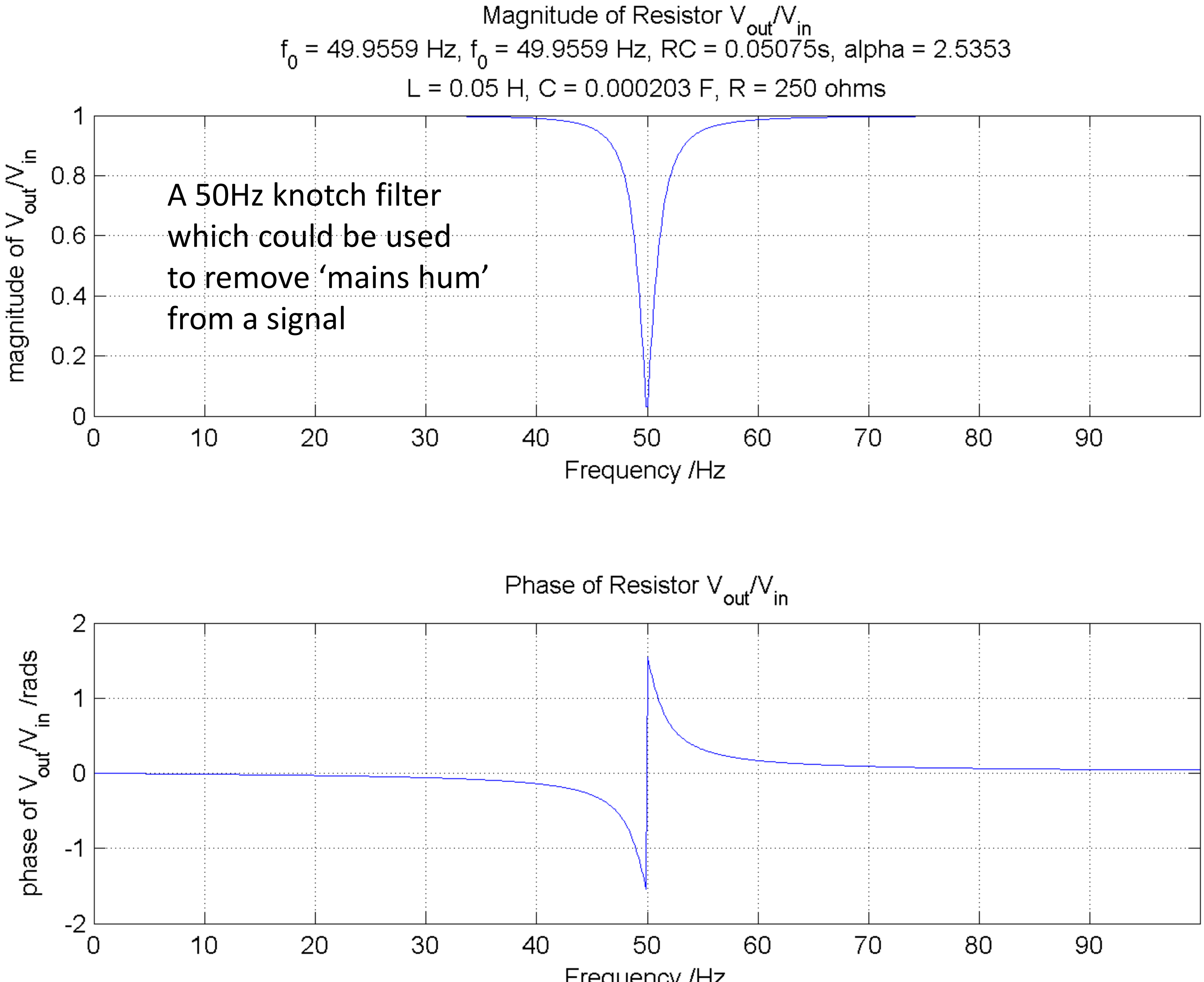
$$f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$\tau = RC$$

