ELECTRICAL CIRCUITS

Q1/ (i) $\boxed{Q = It}$ so $Q = 1.00\times10^{-9} \times 1.00\times10^{-9}$ (C)

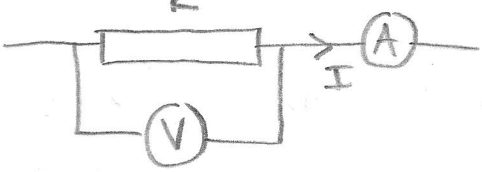
$1.00\,nA$ $\quad$ $1.00\,ns$

$\therefore$ # electrons $= Q/e = \dfrac{1.00\times10^{-18}}{1.602\times10^{-19}} = 6.24$

so $\boxed{\text{Six electrons}}$

(ii) $\boxed{P = I^2R}$ $\therefore$ $I = \sqrt{P/R} = \sqrt{\dfrac{2.0}{10}} = \boxed{0.45\text{ A}}$

$\boxed{V = IR}$ $\therefore$ $V = \boxed{4.47\text{ Volts}}$

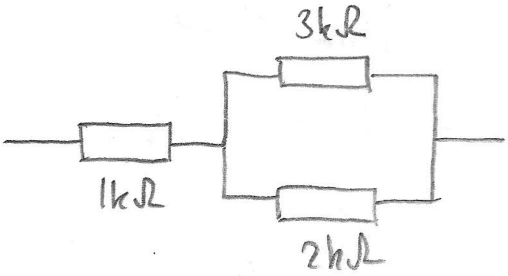


$P = V^2/R$ so if $V \to 2V$, $R \to \frac{1}{2}R$

$P \to \dfrac{(2V)^2}{\frac{1}{2}R} \Rightarrow P \to 8V^2/R$ ie $\boxed{P \to 8P}$

so multiply power dissipated by a factor of eight.

(iii)



$R = 1 + \dfrac{1}{\frac{1}{3}+\frac{1}{2}}$ (kΩ)

$= 2\frac{1}{5}$ kΩ ⟸ Don't express as fractions ⇒ infinite precision!

$= \boxed{2.2\,k\Omega}$

(iv) $eV = E = 1.602\times10^{-19} \times 5000$ (J)

$= \boxed{8.01\times10^{-16}\text{ J}}$

(KE of electron if conversion of electrical energy is 100% efficient)

so $\frac{1}{2}m_e v^2 = eV$ ⟸ Classical KE formula.

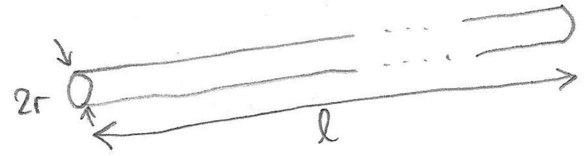
$\therefore$ $v = \sqrt{2eV/m_e}$

$\therefore$ $\dfrac{v}{c} = \sqrt{\dfrac{eV}{\frac{1}{2}m_e c^2}} = \sqrt{\dfrac{1.602\times10^{-19}\times5000}{\frac{1}{2}\times9.109\times10^{-31}\times(2.998\times10^{8})^2}} = \boxed{0.14}$

(so 14% of the speed of light mean relativistic effects probably cannot be ignored).

①

(v)



$2r = 1\ \text{mm}$

$R = 1\,\Omega$

$\rho = 1.68\times10^{-8}\,\Omega\text{m}$ ← resistivity of copper.

$R = \dfrac{\rho l}{\pi r^2}$ so $\boxed{l = \dfrac{\pi R r^2}{\rho}}$

$\therefore\ l = \pi\times 1\times (0.5\times10^{-3})^2 / 1.68\times10^{-8}$

$\boxed{l = 46.7\ \text{m}}$ → calculator memory.

Now $l \to 3l$ and $R \to 0.5\,\Omega$ ie $R \to R/2$

$r = \sqrt{\dfrac{\rho l}{\pi R}}$ so $2r = 2\sqrt{\dfrac{\rho l}{\pi R}}$

$2r = 2\sqrt{\dfrac{\rho\times 3l}{\pi\, R/2}}$

$2r = 2\sqrt{\dfrac{\rho l}{\pi R}}\times\sqrt{6}$

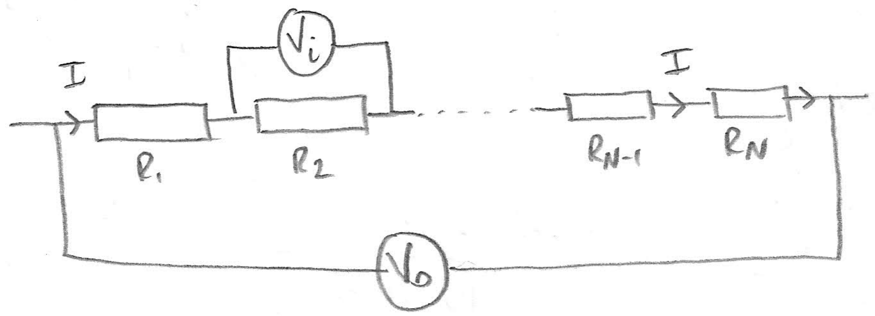
So $2r = \underbrace{1\,\text{mm}}_{\text{previous diameter}}\times\sqrt{6}$

$= \boxed{2.45\ \text{mm}}$

Check by direct substitution:

$2r = 2\times\sqrt{\dfrac{1.68\times10^{-8}\times3\times46.7}{\pi\times0.5}} = \boxed{2.45\times10^{-3}\ \text{m}}$ ✓

(vi)



Kirchoff I (ie conservation of charge) ⇒ current I_i flowing through resistor R_i = I = constant. If this were not the case, charge would be gained or lost in the circuit.

