Electromagnetic induction, generators and transformers

Fleming's Left Hand Rule of electromagnetism tells us that a charge moving in a magnetic field will feel a *force*, which acts *perpendicular* to both the magnetic field direction and the direction of travel of the charge.

Similarly, if a charge carrying conductor is moved through a magnetic field, *an electromotive force (EMF) will be induced across it*, which will result in a flow of charge (i.e. a current). The same result can be achieved by exposing the conductor to a time varying magnetic field.

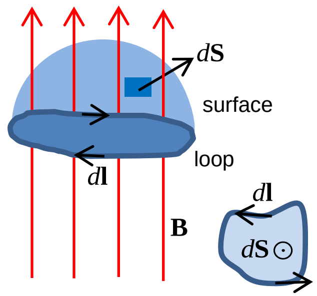
These effects can be described mathematically by **Faraday's Law of Electromagnetic Induction.**

$$\boxed{\varepsilon = -\frac{d\Phi}{dt}}$$

EMF (i.e. a voltage) induced in a ***closed loop*** *of a conductor is proportional to the* ***rate of change of magnetic flux that the loop encloses***

Flux is the integral ("summation", essentially) of the magnetic field *projection* on **vector area elements** of the **surface enclosed by the conductor loop.**

$$\boxed{\Phi = \int_{surface} \mathbf{B} \cdot d\mathbf{S}}$$



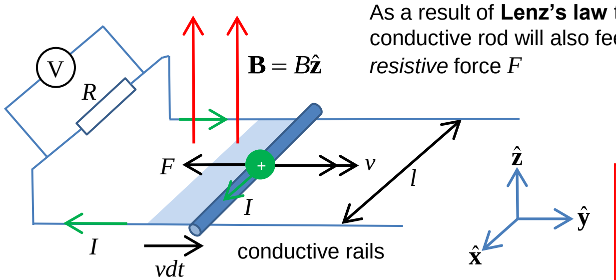
By convention, surface normals are defined as *outward* if the enclosing loop is *anti-clockwise*

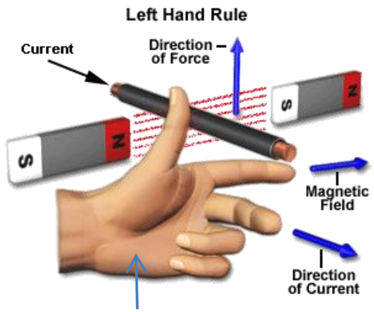
The EMF results from an electric field **E** induced across a conductor. Since *electric field strength is the gradient of the electric potential:*

$$\mathbf{E} = -\nabla\varepsilon \rightarrow \boxed{\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l}}$$

The 'inverse rail gun' is a simple example of **Faraday's Law** in action. In this case the conductor is manually moved along the rails at velocity v. If it moves through a magnetic field as shown, a current will be induced in the circuit and flow through the load R.

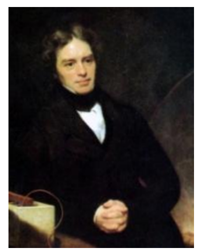
As a result of **Lenz's law** the conductive rod will also feel a *resistive* force F



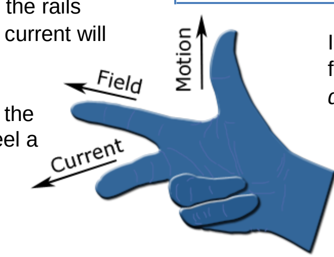


This rule works for finding out the direction of force on a current carrying wire in a magnetic field

But *not* induced currents from motion cutting magnetic field lines! You need the **Right Hand Rule** for this.