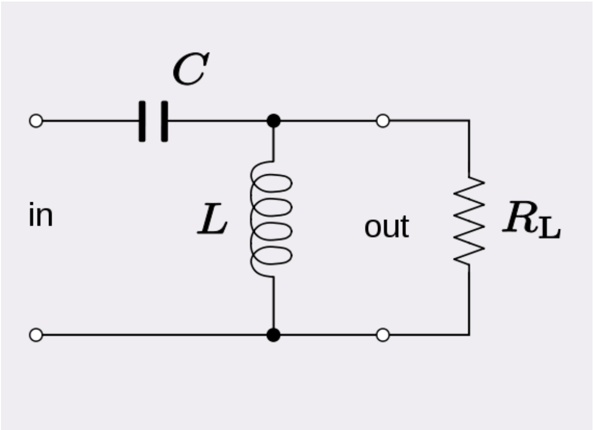
$$\alpha = f_0 \tau$$



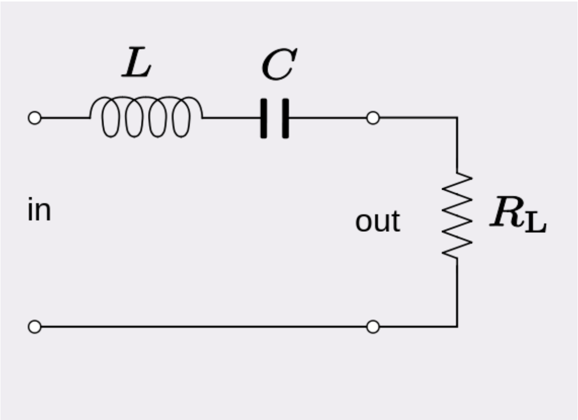



Magnitude of Resistor V_out/V_in
f_0 = 49.9559 Hz, f_0 = 49.9559 Hz, RC = 0.05075s, alpha = 2.5353
L = 0.05 H, C = 0.000203 F, R = 250 ohms
magnitude of V_out/V_in
A 50Hz knotch filter
which could be used
to remove ‘mains hum’
from a signal
Frequency /Hz
Phase of Resistor V_out/V_in
phase of V_out/V_in /rads
Frequency /Hz

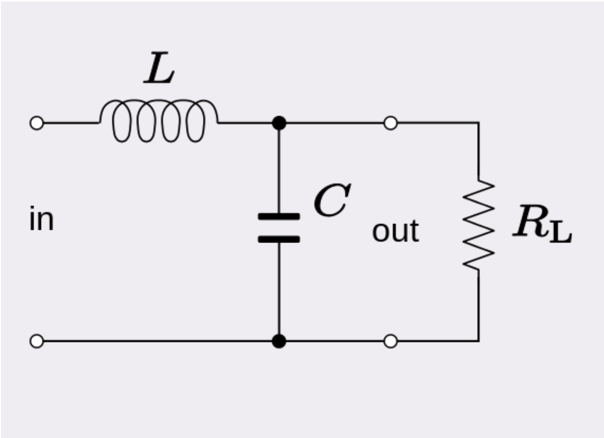
Other types of filter using just R,L,C configurations