So if $R_i = V_i / I_i$ (definition of resistance)

and $V_0 = V_1 + V_2 + \dots + V_N$ {Kirchhoff II or conservation of energy) and $I_i = I$

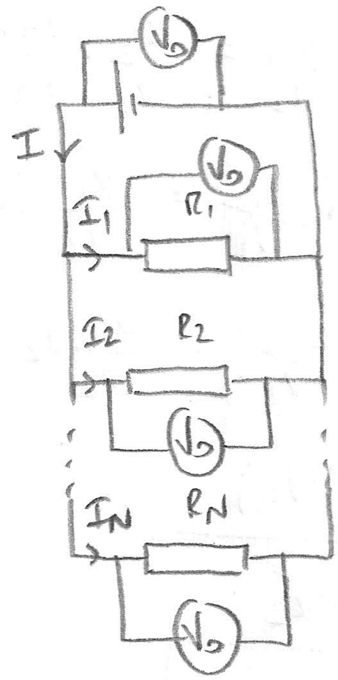
$$\Rightarrow \quad V_0 = I(R_1 + R_2 + \dots R_N)$$

Now total resistance is V_0 / I

$$\therefore \quad \boxed{R_{TOT} = R_1 + R_2 + \dots R_N}$$

So resistances in series <u>add</u>.

(vii)



Each resistor is connected to the same cell, so "feels" the same electric field and $\therefore$ PD across each loop is $= V_0$.

Now current I drawn from cell (or power supply providing voltage V_0) is $\boxed{I = I_1 + I_2 + \dots I_N}$ (Kirchhoff I). Now $I_i = V_i / R_i$ and $V_i = V_0$

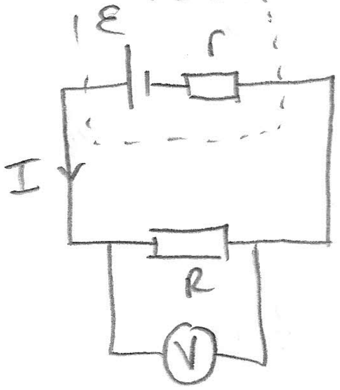
$$\therefore \quad I = V_0 \left(\frac{1}{R_1} + \frac{1}{R_2} + \dots \frac{1}{R_N} \right)$$

If total resistance $R_{TOT} = \frac{V_0}{I} \Rightarrow I/V_0 = \frac{1}{R_{TOT}}$

$$\therefore \quad \boxed{\frac{1}{R_{TOT}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots \frac{1}{R_N}}$$

So resistances <u>add in reciprocals</u> when wired in parallel.

(viii)



ε EMF r internal resistance

Potential divider idea:

$$\boxed{V = \varepsilon \times \frac{R}{R+r}}$$ (Direct ✓)

or: $\varepsilon = Ir + IR$ (KII)

$\therefore \frac{\varepsilon}{r+R} = I$

$V = IR \quad \therefore \quad V = \frac{\varepsilon R}{r+R}$ ✓

So $V_1 = \frac{\varepsilon R_1}{r+R_1}$ (1)

$V_2 = \frac{\varepsilon R_2}{r+R_2}$ (2)

(1)/(2): $\frac{V_1}{V_2} = \frac{R_1}{R_2} \cdot \frac{r+R_2}{r+R_1}$

$\therefore (r+R_1)\frac{V_1}{V_2} = \frac{R_1}{R_2}(r+R_2)$

So $r\left(\frac{V_1}{V_2} - \frac{R_1}{R_2}\right) = \frac{R_1}{R_2}R_2 - R_1\frac{V_1}{V_2}$

$\therefore r = \frac{R_1\left(1 - V_1/V_2\right)}{V_1/V_2 - R_1/R_2}$

$$\therefore \boxed{r = \frac{R_1\left(\frac{V_1}{V_2} - 1\right)}{R_1/R_2 - V_1/V_2}}$$

So $r = \frac{10.0\left(\frac{10.43}{9.88} - 1\right)}{\frac{10.0}{7.0} - \frac{10.43}{9.88}} = \boxed{1.5\ \Omega}$ (1.49 Ω to 3sf)

In (1): $\varepsilon = \frac{V_1(r+R_1)}{R_1} = \frac{10.43(1.49+10)}{10}$

$= \boxed{12.0\ V}$

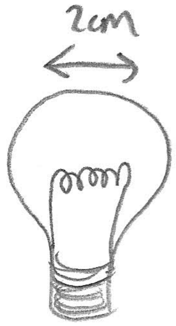
(ix) $R = \rho l / \pi r^2$ $r = 10 \times 10^{-6}$ m (radius of tungsten filament)

$\rho = 5.4 \times 10^{-8}\ \Omega m$

$R = 40\ \Omega$

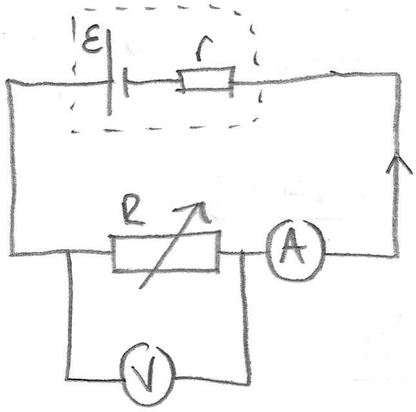
So $l = \dfrac{R \pi r^2}{\rho}$ $\therefore$ $l = \dfrac{40 \times \pi \times (10 \times 10^{-6})^2}{5.4 \times 10^{-8}}$

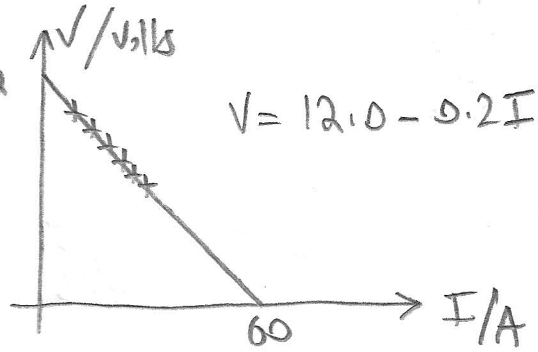
$\boxed{l = 0.233 \text{ m}}$



A 23 cm filament is rather on the long side for a lightbulb (!) so filaments tend to be tightly coiled. The width of a filament bulb might be $\simeq$ 2 cm

(x)





