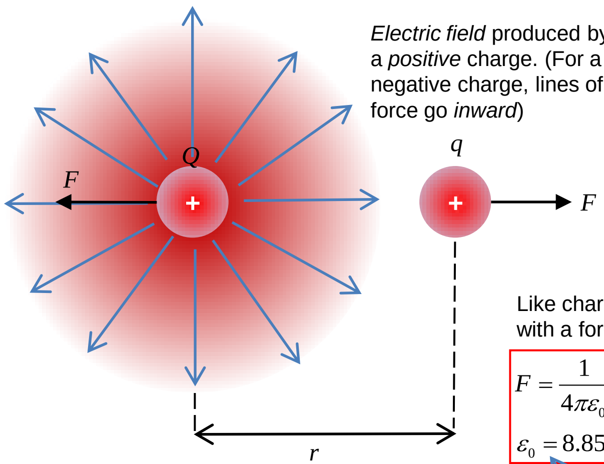
Electric fields

Electric charge is a fundamental characteristic of matter. *Point charges* (i.e. that are infinitesimally small with respect to the characteristic distance scales of interaction) will interact with each other according to *Coulomb's Law*. This is an *inverse-square law*, and directly analogous to *Newton's law of Universal Gravitation*..



Electric field produced by a *positive* charge. (For a negative charge, lines of force go *inward*)

Particle	Charge	Mass /kg	Mass / electron masses
Proton	$+e$	1.6726×10^{-27}	1,836
Neutron	0	1.6749×10^{-27}	1,839
Electron	$-e$	9.1094×10^{-31}	1

$e = 1.6022 \times 10^{-19}\,\mathrm{C}$

Charge is measured in *coulombs*, with the fundamental unit being the charge on the electron.

Charles-Augustin de Coulomb
1736-1806

Like charges will *repel* with a force

$$\boxed{F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2} \qquad \varepsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{Fm}^{-1}}$$

This is Coulomb's Law of Electrostatics

Permittivity of free space

It is instructive to imagine the possible influence of a charge on another, if the second charge could be at *any location in space*. This is the concept of an *electric field* $\mathbf{E}$. At any point in space we can imagine a **vector** describing the *magnitude* and *direction* of the force per unit charge, acting on a charge q.

$$\boxed{\mathbf{F} = q\mathbf{E}}$$

Using *Newton's Second Law*, the charge would accelerate in the direction of $\mathbf{F}$. *The vector field* $\mathbf{E}$ therefore tells us *where a charge would move at a point in space.* This is just like a magnetic field describing the direction that a compass needle would point at any point in space, but also how *strong* the force is at any point.

Carl Friedrich Gauss
1777-1855

Compare the strength of gravity for two protons at separation d with electrostatic repulsion

$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2}$$

$$F_G = \frac{Gm^2}{d^2}$$

$$\frac{F_E}{F_G} = \frac{e^2}{4\pi\varepsilon_0 Gm^2}$$

$$\boxed{\frac{F_E}{F_G} \approx 1.236 \times 10^{36}}$$

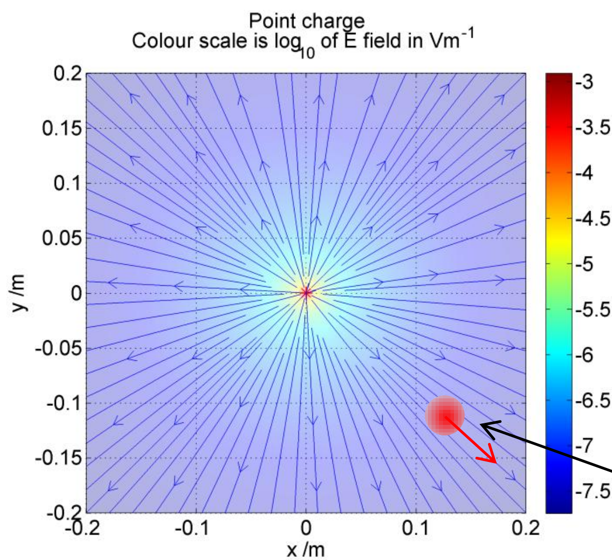
So how does a nucleus stay together?
Over scales of 10^{-15} m the *Strong* and *Weak* **nuclear forces** counteract electrostatic repulsion. Neutrons as well as protons contribute to these forces.

For two electrons:

$$\frac{F_E}{F_G} \approx 1.236 \times 10^{36} \times 1{,}836^2$$

$$\boxed{\frac{F_E}{F_G} \approx 4.165 \times 10^{42}}$$

So electric force is MUCH stronger at an atomic scale.