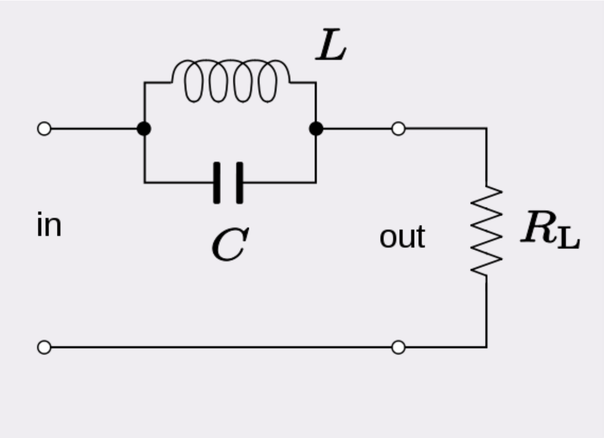
High pass filter



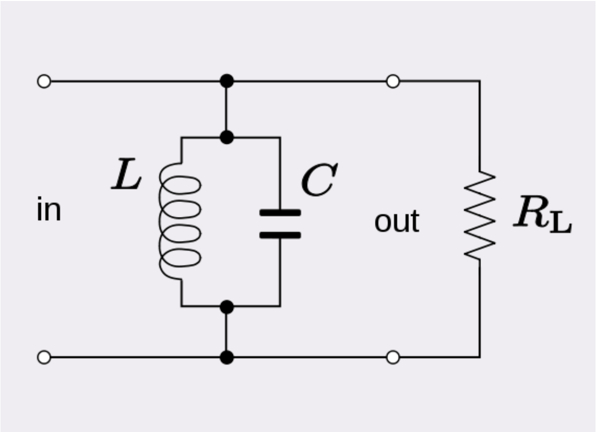
Band pass filter



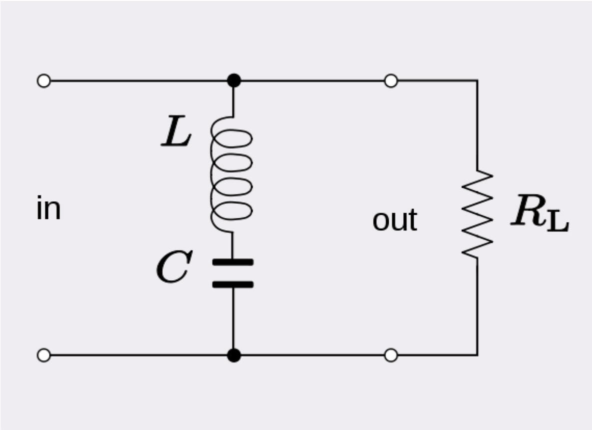
Low pass filter



Band stop filter

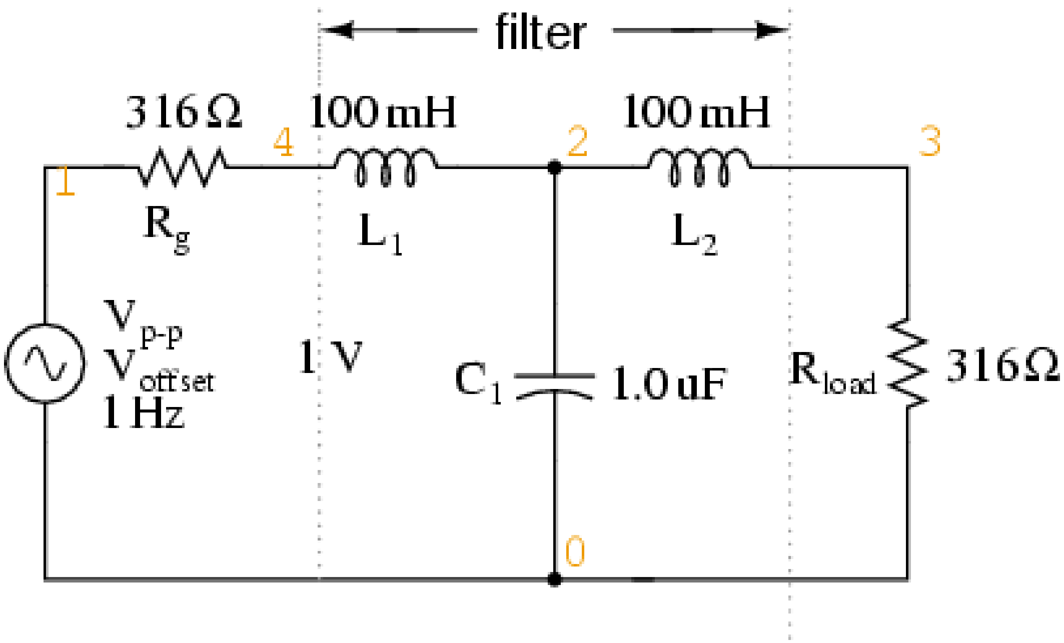


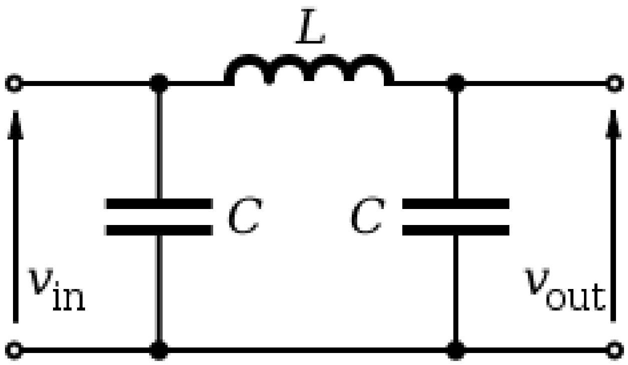
Band pass filter

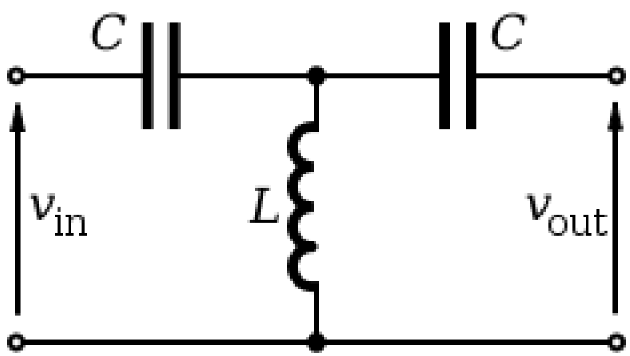


Band stop filter

More complicated L,C,R filter circuits







$$Z_R = R$$

$$Z_C = \frac{1}{i\omega C}$$

$$Z_L = i\omega L$$

Write down equations for the currents in each loop ...
Apply V = IZ ...
Solve simultaneously!

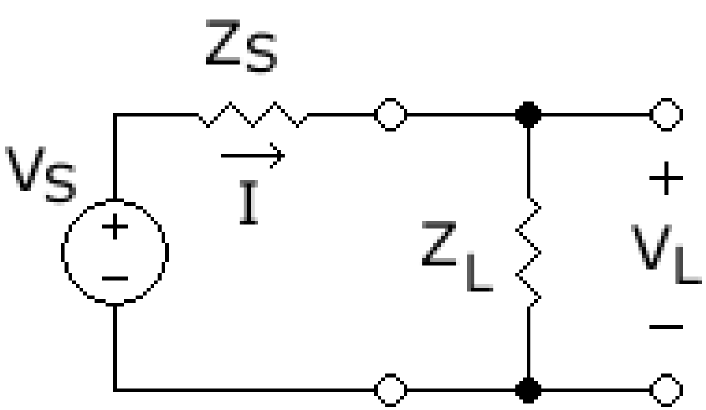
Impedance matching

Note *maximum power is transferred* from input to output if impedances

$$\boxed{Z_L = Z_S^*}$$

if source impedance is *fixed*. (If it is adjustable, then setting $Z_S = 0$ will maximise the power dissipated in the load).

A change in impedance will cause a fraction of the signal to be reflected, which results in a loss of power conveyed to the output



$$Z_L = R_L + iX_L \qquad Z_S = R_S + iX_S$$

$$P_L = I_{rms} R_L = \tfrac{1}{2}|I|^2 R_L$$

$$V_S = I\left(Z_S + Z_L\right) \quad \therefore |V_S|^2 = |I|^2 |Z_S + Z_L|^2$$

$$|Z_S + Z_L|^2 = |R_L + iX_L + R_S + iX_S|^2$$

$$|Z_S + Z_L|^2 = |R_L + R_S + i\left(X_L + X_S\right)|^2$$

$$P_L = \tfrac{1}{2}\frac{|V_S|^2 R_L}{\left(R_S + R_L\right)^2 + \left(X_S + X_L\right)^2}$$

P_L is *maximized* when

$$R_L = R_S \leftarrow$$ See next page!

