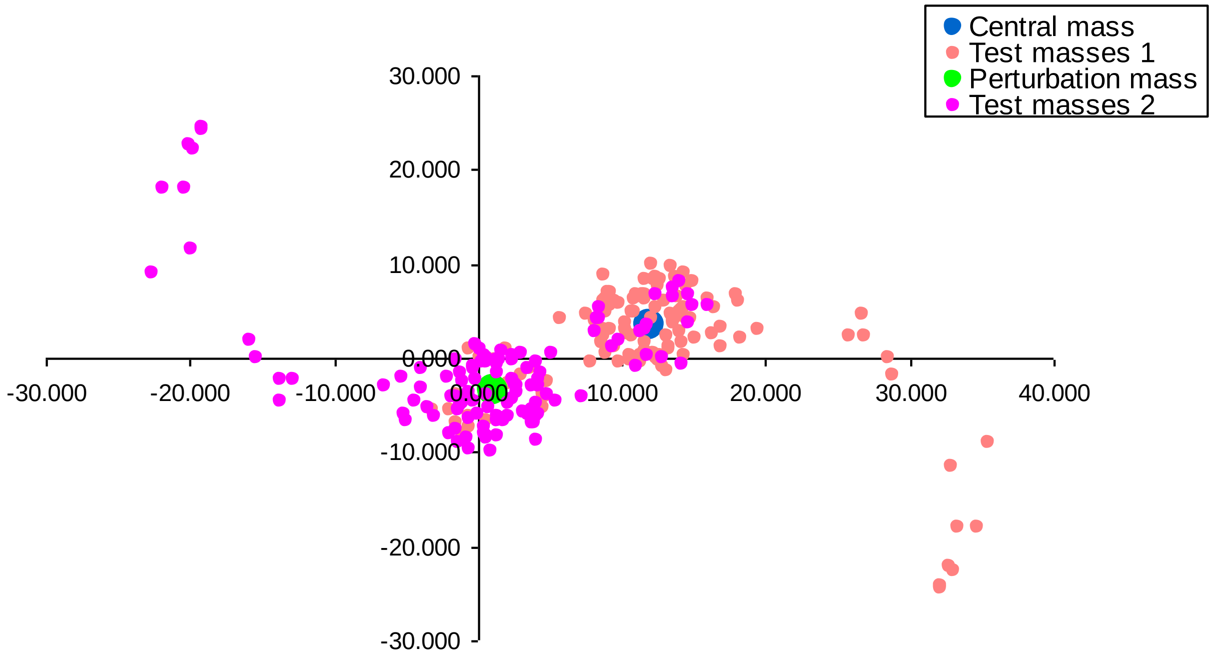
Part II Experimental and Theoretical Physics Computing Project

Interacting Galaxies



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1.1 Abstract

This report describes the computer simulation of a pair of interacting galaxies. Using observations of real galactic structure as a guide, a model for each galaxy was constructed consisting of a central mass and five rings of test particles moving in (initially) anticlockwise circular orbits. The rings (of linearly increasing radii) have number densities 12,18,24,30 and 36. The test particles only interact (via Newtonian gravity) with both central masses. 'Test-Test' interactions have been ignored. This simplification will not allow us to model any collective phenomena caused by test-test interactions but will allow us to compute the problem over an appreciable evolutionary period with fairly limited resources.