The electric field sourced by a point charge Q is given by:

$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}}$$

$\hat{\mathbf{r}}$ is a *unit vector* pointing in the radial direction away from the point charge Q

So the *field lines* due to the point charge radiate outwards, if Q is positive. The field line divergence is indicative of the decay in strength of $\mathbf{E}$ with radius r

The *field lines* indicate the direction the positive charge would move at this point in the electric field due to the charge in the centre of the diagram

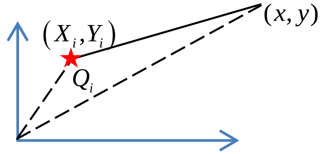
Michael Faraday
1791-1867

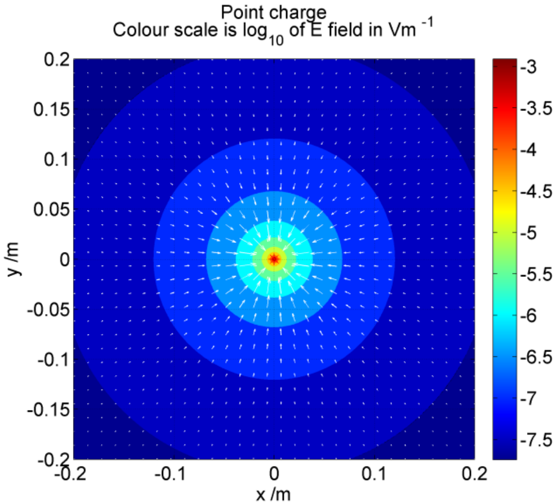
$G = 6.6741 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ Universal Gravitational constant

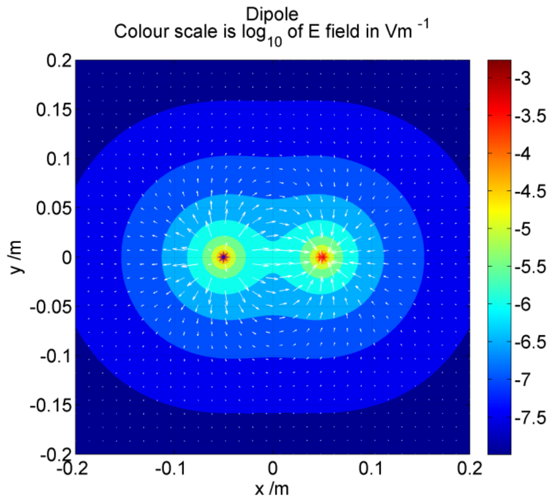
If there is more than one point charge, the vector fields will *superpose*, i.e. add in a vector sense. A **Dipole** is an arrangement of two opposite charges, whereas a **parallel plate capacitor** can be modelled by two lines of opposite point charges, separated by a fixed distance. In all the examples below, a MATLAB computer program has added up the $\mathbf{E}$ fields due to the point charges for any x,y location. Notice the field strength between the capacitor plates appears to be *constant*.

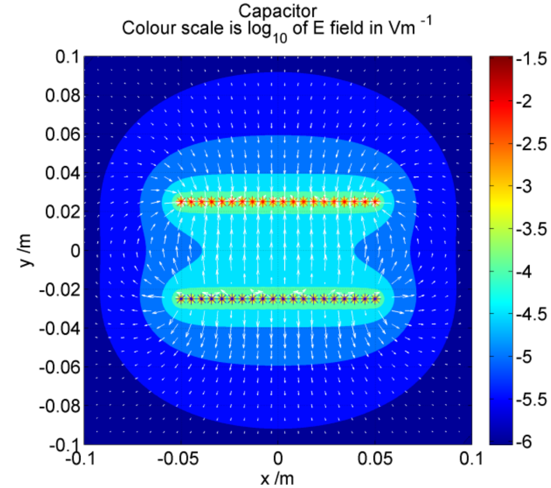
$$\boxed{\mathbf{E}(x,y)=\frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{\left(x-X_i\right)^2+\left(y-Y_i\right)^2}\frac{\hat{\mathbf{x}}\left(x-X_i\right)+\hat{\mathbf{y}}\left(y-Y_i\right)}{\sqrt{\left(x-X_i\right)^2+\left(y-Y_i\right)^2}}}$$