$V = 12.0 - 0.2I$

$\mathcal{E} = Ir + V$

$V = IR$

So $V = \mathcal{E} - Ir$. Comparing with $V = 12.0 - 0.2I$

$\therefore$ $\boxed{\begin{matrix} \mathcal{E} = 12.0 \text{ Volts} \\ r = 0.2\ \Omega \end{matrix}}$

{ You probably won't measure currents more than 2.0 A safely! }

↑

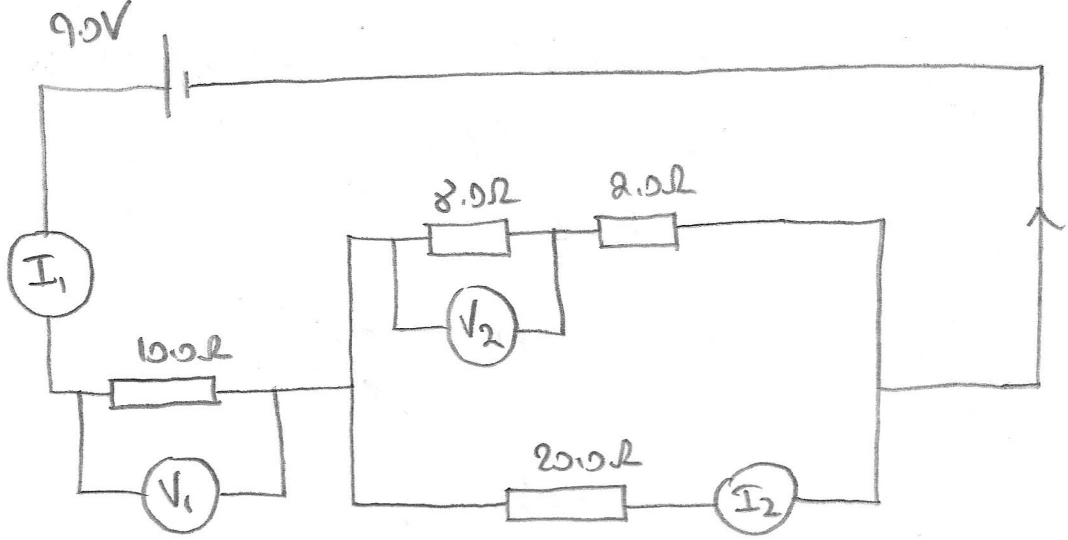
So extrapolate to $V = 0$.

Power dissipated by variable resistor is $I^2 R = P$

Now $I = \dfrac{\mathcal{E}}{R + r}$ so $P = \dfrac{\mathcal{E}^2 R}{(R+r)^2}$ $\left\{ I = \dfrac{12.0}{5 + 0.2} = 2.31\text{A} \right\}$

$\therefore$ when $R = 5.0\ \Omega$: $P = \dfrac{12^2 \times 5}{(5 + 0.2)^2} = \boxed{26.6 \text{ W}}$

Q2



Total resistance is: $R = 10.0 + \dfrac{1}{\dfrac{1}{8+2} + \dfrac{1}{20}}$

$= 16\frac{2}{3}\ \Omega \qquad (16.7\ \Omega)$

keep exact for follow on calcs

$\therefore\ I_1 = \dfrac{9.0}{16\frac{2}{3}}\ \text{(A)} = \boxed{0.54\ A}$

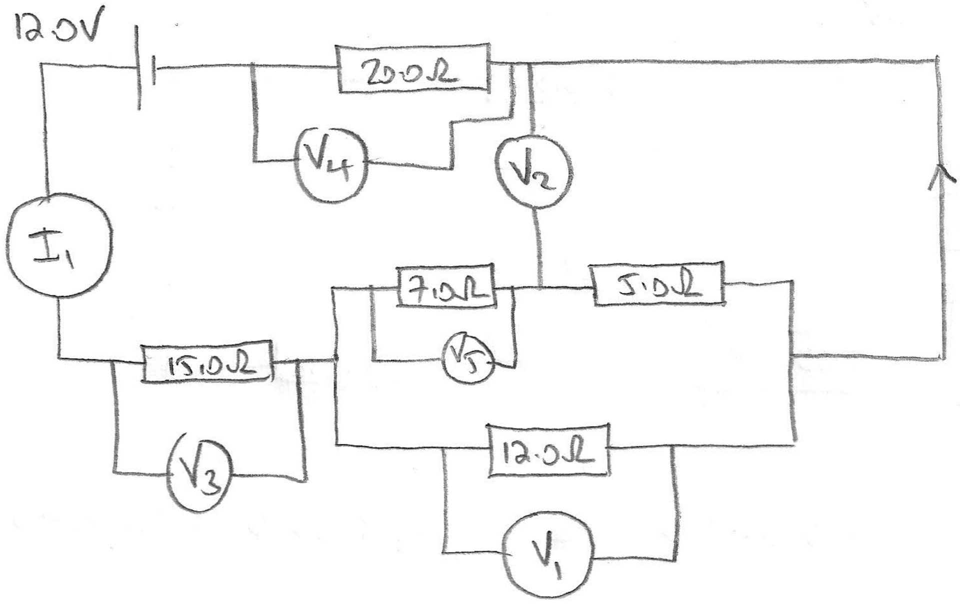
$V_1 = I_1 \times 10.0\ \Omega = \boxed{5.4\ V}$