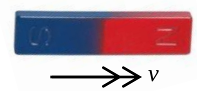


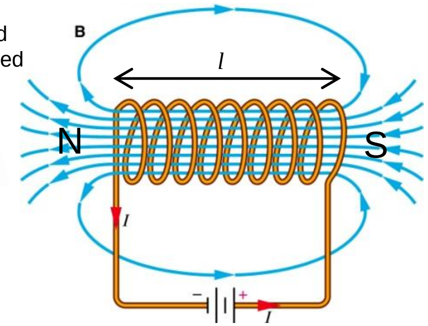
Michael Faraday 1791-1867



We can use **Fleming's Right Handed Rule** to connect Motion, Field and Current in the induction situation

A opposing N pole is induced in a solenoid if a N pole is moved *towards* it. A S pole will be induced if the magnet is moved away.



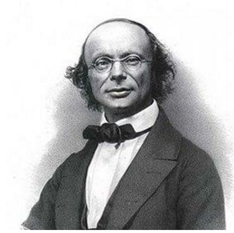


St John Ambrose Fleming 1849-1945

Joseph Henry 1797-1878

$$\boxed{B = \begin{cases} \dfrac{\mu_0 \mu N I}{l} & r < a \\ 0 & r \gg a \end{cases}}$$

Solenoid of N turns in total length l with core of relative permeability μ



Wilhelm Weber 1804-1891

If solenoid has cross sectional area A, flux of magnetic field is

$$BA = -\frac{\mu\mu_0 NIA}{l}$$ (Flux is in units of Webers, Wb)

Therefore flux *linked* by the N loops of coil conductors is

$$NBA = -\frac{\mu\mu_0 N^2 IA}{l}$$

Therefore *back* EMF induced is $\varepsilon = -\frac{d}{dt}(NBA) = -\frac{\mu\mu_0 N^2 A}{l}\frac{dI}{dt} = \boxed{-L\frac{dI}{dt}}$

i.e. the EMF, by **Lenz's Law** will *oppose the change which produced it*

$$\boxed{L = \frac{\mu\mu_0 N^2 A}{l}}$$

Inductance (units of Henrys, H)

Heinrich Lenz 1804-1865

In time dt the change in area swept by the moving conductor is $lvdt$. If our conductor loop follows the current induced (clockwise) then *the vector area* enclosed by the loop points *downwards*. Since **B** points *upwards* the change in magnetic flux is therefore *negative*:

$$d\Phi = -B \times lvdt \quad \therefore \frac{d\Phi}{dt} = -Bvl$$

From Faraday's law this means an EMF: $\boxed{\varepsilon = -\frac{d\Phi}{dt} = Bvl}$

This means an induced current: $\boxed{I = \frac{Bvl}{R}}$

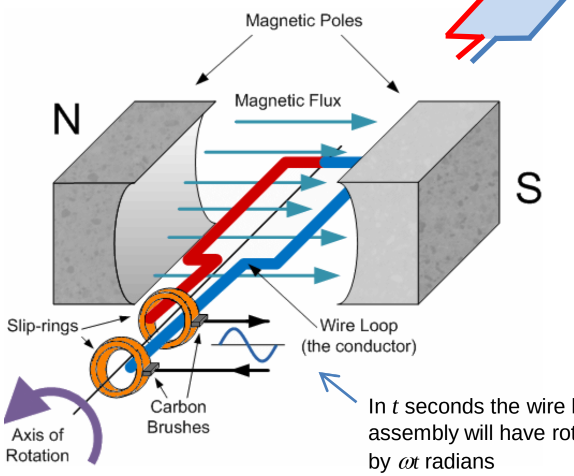
Note from **Fleming's Left Hand Rule**, the induced current will result in the moving conductor feeling a *resistive* force (i.e. in the opposite direction to its velocity) of magnitude: $\boxed{F = BIl}$

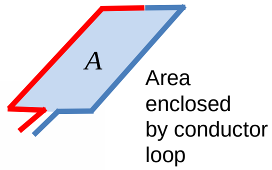
We can also justify Faraday's law using **energy**. The work done /s Fv against the resistive force must equal the power $I\varepsilon$ dissipated in the resistor:

$$\boxed{\begin{aligned} Fv &= BIlv = I\varepsilon \\ \therefore \varepsilon &= Bvl \end{aligned}}$$