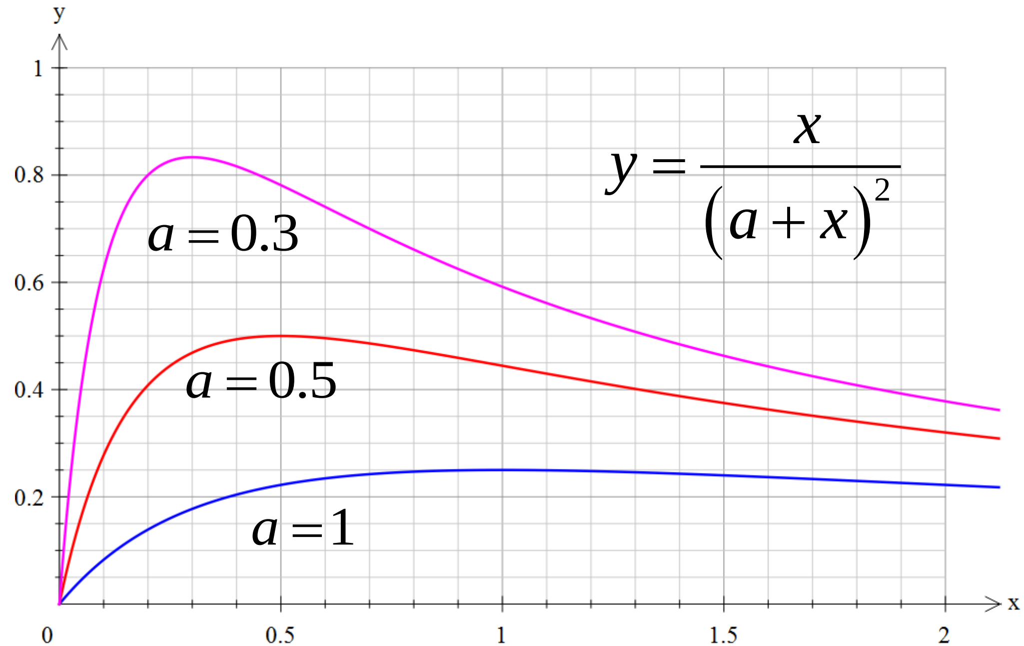
$$X_L = -X_S$$

$$\therefore R_L + iX_L = R_S - iX_S$$

$$\therefore \boxed{Z_L = Z_S^*}$$

Maximizing the load power in the 'power dissipation theorem' is equivalent to maximizing *y* given constant *a* in the equation

$$y = \frac{x}{(a+x)^2}, \quad x > 0$$



$$\frac{dy}{dx} = \frac{(a+x)^2(1) - x\left(2(a+x)\right)}{(a+x)^4}$$

$$\frac{dy}{dx} = \frac{(a+x)(a+x-2x)}{(a+x)^4}$$

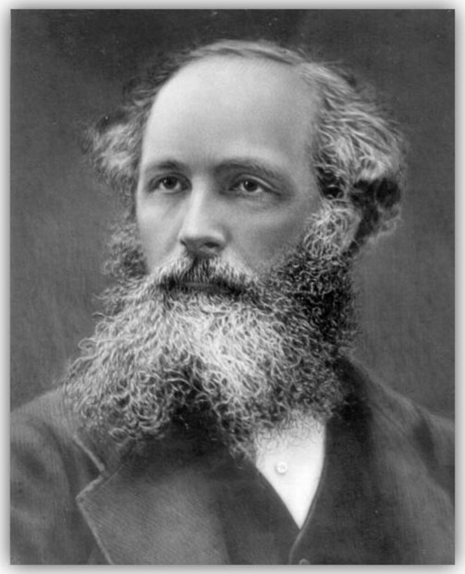
$$\frac{dy}{dx} = \frac{a-x}{(a+x)^3}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=a} = 0$$

Hence maxima at

$$\boxed{\begin{aligned} x &= a \\ y &= \frac{a}{(a+a)^2} = \frac{1}{4a} \end{aligned}}$$

Radiated electromagnetic waves



James Clerk Maxwell
1831-1879

Maxwell developed the experimental work of Faraday and others into a mathematical theory of electric and magnetic fields.

It is a vector theory, encapsulated in four equations which are now know as Maxwell's Equations.

One can combine them to form a wave equation in both the electric and magnetic fields. In each case, the velocity of waves (in 'free space' i.e. a perfect vacuum) is

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$$

Henrich Hertz
1857-1894

i.e. the **speed of light**
$c = 2.998 \times 10^8$ ms^{-1}.

This is *independent* of the frame of reference! ..... A big clue that **Albert Einstein** used to help him develop the *theory of Relativity*.