$\therefore\ I_2 = \dfrac{9 - 5.4}{20.0}\ \text{(A)} = \boxed{0.18\ A}$

$\therefore\ V_2 = (0.54 - 0.18) \times 8.0 \qquad \text{(V)}$

$= \boxed{2.88\ V}$

3/



$$I_1 = \frac{12.0}{R_{TOT}} = \frac{12.0}{15.0 + \dfrac{1}{\dfrac{1}{7+5} + \dfrac{1}{12}} + 20.0} \qquad (A)$$

$$= \frac{12.0}{41} = \boxed{0.293\ A}$$

Total resistance = 41 Ω

$$V_3 = 12.0 \times \frac{15}{41} = \boxed{4.39\ V}$$

$$V_4 = 12.0 \times \frac{20}{41} = \boxed{5.85\ V}$$

$$V_1 = 12.0 - V_3 - V_4 = 12.0 \times \left(1 - \frac{15}{41} - \frac{20}{41}\right)$$

$$= 1\tfrac{31}{41}\ V = \boxed{1.76\ V}$$

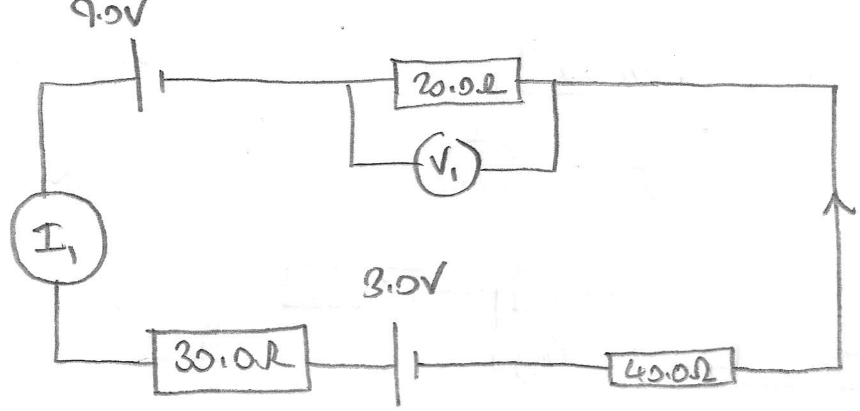
$$V_5 = V_1 \times \frac{7}{7+5} = \boxed{1.02\ V}$$

$$V_2 = 12.0 - V_3 - V_5 - V_4$$

$$= 12.0 - 12 \times \frac{15}{41} - \underbrace{1\tfrac{31}{41} \times \frac{7}{12}}_{V_5} - 12 \times \frac{20}{41}$$

$$= \frac{30}{41}\ V = \boxed{0.73\ V}$$

4/

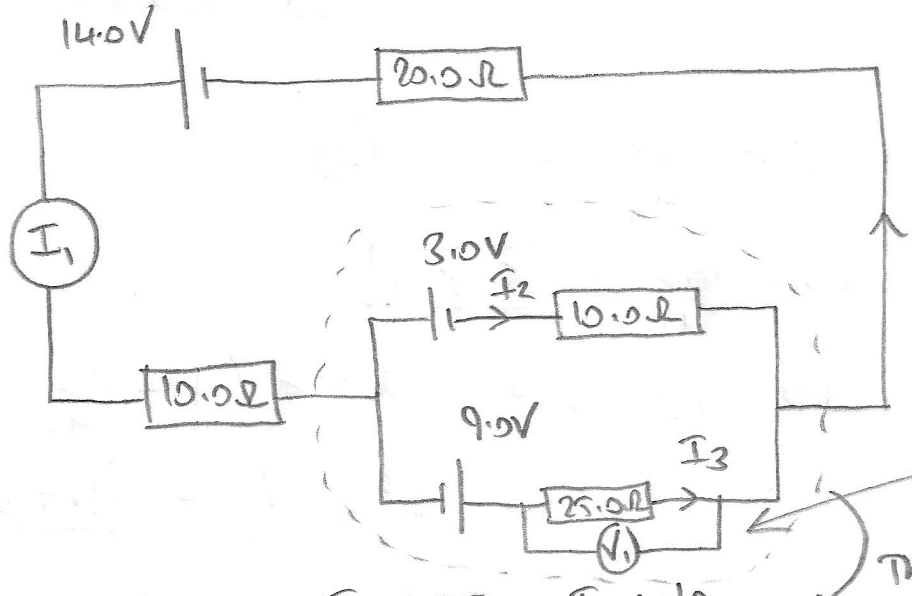


KII: $\underbrace{9.0 - 3.0}_{\text{Net EMF to loop}} = I_1 \times 30.0 + I_1 \times 40.0 + I_1 \times 20.0$

$\Rightarrow \quad I_1 = \dfrac{6.0}{30+40+20} = 0.067\text{ A} = \boxed{66.7\text{ mA}}$

$V_1 = I_1 \times 20.0 = \boxed{1.33\text{ Volts}}$

5/



KI: $\boxed{I_1 = I_2 + I_3}$

! oops. In question, V_1 is across 9.0V too.... will sweet at the end.

This loop

KII: $3.0 + 9.0 = I_3 \times 25 - I_2 \times 10$

↻ + $\quad \Rightarrow \quad 25 I_3 - 10 I_2 = 12$ ①

{Take care to follow direction of I! as you go upside down in a loop!}

KII: (lower loop) ↻+: $\quad 14 + 9 = I_1 \times 10 + I_3 \times 25 + I_1 \times 20$

$\Rightarrow \quad 23 = 30 I_1 + 25 I_3$ ②

So using $I_1 = I_2 + I_3$: $\quad 23 = 30 I_2 + 55 I_3$ ②

$\therefore \quad 75 I_3 - 10 I_2 = 12 \quad ①$

$30 I_2 + 55 I_3 = 23 \quad ②$

$3① + ②: \quad 75 I_3 + 55 I_3 = 36 + 23$

$$I_3 = \frac{59}{130} = \boxed{0.45 A}$$

in ②: $$I_2 = \frac{23 - 55 \times \frac{59}{130}}{30}$$ ← I_3

$$= \boxed{-0.065 A}$$

(so I_2 is in the opposite direction as shown in the diagram)

$\therefore \quad I_1 = I_2 + I_3 = \boxed{0.39 A}$

Now V_1* is actually $-I_3 \times 25.0 + 9.0$ to match the question.

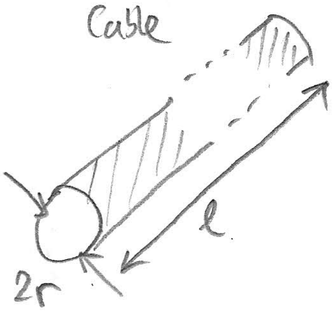
ie $9.0 - \frac{59}{130} \times 25 = \boxed{-2.35 V}$

(Sign doesn't really matter, it depends which way the voltmeter (V_1) is connected. If the 'red' end is on the right, the result is -2.35 V).