μ_0 is the permeability of free space. It is defined to be $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$

Alternating Current (AC) generator





In t seconds the wire loop assembly will have rotated by ωt radians

Flux linked by N windings of conductor loops enclosing area A rotated within uniform magnetic field of strength B is:

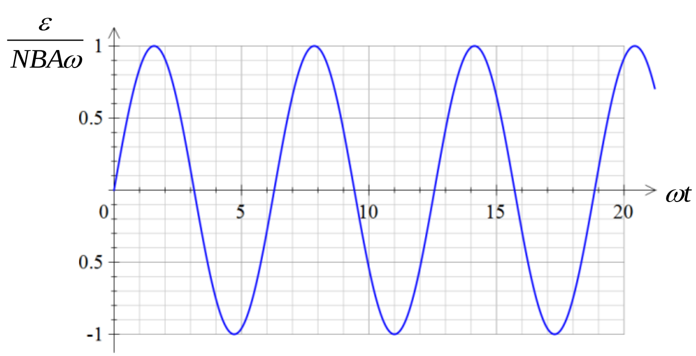
Projection of magnetic field on vector area of wire loop is: $B\cos\omega t$

$$\Phi = NBA\cos\omega t$$

$$\therefore \varepsilon = -\frac{d\Phi}{dt}$$

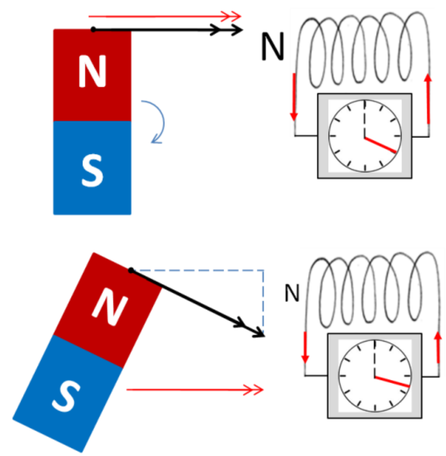
$$\boxed{\varepsilon = NBA\omega\sin\omega t}$$

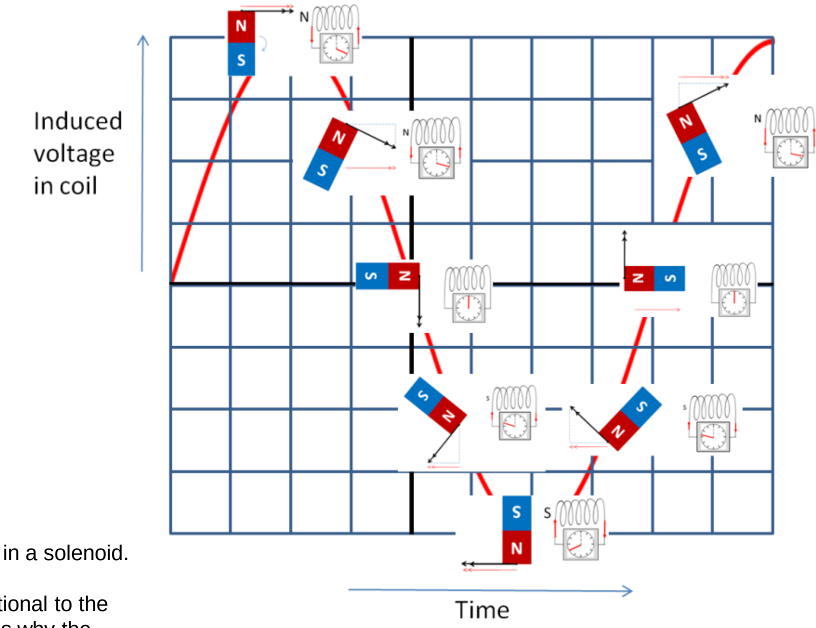
EMF induced in coil



$$\omega = 2\pi f$$

Coil rotation frequency





A **rotating magnet** will induce sinusoidal EMF in a solenoid.