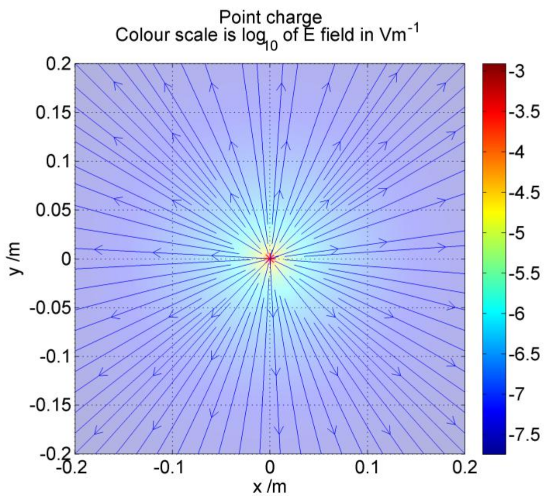
i.e. charge Q_i at fixed location $\left(X_i,Y_i\right)$

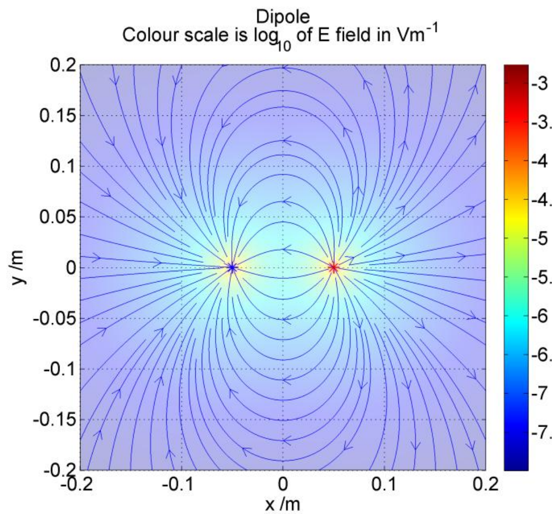


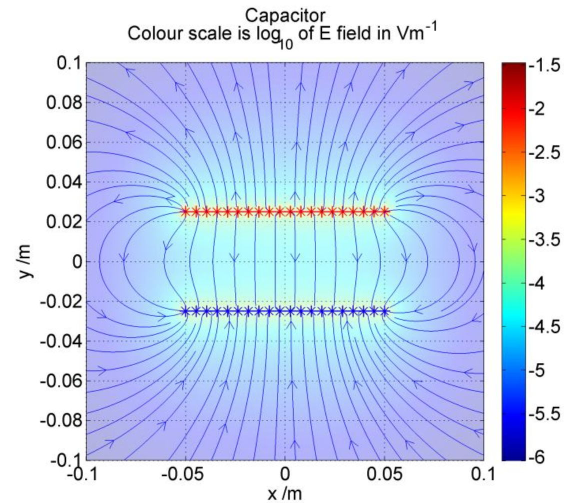












Gauss' Law

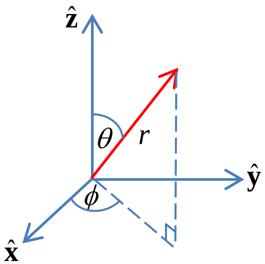
Rather than adding up point charges, we can use Gauss's law to work out the electric field which passes through a surface S which encloses a charge Q

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0} \qquad \mathbf{F} = q\mathbf{E}$$

force on charge q in electric field $\mathbf{E}$

$$Q = \iiint \rho(x,y,z)\,dxdydz$$

charge density



$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

$$d\mathbf{S} = \hat{\mathbf{r}}dA$$

$$dA = r\sin\theta\, d\theta\, d\phi$$

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

$$\int_S \mathbf{E}\cdot d\mathbf{S} = 4\pi r^2 E$$

$$\therefore E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\boxed{\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}}$$

i.e. Gauss's Law is consistent with Coulomb's Law for a point charge

Carl Friedrich Gauss
1777-1855

Electric potential

The work done W in bringing a charge from an infinite distance to a specific position in an electric field is proportional to the *electric potential* V. This quantity can be used in Conservation of Energy calculations The electric potential is the electrical energy per unit charge, i.e. *voltage*.

For the electric potential at radius r from a point charge of strength Q

$$W = \int_\infty^r \mathbf{F}\cdot d\mathbf{r} \qquad V = \frac{W}{q} \qquad d\mathbf{r} = -dr\hat{\mathbf{r}}$$

$$\mathbf{F} = q\mathbf{E} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad \therefore W = -\int_\infty^r \frac{qQ}{4\pi\varepsilon_0 r^2}dr$$

$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\int_r^\infty \frac{1}{r^2}dr \qquad \therefore V = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_r^\infty$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}}$$

For more complex distributions of charges, it is often easier to compute the electric potential since this is a *scalar* function. The resulting electric field can be found from the *gradient* of the potential

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$$\boxed{\mathbf{E} = -\nabla V}$$

This is called the **gradient** ('grad') *vector operator*. In other words, we have to find the slope of the electric potential in each of the x,y,z directions.