* In the diagram here, $V_1 = I_3 \times 25.0$ Volts.

6/ Power cable carries power $\boxed{P = VI}$

loss is $\boxed{kP = I^2 R}$



$k = 0.009$

$V = 750 \times 10^3$ Volts

$P = 2000 \times 10^6$ W.

$l = 1600 \times 10^3$ m

$\rho = 2.8 \times 10^{-8}\ \Omega\text{m}$

$$\boxed{R = \rho \frac{l}{\pi r^2}} \quad \therefore \quad \boxed{r = \sqrt{\frac{\rho l}{\pi R}}}$$

So $I = P/V$

$$\therefore \quad kP = \frac{P^2}{V^2} R$$

$$\therefore \quad \boxed{\frac{kV^2}{P} = R}$$

$$\therefore \quad r = \sqrt{\frac{\rho l}{\pi k V^2 / P}}$$

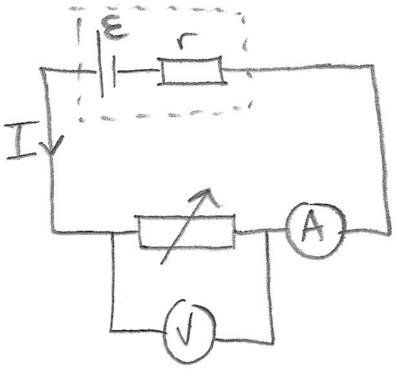
$$\therefore \quad \boxed{r = \sqrt{\frac{P \rho l}{\pi k V^2}}}$$

$$\therefore \quad r = \sqrt{\frac{2000 \times 10^6 \times 2.8 \times 10^{-8} \times 1600 \times 10^3}{\pi \times 0.009 \times (750 \times 10^3)^2}}$$

$$= 0.075 \text{ m} \quad = \boxed{75.1 \text{ mm}}$$

(so a 15 cm diameter cable).

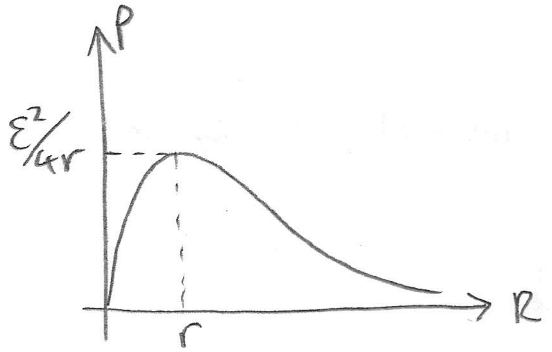
7/



Power dissipated in variable ressor R is $P = I^2 R$

$$I = \frac{\varepsilon}{r+R}$$

$$\therefore \boxed{P = \frac{\varepsilon^2 R}{(r+R)^2}}$$



when $R=0$, $P=0$
$R \to \infty$, $P \to 0$

Stationary point when $\frac{dP}{dR} = 0$

$$\frac{dP}{dR} = \frac{(r+R)^2(1) - R(2(r+R)(1))}{(r+R)^4} \times \varepsilon^2$$

so $\frac{dP}{dR} = 0$ when $(r+R)^2 - 2R(r+R) = 0$

$$(r+R)(r+R-2R) = 0$$

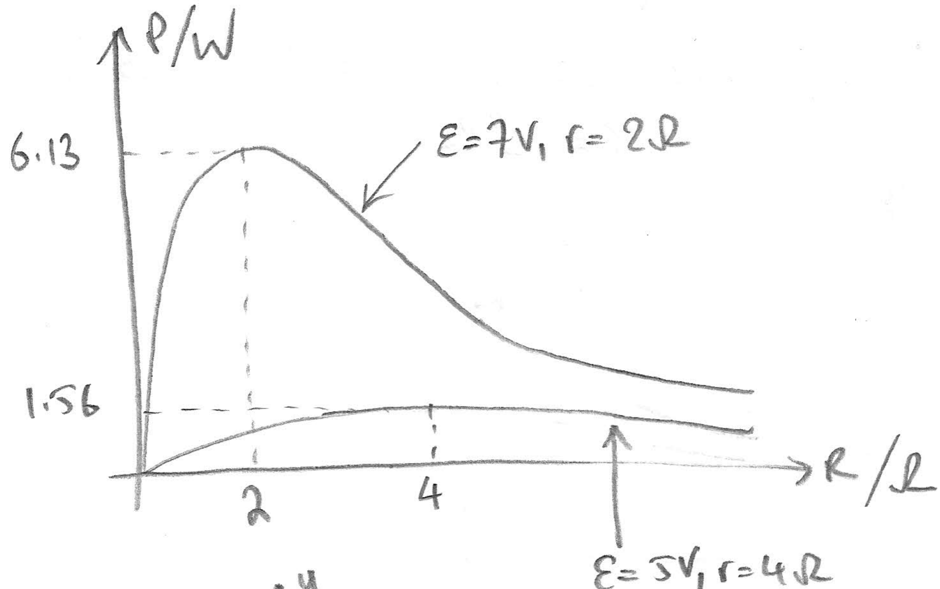
$$(r+R)(r-R) = 0$$

and $\therefore$ since $r, R > 0 \Rightarrow \boxed{r=R}$

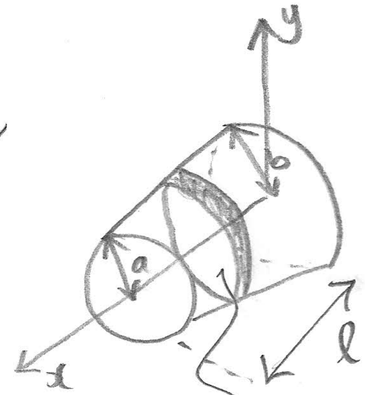
when $r=R$, $P = \frac{\varepsilon^2 r}{(2r)^2} = \boxed{\varepsilon^2/4r}$ ie maximum power dissipated.