The rate of change of flux is essentially proportional to the **velocity** of the magnet **towards** the coil. This is why the maximum deflection of a galvanometer will occur when the magnet is pointing upwards, since all of the velocity is pointing towards the coil.

When the magnet is in the horizontal position, the magnet is moving fully downwards and therefore there is no velocity towards the coil, therefore no change of magnetic flux linked and therefore no EMF induced.

A more efficient AC generator will have coils in several different orientations.

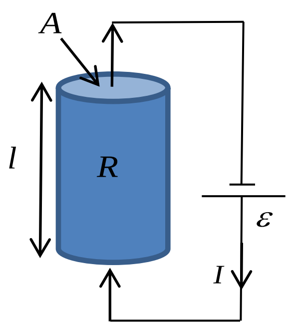
Industrial generators avoid problems with frictional losses in the slip rings by keeping the coils fixed and rotating the magnets instead.

*http://www.electronics-tutorials.ws/accircuits/acp14a.gif

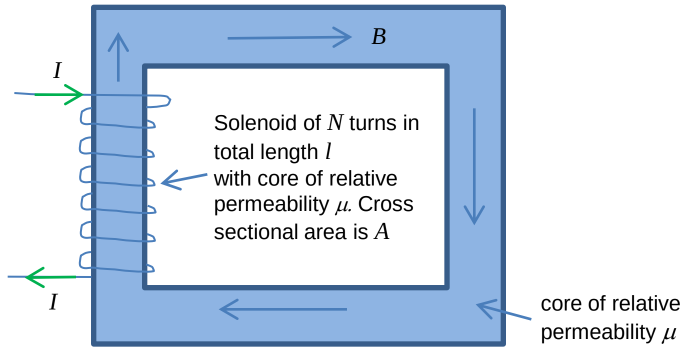
Magnetic circuits. Inductance, Reluctance, Permeance

Just as an EMF drives current round a circuit, we can think of a 'magnetomotive force' (MMF) 'driving' magnetism round a magnetic circuit, which must always comprise of closed loops.*

Electrical circuit



Magnetic circuit



$$\boxed{R = \frac{\rho l}{A}}$$

Resistivity ρ

Conductivity $\sigma = \frac{1}{\rho}$

Resistance R

Conductance $G = \frac{1}{R}$

EMF $\int_{loop} \mathbf{E} \cdot d\mathbf{l} = \varepsilon$

Ohm's Law $\varepsilon = IR$

$$\boxed{\Re = \frac{1}{\mu\mu_0}\frac{l}{A}}$$

Reluctance $\Re$

Permeance $\Lambda = \frac{1}{\Re}$

MMF $\oint_{loop} \mathbf{H} \cdot d\mathbf{l} = \Im$

Hopkinson / Rowland's law $\Im = \Phi\Re$

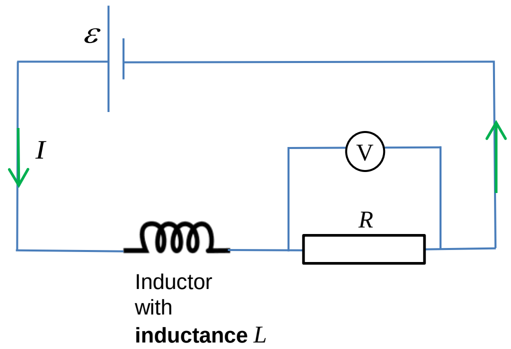
magnetic flux

Note for an N turn solenoid

$$\boxed{\Im = NI}$$

Inductors, Self Inductance and Magnetic Energy

A component in an electrical circuit which is capable of generating a **back-EMF** via induction is called an **inductor**. The symbol for it represents a coil, since this would be the most straightforward geometry for generating a magnetic field.