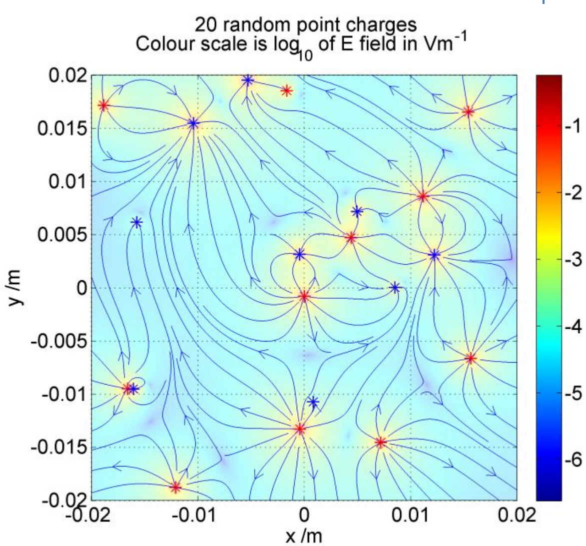
A useful result is that the *line integral* of electric fields from position A to B is the difference in the electric potential, *regardless of the path taken.*

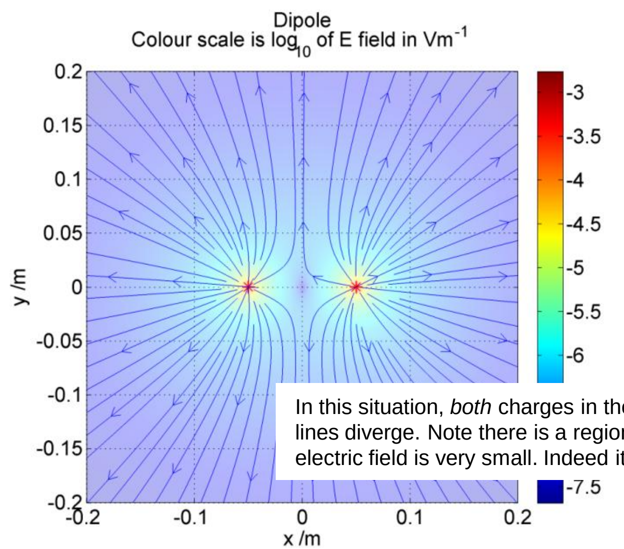
$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B -\nabla V\cdot d\mathbf{l} = \int_A^B -\left(\hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}\right)\cdot\left(\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz\right)$$

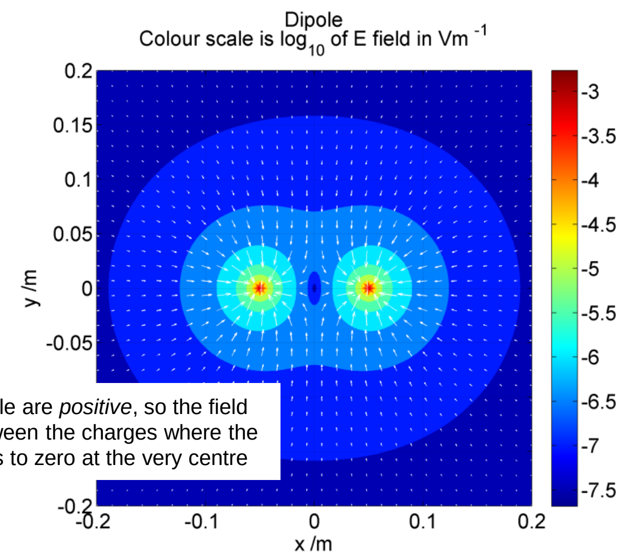
$$\therefore \int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = -\int_A^B dV = V(A) - V(B)$$

$$\boxed{\int_A^B \mathbf{E}\cdot d\mathbf{l} = V(A) - V(B)}$$

Fields that work in this way are called *conservative*.