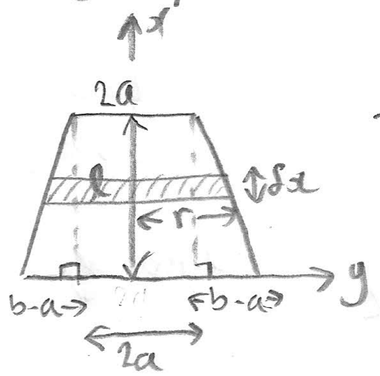
↳ see spreadsheet for $\varepsilon = 5V$, $r = 4\Omega$ vs $\varepsilon = 7V$, $r = 2\Omega$

Max power: $5^2/4 \times 4 = \boxed{1.56W}$ $\quad 7^2/4 \times 2 = \boxed{6.13W}$



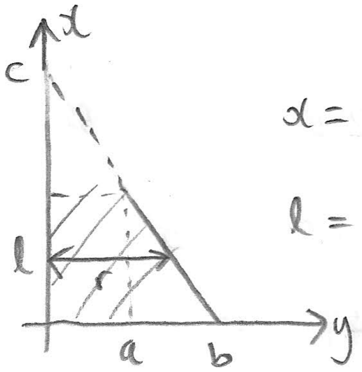
8/





Truncated cone cross section

resistance of cylinder is $\delta R = \dfrac{\rho\,\delta x}{\pi r^2}$



$$x = c - \frac{l}{b-a}\,y$$

$$l = c - \frac{l}{b-a}\,a \qquad \therefore \quad c = l\left(1 + \frac{a}{b-a}\right)$$

$$\therefore \quad c = \frac{l\,(b-a+a)}{b-a}$$

$$c = \frac{lb}{b-a}$$

So $\boxed{x = \dfrac{l\,(b-y)}{b-a}}$

in our case $r = y$

so $\dfrac{(b-a)x}{l} = b - y$

$$\therefore \quad y = r = b - \frac{(b-a)x}{l}$$

$$\therefore \quad \delta R = \frac{\rho\,\delta x}{\pi\left(b - (b-a)x/l\right)^2}$$

(12)

Since resistors add in series, total resistance of the truncated cone is $\int \delta R \Rightarrow R = \int_{x=0}^{l} \frac{\rho\, dx}{\pi\left(b-(b-a)x/l\right)^2}$

$$R = \frac{\rho}{\pi}\int_0^l \frac{dx}{\left(b-(b-a)x/l\right)^2}$$

Now $\frac{d}{dx}\left(\frac{1}{A-Bx}\right) = -\frac{1}{(A-Bx)^2}(-B) = \frac{B}{(A-Bx)^2}$

$$\therefore \int \frac{1}{(A-Bx)^2}\,dx = \frac{1}{B}\,\frac{1}{A-Bx} + C$$

$$\therefore R = \rho/\pi\ \frac{1}{(b-a)/l}\left[\frac{1}{b-(b-a)x/l}\right]_0^l$$

$$\therefore R = \rho/\pi\ \frac{l}{b-a}\left[\frac{1}{b-(b-a)} - \frac{1}{b}\right]$$

$$\therefore R = \rho/\pi\ \frac{l}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right)$$

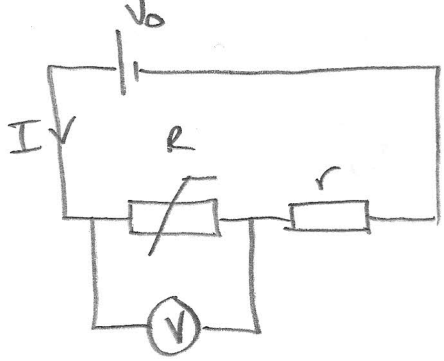
$$\therefore R = \rho/\pi\ \frac{l}{b-a}\left(\frac{b-a}{ab}\right)$$

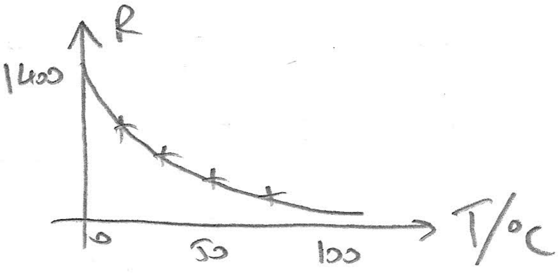
$$\therefore \boxed{R = \frac{\rho l}{\pi a b}}$$ as required.

Does it make sense? well let $a=b \Rightarrow R = \rho l/\pi a^2$
ie resistance of a cylinder ✓

[Note since πab is the cross sectional area of an ellipse, this means an elliptical resistor has the same resistance of a truncated cone!]

9/





let R for thermistor be

$$\boxed{R = R_0 e^{-kT}}$$

(example: $k = 0.0271$, $R_0 = 1183\Omega$)

$0°C < T < 90°C$

T_{min} T_{max}

$$V = V_0 \times \frac{R}{R+r}$$

$$V = \frac{V_0 R_0 e^{-kT}}{R_0 e^{-kT} + r}$$

$$V = \frac{V_0 R_0}{R_0 + r e^{kT}}$$

$$\Delta V = V(T_{min}) - V(T_{max})$$

$$\therefore \Delta V(r) = V_0 R_0 \left(\frac{1}{R_0 + r e^{kT_{min}}} - \frac{1}{R_0 + r e^{kT_{max}}} \right)$$

$$\frac{d\Delta V}{dr} = V_0 R_0 \left(\frac{-e^{kT_{min}}}{(R_0 + r e^{kT_{min}})^2} + \frac{e^{kT_{max}}}{(R_0 + r e^{kT_{max}})^2} \right)$$

so $\frac{d\Delta V}{dr} = 0$ when $\frac{e^{kT_{min}}}{(R_0 + r e^{kT_{min}})^2} = \frac{e^{kT_{max}}}{(R_0 + r e^{kT_{max}})^2}$

$$\Rightarrow \frac{R_0 + r e^{kT_{max}}}{e^{\frac{1}{2}kT_{max}}} = \frac{R_0 + r e^{kT_{min}}}{e^{\frac{1}{2}kT_{min}}}$$

$$\therefore \; r\left(e^{\frac{1}{2}kT_{max}} - e^{\frac{1}{2}kT_{min}}\right) = R_0\left(e^{-\frac{1}{2}kT_{min}} - e^{-\frac{1}{2}kT_{max}}\right)$$

$$\boxed{r = \frac{R_0\left(e^{-\frac{1}{2}kT_{min}} - e^{-\frac{1}{2}kT_{max}}\right)}{e^{+\frac{1}{2}kT_{max}} - e^{+\frac{1}{2}kT_{min}}}}$$