In the circuit on the left, the net EMF applied to the resistor is

$$\varepsilon - L\frac{dI}{dt}$$

Hence by Ohm's Law:

$$\varepsilon - L\frac{dI}{dt} = IR$$

$$\therefore L\frac{dI}{dt} + IR = \varepsilon$$

Consider a solution of the form:

$$I = a\left(1 - e^{-bt}\right)$$

$$\therefore \frac{dI}{dt} = abe^{-bt}$$

$$L\frac{dI}{dt} + IR = \varepsilon \quad \therefore Labe^{-bt} + Ra - Rae^{-bt} = \varepsilon$$

$$ae^{-bt}\left(Lb - R\right) + Ra = \varepsilon$$

The solution fits for all values of time t if

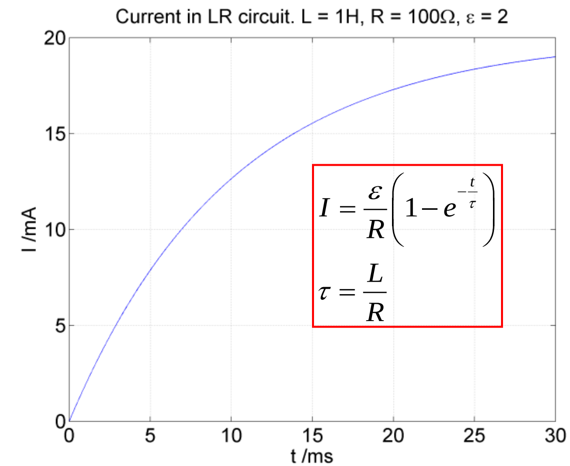
$$Lb - R = 0 \Rightarrow b = \frac{R}{L}$$

$$Ra = \varepsilon$$

Hence: $\boxed{I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)}$

Define characteristic 'current establishment time'

$$\boxed{\tau = \frac{L}{R} \quad \therefore I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{t}{\tau}}\right)}$$



Now power input to circuit is εI

Hence from above: $LI\frac{dI}{dt} + I^2R = \varepsilon I$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}LI^2\right) + I^2R = \varepsilon I$$

Power dissipation in resistor

Input power to circuit

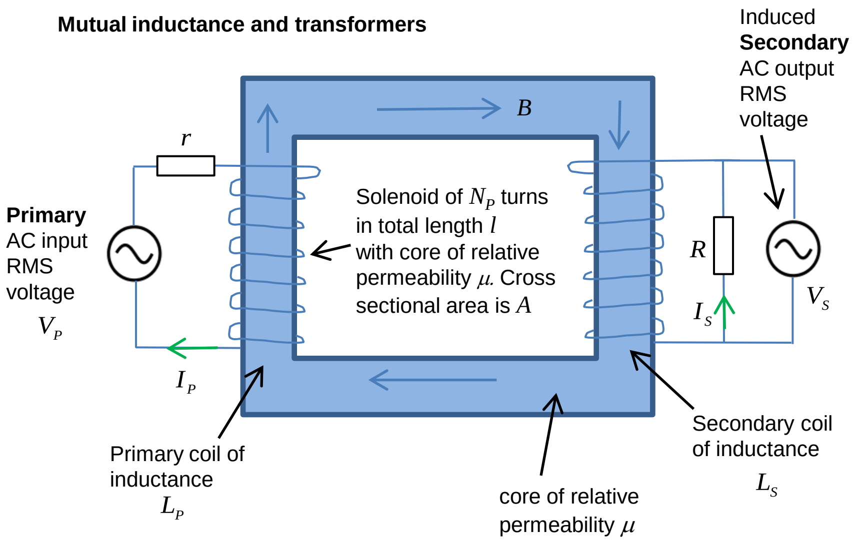
The first term is the **rate of loss of energy in the inductor**, so therefore the **Magnetic Energy** (i.e. the energy stored in the magnetic field of the inductor) is:

$$\boxed{E = \tfrac{1}{2}LI^2}$$

*Unless a magnetic monopole exists

$\mathbf{B} = \mu\mu_0\mathbf{H}$ for isotropic magnetic media

Mutual inductance and transformers



For *coupled* inductors, connected within a magnetic circuit mediated by a soft magnetic core such as iron (i.e. where it will lose its magnetism once current is switched off in the solenoids), if we apply Kirchoff's Second Law:

Applied EMF minus back EMF + EMF due to mutual induction*

$$\boxed{\begin{aligned} &V_P - L_P\frac{dI_P}{dt} + M\frac{dI_S}{dt} = I_P r \\ &M\frac{dI_P}{dt} - L_S\frac{dI_S}{dt} = I_S R = V_S \end{aligned}}$$

Primary solenoid

Secondary solenoid

M is the **mutual inductance** between the solenoid coils. Let us consider steady state solutions to the coupled differential equations above. It makes sense that the secondary AC output is at the same frequency, but different phase and amplitude as the input.

Actual values will be the *real parts* of these

$$V_P = V_0 e^{i\omega t}$$
$$I_P = \frac{V_0}{r} A e^{i\omega t}$$
$$I_S = \frac{V_0}{R} B e^{i\omega t}$$

Note constants A and B will be **complex numbers** to account for phase differences

Substituting these into Kirchoff's Second Law and dividing by $V_0 e^{i\omega t}$

$$1 - \frac{L_P i\omega A}{r} + \frac{Mi\omega B}{R} = A$$
$$\frac{Mi\omega A}{r} - \frac{L_S i\omega B}{R} = B$$

$$\therefore B = \frac{iM\omega A}{r\left(1+\frac{L_S i\omega}{R}\right)} = \frac{iM\omega RA}{r(R+L_S i\omega)}$$

Substituting for B in the Primary coil equation

$$1 - \frac{L_P i\omega A}{r} + \frac{Mi\omega}{R}\left(\frac{iM\omega RA}{r(R+L_S i\omega)}\right) = A$$
$$1 - \frac{L_P i\omega A}{r} - \frac{M^2\omega^2 A}{r(R+L_S i\omega)} = A$$
$$\therefore 1 = A\left(1 + \frac{L_P i\omega}{r} + \frac{M^2\omega^2}{r(R+L_S i\omega)}\right)$$

$$\boxed{\begin{aligned} &\therefore A = \left(1 + \frac{L_P i\omega}{r} + \frac{M^2\omega^2}{r(R+L_S i\omega)}\right)^{-1} \\ &\therefore B = \frac{iM\omega RA}{r(R+L_S i\omega)} \end{aligned}}$$

The **power** dissipated in the loads r and R is given by**

$$\boxed{\begin{aligned} P_P &= \tfrac{1}{2}\mathrm{Re}\left(I_P r \times I_P^*\right) \\ P_S &= \tfrac{1}{2}\mathrm{Re}\left(I_S R \times I_S^*\right) \end{aligned}}$$

It can be shown that the **Ideal Transformer Equation** is applicable in a high frequency, low secondary current and high coupling situation:

$$\omega \gg 1 \qquad R \gg r \qquad k \to 1$$

$$\boxed{\left|\frac{V_P}{V_S}\right| \to \sqrt{\frac{L_P}{L_S}} = \frac{N_P}{N_S}}$$

in our case since the coil lengths and areas are the same

**

$$P = \overline{\mathrm{Re}(V) \times \mathrm{Re}(I)}$$
$$V = V_R + iV_I$$
$$I = I_R + iI_I$$
$$VI^* = (V_R + iV_I)(I_R - iI_I)$$
$$VI^* = V_R I_R + V_I I_I + i(V_I I_R + V_R I_I)$$
$$\therefore \boxed{\tfrac{1}{2}\mathrm{Re}(VI^*) = \tfrac{1}{2}\overline{V_R I_R} + \tfrac{1}{2}\overline{V_I I_I}}$$

The inductances are given by (see page 1):

$$\boxed{L_P = \frac{\mu\mu_0 N_P^2 A}{l} \qquad L_S = \frac{\mu\mu_0 N_S^2 A}{l} \qquad \therefore \frac{L_P}{L_S} = \left(\frac{N_P}{N_S}\right)^2}$$