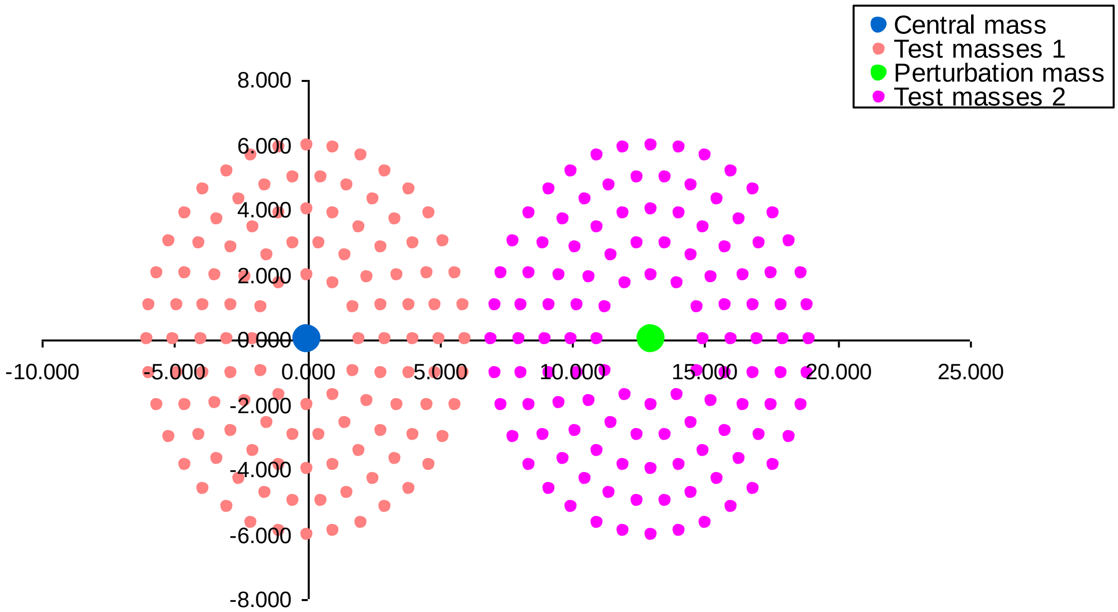
System evolution was computed for parabolic and mutually circular orbit initial conditions using the Verlet Integration Method (with a time step of 0.01s) and Newtonian equations of motion scaled to be spatially dimensionless. Inter-galaxy test mass exchange via 'bridge' like structures was observed in both scenarios resulting in a (possibly) equilibrium mass distribution. 'Tidal tails' were also observed to form from outlying test particles, consistent with the results of *Toomre and Toomre*[*] and known astronomical observations.

[*] See Reference 1.

2.1 Introduction

This project details the computer simulation of two gravitationally interacting galaxies. We will label these "central" and "perturbation" by virtue of the initial conditions described below.

Cosmological observations have determined many galaxies to have a disc-like structure with a bulging sphere of condensed stars at the centre. We will therefore model our galaxies as a large central mass with many, effectively massless 'test' particles distributed in a circular disc around it. To avoid excessive computation 120 test particles will be placed in initially circular orbits around the central masses as shown below.



We will only consider interactions between the central masses and the dynamics of the test particles due to the gravitational attraction of the two central masses. We will ignore test-test interactions.
Although we will lose insight into possible collective phenomena exhibited by mutually attracting test particles, the essence of the interaction dynamics should be preserved. This simplification should dramatically reduce the computational resources required for the simulation and hence for a given amount of real time a greater degree of system evolution will be observable.

3. Analysis

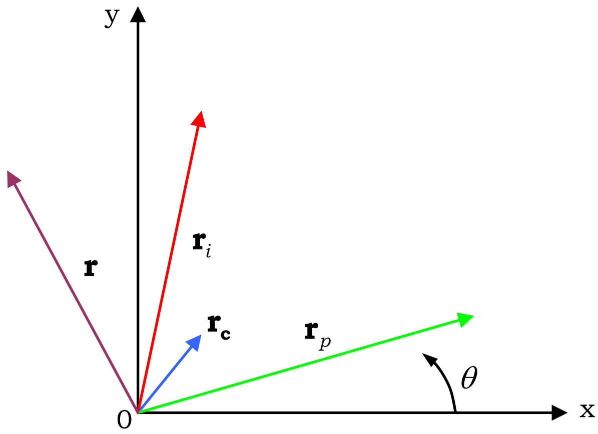
3.1 System Dynamics and numerical trajectory solution

Classical Newtonian mechanics will be used to model the gravitational interactions and the resultant dynamics of the particles involved.