Guglielmo Marconi
Wireless transmission pioneer
1874-1937

Permittivity & permeability

Coulomb's Law of force (F) between two charges (Q_1,Q_2) separated by distance r

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}$$

$$\boxed{\varepsilon_0 = 8.85418782 \times 10^{-12}\,\mathrm{kg^{-1}m^{-3}s^{4}A^{2}}}$$

Charles-Augustin de Coulomb
1736-1806

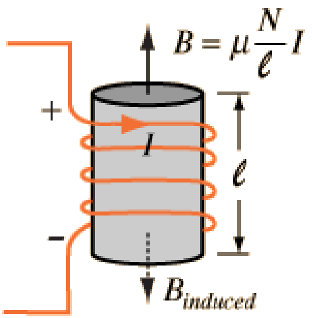
$$B = \mu\mu_0\frac{NI}{l}$$

Magnetic field strength inside a *solenoid* of N turns and cross section A

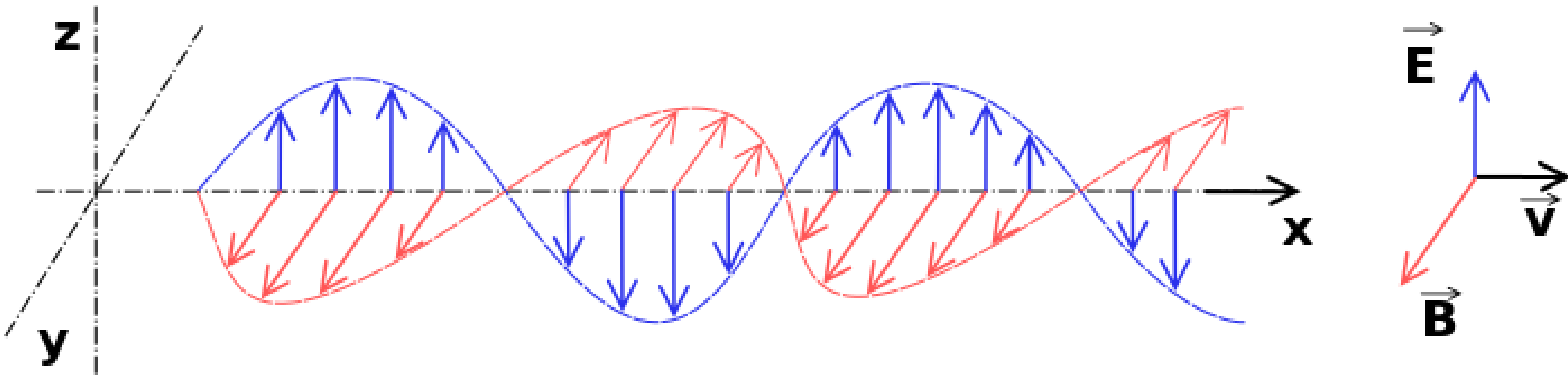
Joseph Henry
1797-1878

$$L = \frac{\mu\mu_0 N^2 A}{l}$$

Inductance of a coil of N turns and cross section A



$$\boxed{\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kgms^{-2}A^{-2}}}$$