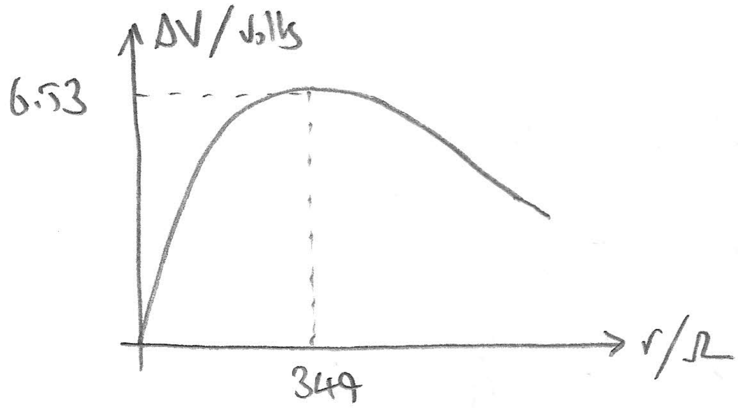
This is the optimum r which maximizes ΔV.

So for our numbers: $$r = 1183\,\Omega \times \frac{\left(1 - e^{-\frac{1}{2}0.0271\times 90}\right)}{e^{\frac{+0.0271\times 90}{2}} - 1}$$

$$= \boxed{349\,\Omega}$$

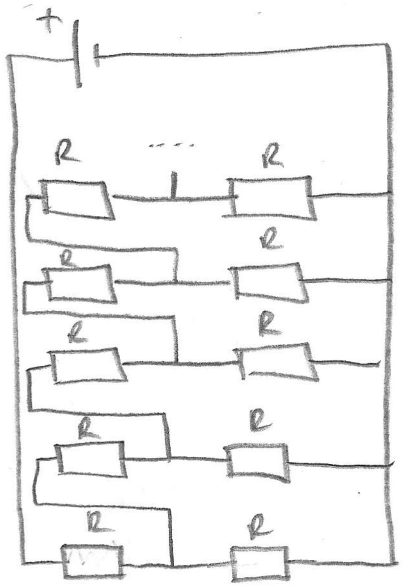
See spreadsheet for a plot of $\Delta V(r)$.

(if $V_0 = 12$ Volts).



Q10/

Resistance of the first circuit is:



$$R_{TOT} = R + \cfrac{1}{\cfrac{1}{R} + \cfrac{1}{R + \cfrac{1}{\cfrac{1}{R} + \cfrac{1}{R + \cfrac{1}{\cdots}}}}}$$

$$= R\left(1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}\right)$$

A "continued fraction"

Now if $\phi = (\ldots)$

$\phi = 1 + 1/\phi$ $\quad \therefore \quad \phi^2 = \phi + 1$

$$\therefore \quad \phi^2 - \phi - 1 = 0$$

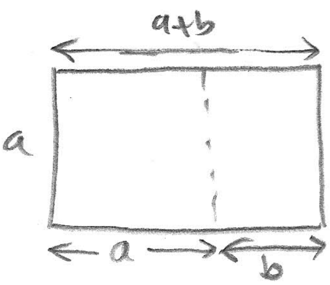
$$\left(\phi - \frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0$$

$$\left(\phi - \frac{1}{2}\right)^2 = 5/4$$

$$\phi = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

Now clearly $\phi > 0$ so $\boxed{\phi = \frac{1+\sqrt{5}}{2}}$ The Golden Ratio!

$\simeq 1.618$



$(a \geq b)$

Golden ratio when $\frac{a+b}{a} = \frac{a}{b} = \phi$

$\therefore \quad 1 + \frac{1}{\phi} = \phi$

which is the same as above.

ie $\boxed{\phi = \frac{1+\sqrt{5}}{2}}$

Resistance of the second circuit is:

$$R_{TOT} = R\left(3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}}}}}\right)$$

$$\approx 3.14151\,R$$

$$\approx \boxed{\pi R} \qquad (\pi = 3.14159265359\ldots)$$

Resistance of the third circuit is:

$$R_{TOT} = R\left(6 + \cfrac{1}{2 + \cfrac{1}{12 + \cfrac{1}{2 + \cfrac{1}{12 + \cdots}}}}\right)$$