We will record the positions of all particles in the system using 2D vectors and a Cartesian basis. Using the subscripts *c*, *p*, and *i* for central, perturbation and test masses respectively we can write down Newton's 2nd law to describe the motion of the system.

$$m_c \ddot{\mathbf{r}}_\mathbf{c} = Gm_p m_c \frac{(\mathbf{r}_p - \mathbf{r}_\mathbf{c})}{|\mathbf{r}_p - \mathbf{r}_\mathbf{c}|^3} \qquad (1)$$

$$m_p \ddot{\mathbf{r}}_\mathbf{p} = Gm_p m_c \frac{(\mathbf{r}_c - \mathbf{r}_p)}{|\mathbf{r}_c - \mathbf{r}_p|^3} \qquad (2)$$



$$m_i \ddot{\mathbf{r}}_\mathbf{i} = Gm_i m_c \frac{(\mathbf{r}_c - \mathbf{r}_i)}{|\mathbf{r}_c - \mathbf{r}_i|^3} + Gm_i m_p \frac{(\mathbf{r}_p - \mathbf{r}_i)}{|\mathbf{r}_p - \mathbf{r}_i|^3} \qquad (3)$$

For computational purposes it would be preferable to work with units that take 'sensible' values, i.e. within the range 0.1 – 10. If the numbers generated by the problem far exceed or are much smaller than this, computational errors resulting from the finite size of storable numbers in a computer could yield false and possibly unpredictable results. In this instance we can get around the problem entirely by defining a length scale such that our working variables are dimensionless.

Consider the scaling relation $r = (Gm_c)^{1/3} r'$. By substitution into (1) we find: $\ddot{\mathbf{r}}'_\mathbf{c} = \frac{m_p}{m_c} \frac{(\mathbf{r}'_p - \mathbf{r}'_\mathbf{c})}{|\mathbf{r}'_p - \mathbf{r}'_\mathbf{c}|^3}$. i.e. our new working variable $\mathbf{r}'$ is dimensionless. Hence for purposes of computation we will use this dimensionless scale and then, if required, we could infer from our data real astrophysical quantities by multiplying by $(Gm_c)^{1/3}$.

Total energy is also a quantity that we will need to use in this problem and thus we need to know it's scaling relation. (And check the scaling above leads to a dimensionless energy).

Since test masses are deemed negligible compared to central and perturbation masses we can write down a greatly simplified expression for the total energy of the system.

$$E_{TOT} = \tfrac{1}{2} m_c v_c^2 + \tfrac{1}{2} m_p v_p^2 - \frac{G m_p m_c}{\left|\mathbf{r_p} - \mathbf{r_c}\right|} \qquad (4)$$

Noting that in our dimensionless scale the velocity v is given by $v = v'(Gm_c)^{1/3}$ we find:

$$\frac{E_{TOT}}{m_c (Gm_c)^{2/3}} = \tfrac{1}{2} v_c'^2 + \tfrac{1}{2} \frac{m_p}{m_c} v_p'^2 - \frac{m_p}{m_c} \frac{1}{\left|\mathbf{r_p'} - \mathbf{r_c'}\right|} \qquad (5)$$

We can thus identify $\frac{E_{TOT}}{m_c (Gm_c)^{2/3}}$ as the dimensionless total energy E'_{TOT}.

The set of dimensionless equations that describe the classical time evolution of our system are therefore as follows:

$$\ddot{\mathbf{r}}_\mathbf{c}' = \frac{m_p}{m_c} \frac{(\mathbf{r}_p' - \mathbf{r}_\mathbf{c}')}{\left|\mathbf{r}_p' - \mathbf{r}_\mathbf{c}'\right|^3} \qquad (6) \qquad\qquad \ddot{\mathbf{r}}_\mathbf{p}' = \frac{(\mathbf{r}_c' - \mathbf{r}_p')}{\left|\mathbf{r}_c' - \mathbf{r}_p'\right|^3} \qquad (7)$$