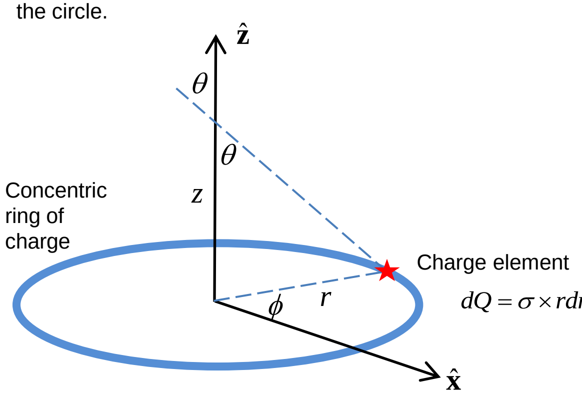




In this situation, *both* charges in the dipole are *positive*, so the field lines diverge. Note there is a region between the charges where the electric field is very small. Indeed it tends to zero at the very centre

Electric field strength above an infinite conducting plane

Consider an infinite conducting plane with charge per unit area σ. The electric field at a perpendicular distance z from the plane must only be in the z direction. Why? If we consider concentric rings of charge contributing to the electric field, by symmetry we can clearly see that any field in the x,y direction will be opposed by an equal and opposite signed field from a charge at the diametric opposite in the circle.



$$dQ = \sigma \times r dr d\phi$$

$$E_x = E_y = 0$$

$$E_z = \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{\sigma r d\phi dr}{4\pi\varepsilon_0\left(r^2+z^2\right)}\cos\theta$$

$$\sqrt{r^2+h^2}\cos\theta = z$$

$$\therefore E_z = \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{z\sigma r d\phi dr}{4\pi\varepsilon_0\left(r^2+z^2\right)^{\frac{3}{2}}}$$

$$E_z = \frac{2\pi\sigma z}{4\pi\varepsilon_0}\int_{r=0}^{\infty}\frac{rdr}{\left(r^2+z^2\right)^{\frac{3}{2}}}$$

$$E_z = \frac{\sigma z}{2\varepsilon_0}\left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^{\infty} \longleftarrow \int \frac{rdr}{\left(r^2+z^2\right)^{\frac{3}{2}}} = -\frac{1}{\sqrt{r^2+z^2}}+c$$

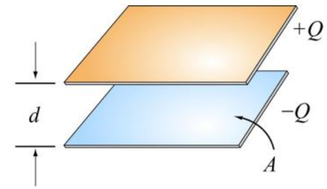
$$E_z = \frac{\sigma z}{2\varepsilon_0}\left[0-\left(-\frac{1}{z}\right)\right]$$

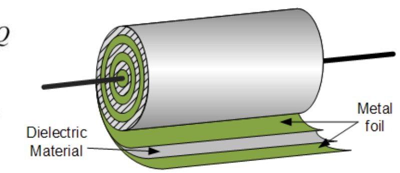
$$\boxed{E_z = \frac{\sigma}{2\varepsilon_0}}$$

So the electric field strength above an infinite conductive plane with charge per unit area σ is a *constant*

Parallel plate capacitor

A parallel plate capacitor can be modelled as two infinite charges planes separated by distance d, separated by an insulating dielectric, and rolled up like a Swiss cheese.





Noting there are two planes (with opposing but equal charges Q)

$$\sigma = \frac{Q}{A}$$

$$E = 2\times\frac{\sigma}{2\varepsilon_0}$$

$$\boxed{E = \frac{Q}{\varepsilon_0 A}}$$

A real capacitor will have finite area A. Let us assume this is much larger than the plate separation d so the 'infinite plane' argument has some validity

i.e. a constant electric field between the plates
We can obtain the same result using Gauss' law:

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

$$EA = \frac{Q}{\varepsilon_0} \longleftarrow$$ Assuming the electric field is both constant and perpendicular to the conducting plane surface

$$\boxed{E = \frac{Q}{\varepsilon_0 A}}$$

The electric potential is:

$$V = \int_0^d E dx$$

$$\boxed{V = \frac{Qd}{\varepsilon_0 A}}$$

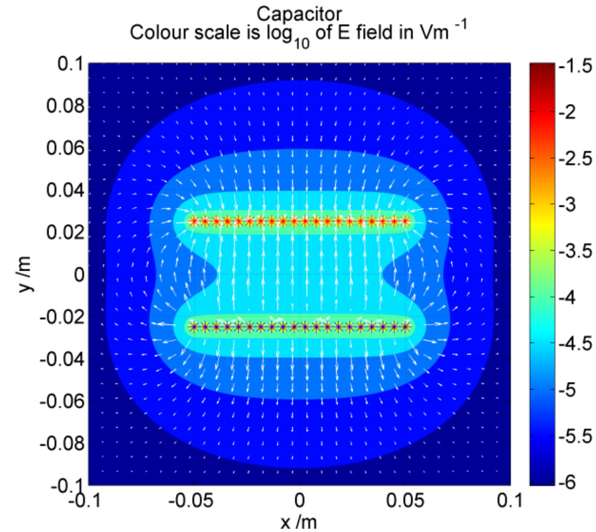
$$\therefore \boxed{E = \frac{V}{d}}$$

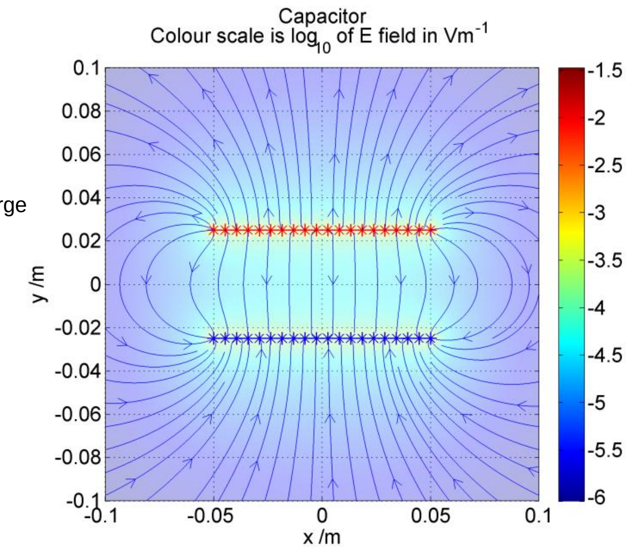
We can define the *Capacitance* (units are Farads) as the ratio of charge stored on the plates to voltage between them

$$Q = CV$$

$$C = \frac{Q}{V}$$

Hence: $\boxed{C = \frac{\varepsilon_0 A}{d}}$

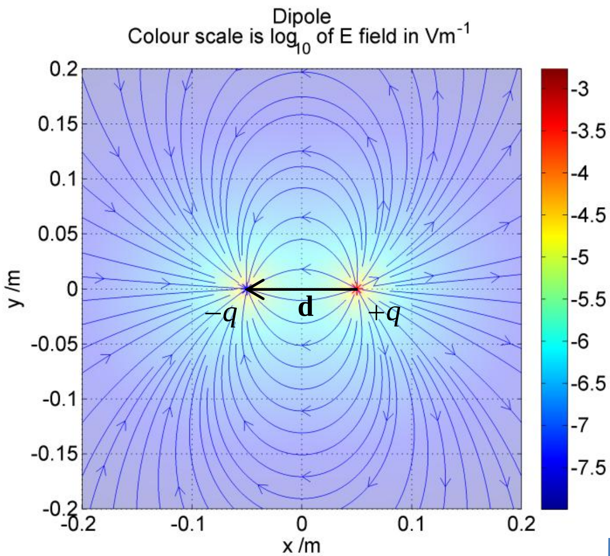




Electric field of a dipole*

$$\boxed{\begin{aligned} V(r) &\approx \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2} \\ \mathbf{E}(r) &\approx \frac{3\left(q\mathbf{d}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - q\mathbf{d}}{4\pi\varepsilon_0 r^3}\end{aligned}}$$

d is the vector separation of opposing charges of magnitude q



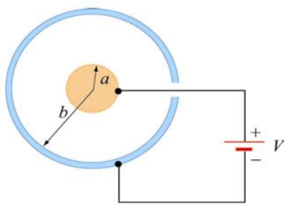
Capacitance of a charged sphere

Consider a charged sphere of charge Q placed inside a hollow conducting sphere of charge $-Q$

$$V = V_a - V_b = \int_a^b E dr = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_a^b$$

$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

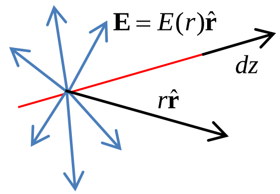
$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 ab}{b-a}}$$



if $b \gg a$

$$\boxed{C = 4\pi\varepsilon_0 a}$$

Electric field of a line charge



Charges on a conducting line will distribute themselves until there is no net electric field along the line. One expects this steady state situation to happen rapidly.

Therefore the electric field is perpendicular to the line, and will only vary with radial distance from the line, assuming the line is both long and straight.