Define a dimensionless **coupling constant** from which we can define the **mutual inductance**

$$\boxed{M = k\sqrt{L_P L_S}} \qquad k \geq 0$$

Now consider the quantity

$$E = \tfrac{1}{2}L_P I_P^2 + \tfrac{1}{2}L_S I_S^2 + MI_S I_P$$
$$\therefore \frac{dE}{dt} = L_P I_P \frac{dI_P}{dt} + L_S I_S \frac{dI_S}{dt} + MI_S \frac{dI_S}{dt} + MI_P \frac{dI_P}{dt}$$

which is essentially *the rate of power flowing in the inductors*. This must be greater than zero, hence:

$$\tfrac{1}{2}L_P I_P^2 + \tfrac{1}{2}L_S I_S^2 + MI_S I_P \geq 0$$
$$\tfrac{1}{2}\begin{pmatrix} I_P & I_S \end{pmatrix}\begin{pmatrix} L_P & M \\ M & L_S \end{pmatrix}\begin{pmatrix} I_P \\ I_S \end{pmatrix} \geq 0$$
$$\therefore \begin{vmatrix} L_P & M \\ M & L_S \end{vmatrix} \geq 0$$
$$\therefore L_P L_S - M^2 \geq 0$$
$$\therefore M \leq \sqrt{L_P L_S}$$
$$\therefore \boxed{0 \leq k \leq 1}$$

So the maximum value for the coupling constant is *unity*.

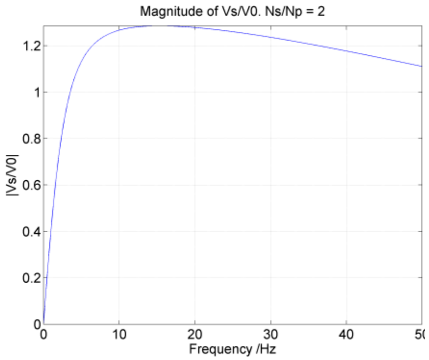
Simplistic derivation of ideal transformer equation

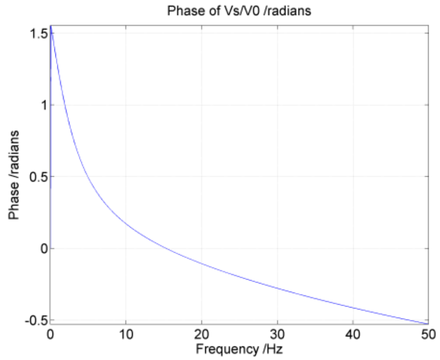
Magnetic flux linked by primary and secondary coils is:

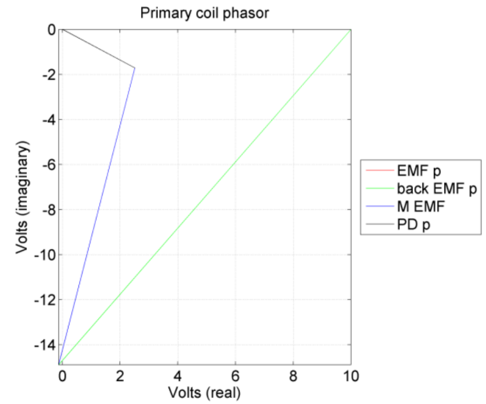
$$\Phi_S = N_S BA; \quad \Phi_P = N_P BA$$
$$V_S = -\frac{d\Phi_S}{dt} = -N_S A\frac{dB}{dt}$$
$$V_P = -\frac{d\Phi_P}{dt} = -N_P A\frac{dB}{dt}$$
$$\therefore \boxed{\frac{V_S}{V_P} = \frac{N_S}{N_P}}$$