let $x = 12 + \cfrac{1}{2 + \cfrac{1}{12 + \cfrac{1}{2 + \cdots}}}$

$\therefore \left(\ldots\right) = x - 6 = 12 + \cfrac{1}{2 + 1/x} - 6$

$\therefore$

$$x = 12 + \frac{1}{2 + \frac{1}{x}}$$

$$x = 12 + \frac{x}{2x+1}$$

$$(2x+1)x = 12(2x+1) + x$$

$$2x^2 - 24x - 12 = 0$$

$$x^2 - 12x - 6 = 0$$

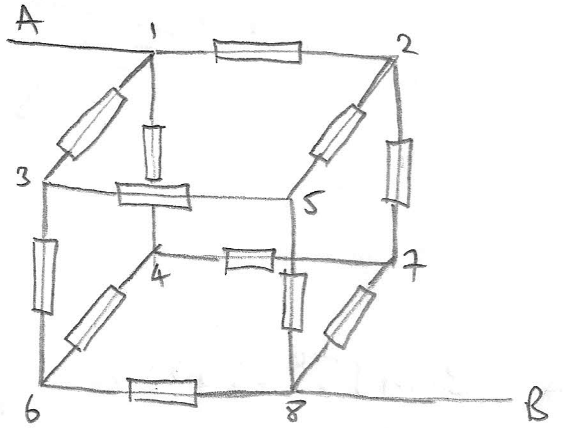
$$(x-6)^2 - 42 = 0$$

$$\therefore x = 6 \pm \sqrt{42}$$

$x > 0$ so $x - 6 = \boxed{\sqrt{42}}$

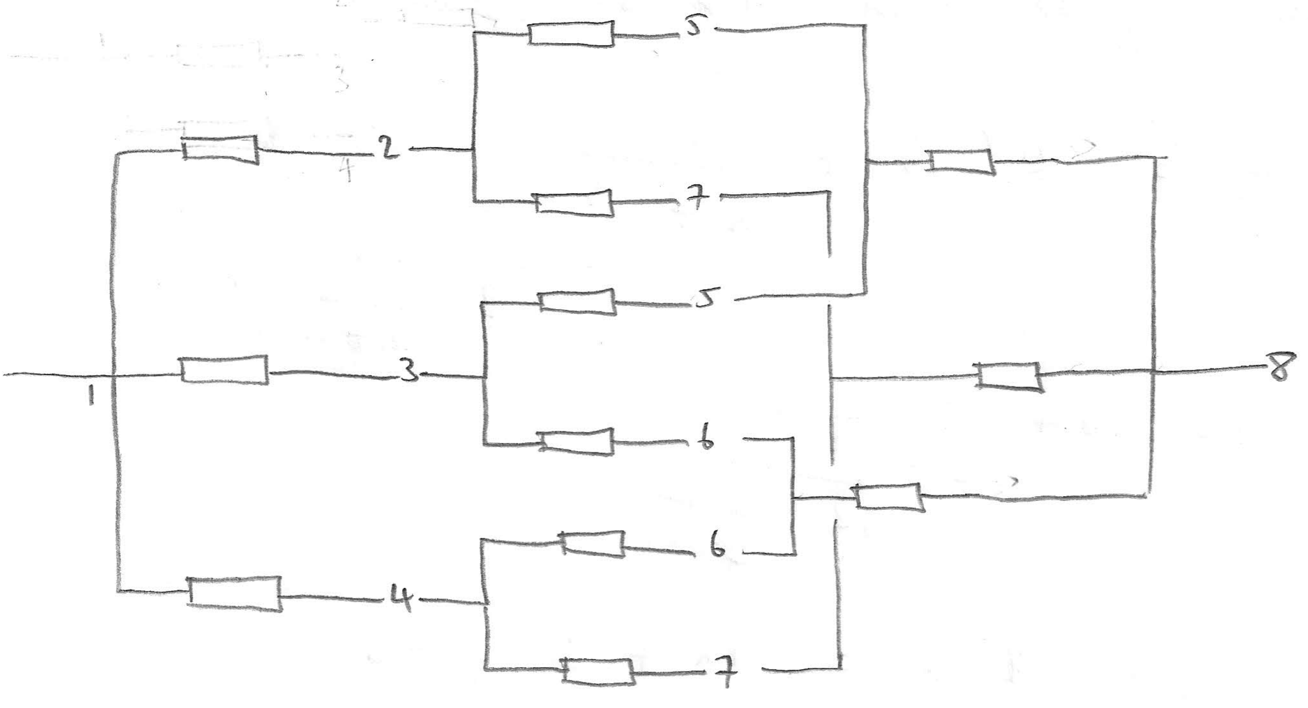
In "Hitch Hiker's Guide to the Galaxy" "42" is the answer to the "Ultimate Question", pondered by the mega-computer "Deep Thought". So the circuit "at the root of Deep Thought" should have a resistance of $\sqrt{42}$ R !

Q11



Consider a cube of identical resistors. Each has a resistance of 1Ω.

To find the resistance between A and B we need to redraw the circuit in 2D!

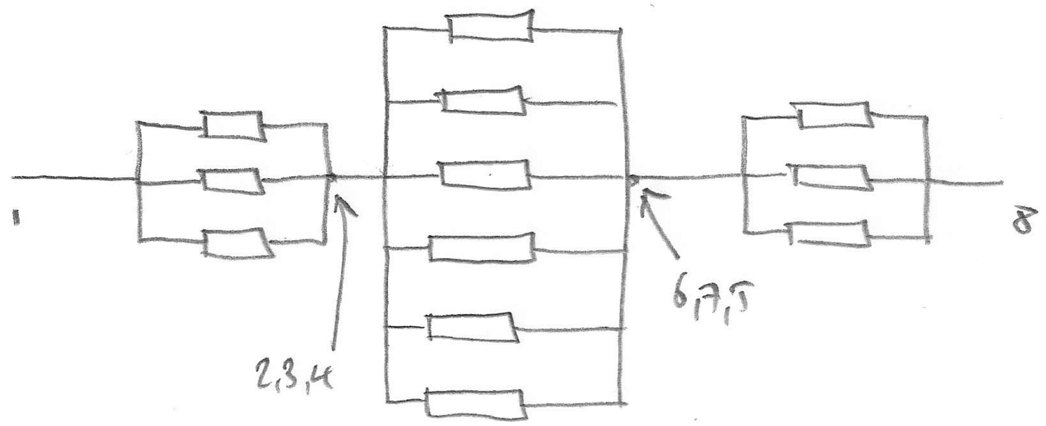


Now this is tricky to consider as a connection of parallel circuits because of the way the second set of vertices cross-link.

And so a trick is needed! (Note this only works because all the resistors are the same, without this you would have to solve a set of simultaneous equations for currents in each wire, using KI and KII).

The trick is, by symmetry, voltages at 2,3,4 ARE THE SAME. Ditto for 6,7,5.

∴ we can consider the following equivalent circuit:



The total resistance is therefore:

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \boxed{5/6} \quad (\Omega)$$

Extension: Write a computer program to solve the 12 equations (simultaneous) (KII, KI) for currents I_i through a set of different resistors wired like this, given a PD V_{AB} between A and B. Use this information to find the total current I_{AB} and hence find $R_{AB} = V_{AB}/I_{AB}$.

AR 22/7/20.