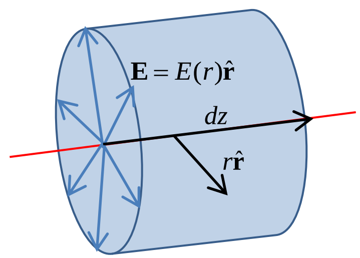
Let the charge per unit length be λ

Gauss's Law

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

$$2\pi r E dz = \frac{\lambda dz}{\varepsilon_0}$$

$$\therefore \boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$



If the line charge was actually a cylindrical conductor of radius a, and this was placed inside a hollow cylindrical conductor of radius b such that $b > a$, the voltage between the outer and inner conductors would be

$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = V_a - V_b$$

$$V = V_a - V_b = \int_a^b E dr = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 r}dr$$

$$\therefore \boxed{V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)}$$

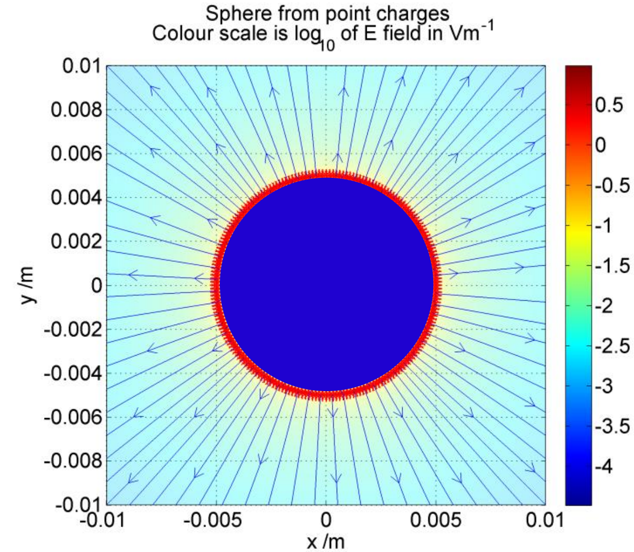
This means the *capacitance* per unit length of a coaxial cable is:

$$\boxed{C = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

Note the plot on the right is only an approximation. Instead of a sphere there is a *ring* of positive charges. While these will result in a zero field strength at the centre, this is not the case for all positions inside the ring.
This shortcoming has been 'overcome' (!) by setting the field strength to be zero inside the sphere.

For a realistic simulation of a charged sphere, charges would have to be uniformly distributed on the surface. This is a *three dimensional distribution*. Due to the inverse square law, this means the field, due to all the charges, is zero everywhere inside the conductor.

Electric field outside a charged sphere of charge Q



The charged sphere will exhibit general features of *any hollow conductor*. Unless it is *polarized* via the application of an external electric field, the charges will tend to distribute uniformly on the conductor surface. If this was not the case, any inhomogeneities would cause charge to move. Since charge is highly mobile in a conductor, we may assume a steady state is rapidly attained.

Gauss's Law – *outside* the conductor

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

$$E\times 4\pi r^2 = \frac{Q}{\varepsilon_0} \quad \therefore \boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

i.e. a spherical conductor looks just like a point charge

Gauss's Law – *inside* the conductor

$$\int_S \mathbf{E}\cdot d\mathbf{S} = 0$$

$$E\times 4\pi r^2 = 0 \quad \therefore \boxed{E = 0}$$

i.e. there is **no electric field inside a hollow conductor**. This is why a conductive container can be used as an effective shield of EM radiation. We call this a *Faraday Cage.*

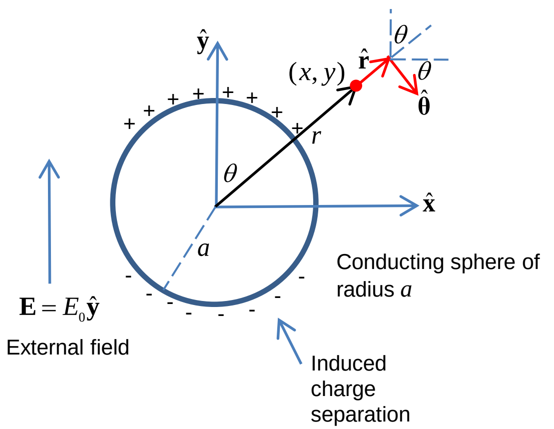
*This shall be justified in another Eclecticon note

Conductive sphere in a uniform electrical field

A conductive sphere placed in a uniform electric field will be polarized by it. In other words, charges will be separated on the surface of the sphere to align with the field.

Note unless the sphere was initially charged, the total charge must still sum to zero. Either way, although the distribution of charge is modified by the external field, the total amount on the sphere must remain the same.

To determine the electrical field outside the sphere (if we assume a hollow conductor, then the field must be exactly zero inside) we shall make a *sensible guess* that it comprises of something which looks like a *dipole*, superimposed upon a uniform field.