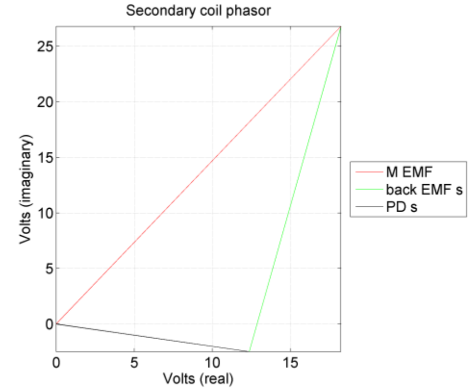
i.e. via Faraday's Law

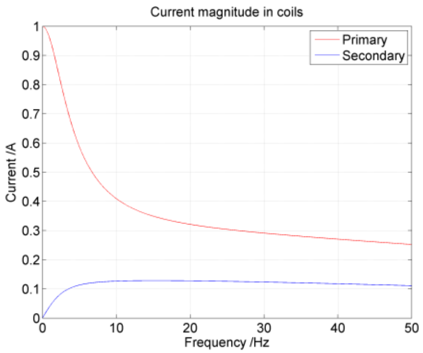
*Here we use the result EMF = Inductance x rate of change of current

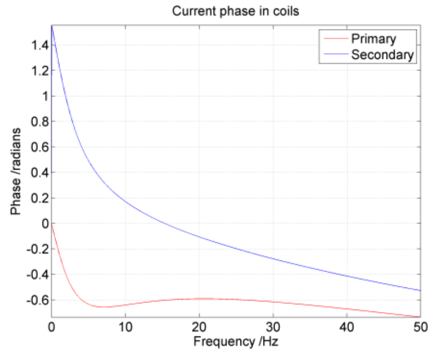


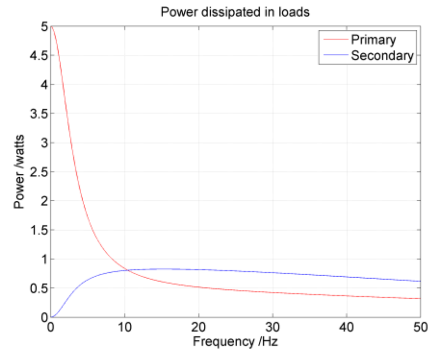












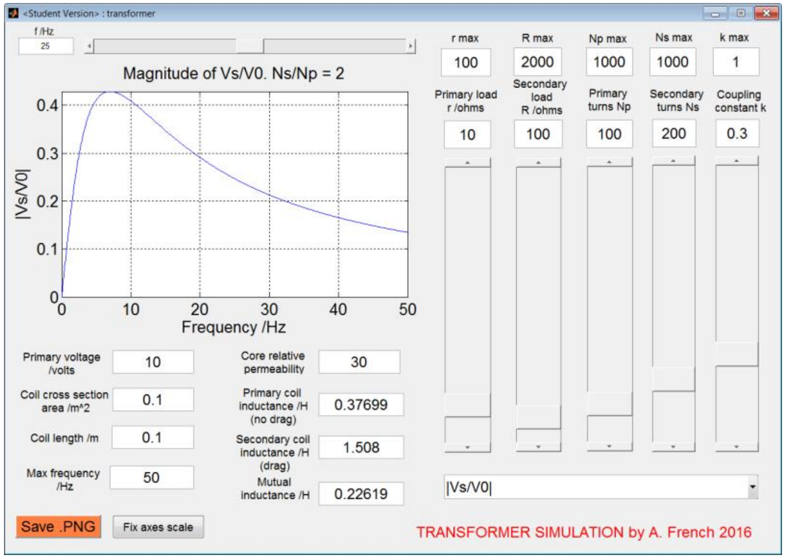
MATLAB model of a transformer.
The **phasor diagram** is a plot of the real and imaginary parts of the various EMF and voltage components of the equations

$$V_P - L_P \frac{dI_P}{dt} + M \frac{dI_S}{dt} = I_P r$$

Primary solenoid

$$M \frac{dI_P}{dt} - L_S \frac{dI_S}{dt} = I_S R = V_S$$

Secondary solenoid



Significant deviation from the **Ideal Transformer Equation** when k is much less than unity

Much better agreement when the secondary load resistance R is much higher than the primary load r, and k is unity