Analysis of experimental errors

[**Progression**: (i) *Upper and Lower Bounds*, (ii) how and when to add *power-scaled percentage errors*, (iii) *Law of Errors* i.e. when to combine gradient scaled errors in *quadrature*, using data statistics].

All measurements will be subject to experimental error. To **quantify** such errors we might do one of two things: The first is to guess the *largest and smallest possible values the measurements could take.* These are called the **upper and lower bounds.** The second might be to perform a **statistical analysis** of measured data, if lots of repeats are performed.

Note these notes only concern **random error** rather than **systematic** (offset) error. i.e. ***precision*** and not ***accuracy***.

For example: a measurement of x = 11mm using a ruler with marks every mm would have bounds between 10.5mm and 11.5mm. Given the rules of rounding numbers, we could write this an *inequality*:

$$10.5\text{mm} \le x < 11.5\text{mm}$$

The upper and lower bounds idea can be readily extended to determine the largest and smallest possible values of a *formula* involving quantities which themselves are subject to upper and lower bounds.

Example (we'll ignore units for brevity): $1.23 \le x < 4.56 \quad 3.21 \le y < 6.54$

$$\therefore 1.23 \times 3.21 \le xy < 4.56 \times 6.54$$

$$\therefore 3.95 \le xy < 29.82$$

$$\therefore \frac{3.21}{4.56} < \frac{y}{x} < \frac{6.54}{1.23} \quad \therefore 0.70 < \frac{y}{x} < 5.32$$

Note in the *division* case, the mixture of upper and lower bounds means we can't have a 'less than equals' to represent the lower limit.

For a formula that is a *product*, we can derive a useful 'rule of thumb' for the **combination of percentage errors**. Let us assume that the quantities x,y are positive, have symmetric upper and lower bounds, and that the range of x and y is much smaller than the middle of the range for each.

$x = \bar{x} \pm \Delta x, \ \ y = \bar{y} \pm \Delta y$ where $\bar{x}$ is the 'middle of the range' $\bar{x} - \Delta x \le x < \bar{x} + \Delta x$. Also assume: $\Delta x \ll \bar{x}, \ \ \Delta y \ll \bar{y}$

The upper and lower bounds of product $z = xy$ are defined by: $\bar{z} \pm \Delta z = (\bar{x} \pm \Delta x)(\bar{y} \pm \Delta y) = \bar{x}\,\bar{y} \pm \Delta x\,\bar{y} \pm \Delta y \bar{x} + \Delta x \Delta y \approx \bar{x}\,\bar{y} \pm (\Delta x\,\bar{y} + \Delta y \bar{x})$ i.e. ignore $\Delta x \Delta y$

Hence if : $\bar{z} = \bar{x}\,\bar{y}$ $\quad \therefore \Delta z = \Delta x\,\bar{y} + \Delta y \bar{x} \Rightarrow \dfrac{\Delta z}{\bar{z}} = \dfrac{\Delta x}{\bar{x}} + \dfrac{\Delta y}{\bar{y}}$

Upper and Lower Bounds 'rule of thumb':
Percentage error of a product are the sum of the percentage errors of the quantities that are multiplied

Example: $x = 9.8 \pm 0.7 \quad y = 6.5 \pm 0.4$ $\quad \bar{z} = \bar{x}\,\bar{y} = 9.8 \times 6.5 = 63.7$

$$\frac{\Delta z}{\bar{z}} = \frac{\Delta x}{\bar{x}} + \frac{\Delta y}{\bar{y}} = \frac{0.7}{9.8} + \frac{0.4}{6.5} = \tfrac{121}{910} \approx 13.3\% \quad \therefore \Delta z = \tfrac{121}{910} \times 63.7 = 8.47$$

0.7/9.8 = 7.1% 0.4/6.5 = 6.2% Therefore: $z = 63.7 \pm 8.5$ i.e. we add the percentage errors: 7.1% + 6.2% = 13.3%

Powers of x or y can be treated like *multiples of various sub-products,* so our **rule of thumb for upper and lower bounds** generalizes to **'an addition of power weighted % errors'**

$I = 9.8 \pm 0.7 \quad R = 6.5 \pm 0.4 \quad P = I^2 R$

$\bar{P} = 9.8^2 \times 6.5 = 6.2 \times 10^2$

$\Delta P = 9.8^2 \times 6.5 \times \left(\dfrac{2 \times 0.7}{9.8} + \dfrac{0.4}{6.5} \right) = 1.3 \times 10^2$

$\therefore P = (6.2 \pm 1.3) \times 10^2$

$$z = kx^a y^b \quad \bar{z} = k\bar{x}^a \bar{y}^b$$
$$\frac{\Delta z}{\bar{z}} = \frac{a\Delta x}{\bar{x}} + \frac{b\Delta y}{\bar{y}}$$

Note we have a problem if the powers are *negative*.
'Common sense dictates' the sensible way forwards is simply to *ignore* the signs of powers a or b ...