Imagine a dipole corresponding to two opposing charges placed at the top and bottom of the sphere

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{d} = 2a\hat{\mathbf{y}} \quad \therefore \mathbf{d}\cdot\hat{\mathbf{r}} = 2a\cos\theta \quad \therefore V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2}$$

Let us assume the 'sphere-dipole field' has a similar expression, where 'polarization' p is to be found:

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

The electric potential a long way from the sphere must relate to the electric field by

$$E_0 = -\frac{\partial V}{\partial y}$$

$$\therefore V_{uniform} = -E_0 y = -E_0 r\cos\theta$$

The full potential is therefore, for $r \geq a$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} - E_0 r\cos\theta}$$

We can therefore evaluate the electric field. Given the form of V it is most convenient to do this in *plane polar coordinates**

$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = E_r\hat{\mathbf{r}} + E_\theta\hat{\boldsymbol{\theta}}$$

$$E_r = -\frac{\partial V}{\partial r} = E_0\cos\theta + \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -E_0\sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

Now there cannot be any tangential electric fields at the surface of the conductor. If there were, then the charges would move.

$$\therefore E_\theta(r = a) = 0$$

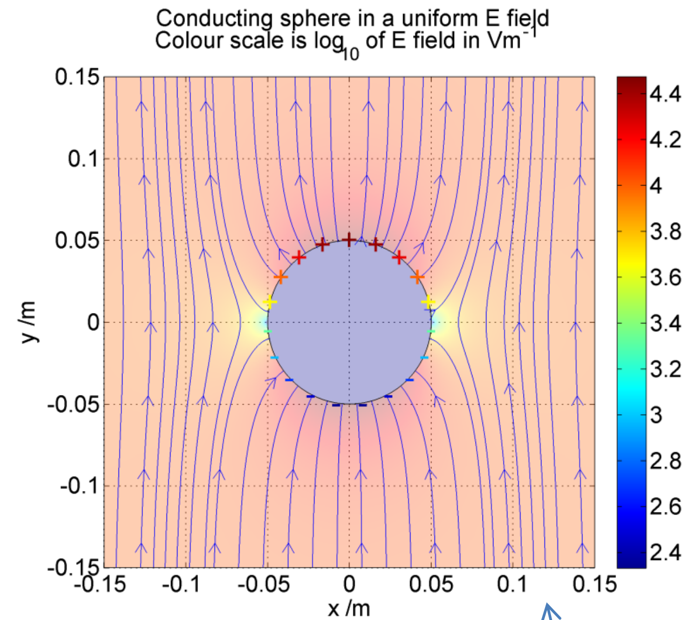
$$\therefore -E_0\sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 a^3} = 0$$

$$\therefore \boxed{p = 4\pi\varepsilon_0 a^3 E_0}$$

Hence: $r \geq a$

$$\boxed{E_r = \left(1 + \frac{2a^3}{r^3}\right)E_0\cos\theta \qquad E_\theta = \left(\frac{a^3}{r^3} - 1\right)E_0\sin\theta}$$

$r < a$ $\quad \boxed{E_r = E_\theta = 0}$



Simulation of the electric field around a conducting sphere

$$\varepsilon_0 = 8.85\times10^{-11}\,\mathrm{Fm}^{-1}$$

$$E_0 = 10^4\,\mathrm{Vm}^{-1}$$

$$a = 0.05\mathrm{m}$$

Note to aid in plotting this field

$$\boxed{\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta \qquad \hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta - \hat{\mathbf{y}}\sin\theta}$$

$$\boxed{r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{x}{y}\right)}$$

Gauss' Law states

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

$$\therefore E_r(r = a) = \frac{\sigma}{\varepsilon_0}$$

where σ is the charge per unit area on the sphere surface.

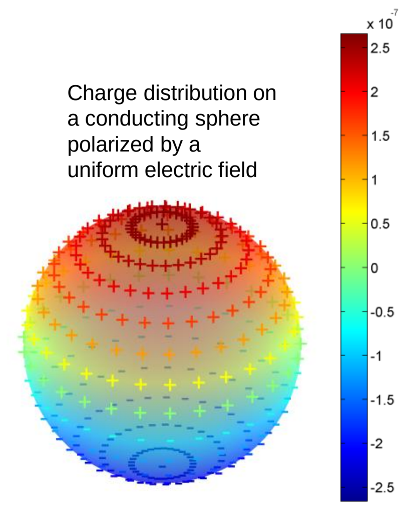
Hence:

$$E_r = \left(1 + \frac{2a^3}{r^3}\right)E_0\cos\theta$$

$$\therefore E_r(r = a) = 3E_0\cos\theta$$

$$\therefore \boxed{\sigma = 3\varepsilon_0 E_0\cos\theta}$$

Charge distribution on a conducting sphere polarized by a uniform electric field



* $\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}$