$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

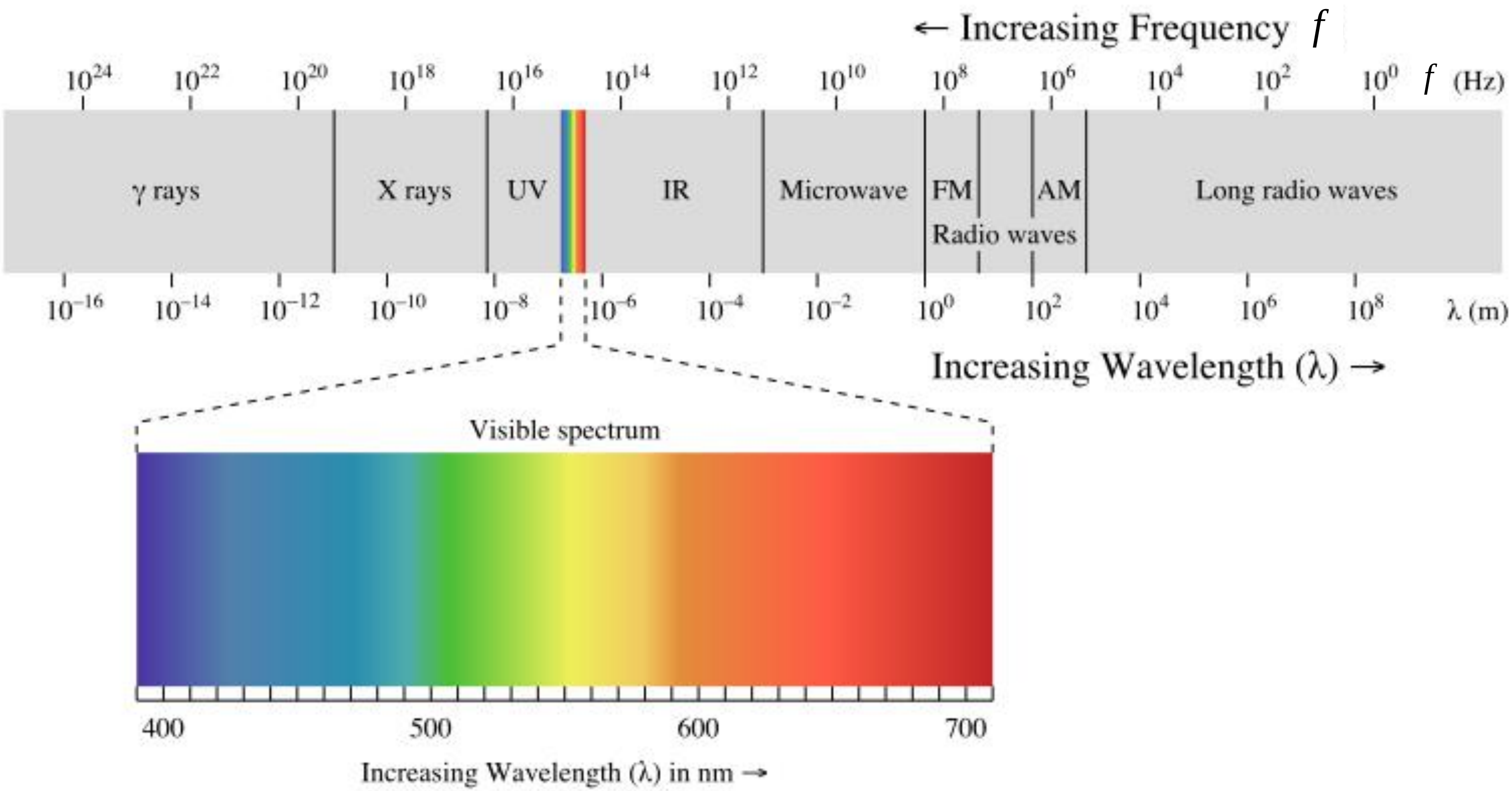
$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Wave equations for electric fields **E** and magnetic fields **B**

c = 2.998 x 10^8 ms^{-1} independent of any coordinate system! *So no matter how fast you are moving, electromagnetic waves always propagate at the same speed .....*

The Electromagnetic Spectrum



$$\boxed{c = f\lambda}$$

Differential form:		Integral form:	
$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$	(7.50)	$\underset{\text{closed surface}}{\oint} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{1}{\epsilon_0} \underset{\text{volume}}{\int} \rho \, \mathrm{d}\tau$	(7.51)
$\nabla \cdot \boldsymbol{B} = 0$	(7.52)	$\underset{\text{closed surface}}{\oint} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0$	(7.53)
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	(7.54)	$\underset{\text{loop}}{\oint} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$	(7.55)
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$	(7.56)	$\underset{\text{loop}}{\oint} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I + \mu_0 \epsilon_0 \underset{\text{surface}}{\int} \frac{\partial \boldsymbol{E}}{\partial t} \cdot \mathrm{d}\boldsymbol{s}$	(7.57)

	Equation (7.51) is "Gauss's law"	$\mathrm{d}\boldsymbol{s}$	surface element
	Equation (7.55) is "Faraday's law"	$\mathrm{d}\tau$	volume element
$\boldsymbol{E}$	electric field	$\mathrm{d}\boldsymbol{l}$	line element
$\boldsymbol{B}$	magnetic flux density	Φ	linked magnetic flux ($= \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}$)
$\boldsymbol{J}$	current density	I	linked current ($= \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{s}$)
ρ	charge density	t	time

Maxwell's Equations (from Woan, *The Cambridge Handbook of Physics Formulas*)