... But can we be a little more rigorous mathematically? Yes we can!

A direct **upper and lower bound analysis** is useful when formulae are *not products, or polynomials*. In these situations, the *special case* of adding power-weighted percentage errors is *inappropriate and should not be attempted*. One must be mindful of the conditions which must be satisfied in order for the 'rule of thumb' to be valid. (i.e. small errors relative to average values, symmetric errors, polynomial formulae).

Examples:

i.e. **add absolute errors for *either* addition or subtraction**

$$x = 9.8 \pm 0.2 \quad y = 7.3 \pm 0.3 \quad z = x - y$$
$$z_{\min} = 9.6 - 7.6 \quad z_{\max} = 10.0 - 7.0$$
$$\therefore \; 2.0 < z < 3.0$$
$$\therefore z = (9.8 - 7.3) \pm (0.2 + 0.3)$$
$$\therefore \boxed{z = 2.5 \pm 0.5}$$

$$x = 0.9 \pm 0.2 \quad y = 2.5 \pm 0.3 \quad z = \tan x - y$$
$$z_{\min} = \tan(0.7) - 2.8 = -1.96 \quad z_{\max} = \tan(1.1) - 2.2 = -0.24$$
$$\therefore \boxed{-2.0 < z < -0.2}$$

Take care when a lower bound means the *most negative*

$$x = 1.8 \pm 0.1 \quad y = 3.5 \pm 0.2 \quad z = \frac{\log_{10} y}{2^x}$$
$$z_{\min} = \frac{\log_{10} 3.3}{2^{1.9}} = 0.14 \quad z_{\max} = \frac{\log_{10} 3.7}{2^{1.7}} = 0.17$$
$$\therefore \boxed{0.14 < z < 0.17}$$

However, is an upper and lower bound calculation <u>always</u> the most appropriate thing to do? *What we really want* is to estimate the **uncertainty** in a quantity $z(x,y....)$ based upon ***estimates of the uncertainties*** in $x,y....$, which we can **calculate from experimental data** using standard statistical methods such **mean average** and **standard deviation.**

The standard approach to this problem* is what is know as ***The Law of Error Propagation***. Let us assume: $x = \bar{x} \pm \sigma_x, \; y = \bar{y} \pm \sigma_y, ...$ and all $x,y....$ variables are **Normally distributed** with means $\bar{x}, \bar{y}, ...$ and standard deviations $\sigma_x, \sigma_y ...$

*If the standard deviations are **small** compared to the magnitude of the mean values,* **The Law of Error Propagation** states:

$$\sigma_z^2 = \left(\sigma_x \left. \frac{\partial z}{\partial x} \right|_{\bar{x},\bar{y}...} \right)^2 + \left(\sigma_y \left. \frac{\partial z}{\partial y} \right|_{\bar{x},\bar{y}...} \right)^2 + ...$$

Examples:

$$z = ax - by$$
$$\bar{z} = a\bar{x} - b\bar{y}$$
$$\sigma_z^2 = (a\sigma_x)^2 + (-b\sigma_y)^2$$
$$\therefore \boxed{\sigma_z = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2}}$$

So for **arithmetic combinations,** uncertainties ***add in quadrature.***

i.e. we take a weighted sum of the squares of the uncertainties, and then square root.

$$z = kx^3 y^{-\frac{1}{3}}$$
$$\bar{z} = k\bar{x}^3 \bar{y}^{-\frac{1}{3}}$$
$$\sigma_z^2 = \left(3k\bar{x}^2\bar{y}^{-\frac{1}{3}}\sigma_x\right)^2 + \left(-\tfrac{1}{3}k\bar{x}^2\bar{y}^{-\frac{4}{3}}\right)^2$$
$$\therefore \boxed{\frac{\sigma_z^2}{\bar{z}^2} = \left(\frac{3\sigma_x}{\bar{x}}\right)^2 + \left(\frac{\frac{1}{3}\sigma_y}{\bar{y}}\right)^2}$$

For **polynomial expressions**, we combine **the power-weighted percentage errors *in quadrature***, *rather than simply adding them as in the upper and lower bound analysis.*

As expected, the estimated uncertainty using the **Law of Errors** is *smaller* than for upper-and-lower bounds. It is *good news* that a cruder calculation should yield a *larger* estimate of uncertainty rather than the converse!

Upper and lower bound

$$I = 9.8 \pm 0.7 \quad R = 6.5 \pm 0.4 \quad P = I^2 R$$
$$\bar{P} = 9.8^2 \times 6.5 = 6.2 \times 10^2$$
$$\Delta P = 9.8^2 \times 6.5 \times \left(\frac{2 \times 0.7}{9.8} + \frac{0.4}{6.5} \right) = 1.3 \times 10^2$$
$$\therefore \boxed{P = (6.2 \pm 1.3) \times 10^2}$$

Law of Errors

$$I = 9.8 \pm 0.7 \quad R = 6.5 \pm 0.4 \quad P = I^2 R$$
$$\bar{P} = 9.8^2 \times 6.5 = 6.2 \times 10^2$$
$$\Delta P = 9.8^2 \times 6.5 \times \sqrt{\left(\frac{2 \times 0.7}{9.8} \right)^2 + \left(\frac{0.4}{6.5} \right)^2} = 97.10..$$
$$\therefore \boxed{P = (6.2 \pm 1.0) \times 10^2}$$

To determine ***unbiased estimates*** of the means and standard deviations of variables using experimental data *samples* (i.e. N independent measurements of each parameter)

A good use of a spreadsheet! →

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \qquad \bar{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

The $N - 1$ is used since the mean estimate is used in the calculation of the standard deviation. i.e. the *actual* population mean is not known *a priori*.

*See Barlow *A Guide to the Use of Statistical Methods in the Physical Sciences* pp55-59

Justification of the Law of Errors (based upon Barlow's analysis*)

i.e. there is **no correlation** between them – you can't for example express $y = g(x)$

Let $z = f(x, y....)$ where $x, y...$ are *independent random variables that are normally distributed.* Let's consider *small deviations* from the mean values of the inputs and hence approximate z in this limit using a first-order **Taylor expansion:**

$$z = f(x, y...) \approx f(\bar{x}, \bar{y}...) + (x - \bar{x})\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y},...} + (y - \bar{y})\left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y},...} + ...$$

The uncertainty in z, i.e. its *standard deviation* σ, is the **square root** of its *variance* $V[z]$:

$$V[z] \approx \left(\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y},...}\right)^2 V[x] + \left(\left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y},...}\right)^2 V[y] + ...$$

This is because: $f(\bar{x}, \bar{y}...) - \bar{x}\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y},...} - \bar{y}\left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y},...} - ...$ and: $V[ax + b] = a^2 V[x]$

are *constants* – i.e. don't vary with x or y

Hence: $V[x, y, ...] = \sigma^2_{x,y...}$

$$\therefore \sigma_z^2 \approx \left(\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y},...}\right)^2 \sigma_x^2 + \left(\left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y},...}\right)^2 \sigma_y^2 + ...$$

i.e. the **Law of Error Propagation**

Expectation means 'mean average'

$$E[x] = \bar{x} = \tfrac{1}{N}\sum_{i=1}^{N} x_i$$

for a discrete set of data samples.

Hence: $E[ax + b] = aE[x] + b$

Variance

$$V[x] = E\left[(x - \bar{x})^2\right]$$
$$= E[x^2 - 2x\bar{x} + \bar{x}^2]$$
$$= E[x^2] - 2\bar{x}E[x] + \bar{x}^2$$
$$= E[x^2] - \bar{x}^2$$

Variance of a scaled variable

$$V[x] = E\left[(x - E[x])^2\right]$$
$$\therefore V[ax + b] = E\left[(ax + b - E[ax + b])^2\right]$$
$$\therefore V[ax + b] = E\left[(ax + b - aE[x] - b)^2\right] = E\left[a^2(x - E[x])^2\right]$$
$$\therefore V[ax + b] = a^2 E\left[(x - E[x])^2\right]$$
$$\therefore V[ax + b] = a^2 V[x]$$

*See Barlow *A Guide to the Use of Statistical Methods in the Physical Sciences* pp55-59