$$\ddot{\mathbf{r}}_\mathbf{i}' = \frac{m_p}{m_c} \frac{(\mathbf{r}_p' - \mathbf{r}_i')}{\left|\mathbf{r}_p' - \mathbf{r}_i'\right|^3} + \frac{(\mathbf{r}_c' - \mathbf{r}_i')}{\left|\mathbf{r}_c' - \mathbf{r}_i'\right|^3} \qquad (8)$$

$$E'_{TOT} = \tfrac{1}{2} v_c'^2 + \tfrac{1}{2} \frac{m_p}{m_c} v_p'^2 - \frac{m_p}{m_c} \frac{1}{\left|\mathbf{r_p'} - \mathbf{r_c'}\right|} \qquad (9)$$

We will solve the first three (2nd order ordinary differential equations) using the *Verlet Integration Method*[1]. (VIM)

If h is a small unit of time progression, i.e. $t_{n+1} = t_n + h$:

$$\mathbf{r}'(t+h) = \mathbf{r}'(t) + \dot{\mathbf{r}}'(t)h + \tfrac{1}{2}\ddot{\mathbf{r}}'(t)h^2 \qquad (10)$$

$$\dot{\mathbf{r}}'(t+h) = \dot{\mathbf{r}}'(t) + \tfrac{1}{2} h\left(\ddot{\mathbf{r}}'(t) + \ddot{\mathbf{r}}'(t+h)\right) \qquad (11)$$

Error in this method is $O(h^4)$ and thus is particularly suited to this problem, i.e where longer time steps are desirable to quickly compute how the system evolves.

[1] See Appendix for derivation.

3.2 Initial conditions

As illustrated above we will centre our 2D Cartesian basis on the initial position of the central mass. The perturbation galaxy will start it's motion from the x axis (i.e. have no y component) to simplify the following analysis. (Note since we have an arbitrary choice of axis positioning, this does in fact represent a perfectly general scenario).

We will apply more specific conditions in the distribution of test masses and the velocities of each particle. We will aim to achieve the following:

(i) Initially circular orbits of test masses around their respective galaxy centre. We will start with both galaxies rotating anticlockwise though this could be generalised to investigate orientation dependant behaviour.

(ii) The perturbation test masses shall also move with the same initial velocity as the perturbation mass. i.e. the entire perturbation galaxy shall move, and rotate as a whole initially.

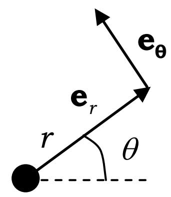
(iii) The perturbation mass shall be initially equal to the central mass. However, if the mass ratio could be alterable this could provide the added feature of simulating a collapsing galaxy or even a 'black hole.'

(iv) Parabolic ($E'_{TOT} = 0$) orbits of the central and perturbation masses should be simulated as well as mutual circular orbits about their centre of mass.

Initial test mass distribution will be evenly spaced rings.

Ring	*Density*	*Radius*
1	12	2
2	18	3
3	24	4
4	30	5
5	36	6

To achieve initially circular test mass orbits let us consider the acceleration vector in 2D polar co-ordinates.



$$\ddot{\mathbf{r}} = \mathbf{e_r}\left(\ddot{r} - r\dot{\theta}^2\right) + \mathbf{e}_\theta\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \qquad (11)$$

For a particle in an inverse square orbit at a radius r from a "central" mass; using dimensionless variables as before (and dropping the ' from now on)

$$\mathbf{r} = r\mathbf{e_r} \qquad (13) \qquad \mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta \qquad (14)$$

$$\ddot{\mathbf{r}} = -\frac{1}{r^2}\mathbf{e_r} \qquad (15)$$

For a circular orbit $\ddot{r} = \dot{r} = 0$ and hence by equating $\mathbf{e_r}$ coefficients of (11) and (15)

$$\dot{\theta} = \pm\frac{1}{r^{\frac{3}{2}}} \qquad (16)$$

Now in Cartesian co-ordinates $\mathbf{r} = \mathbf{e_x} r\cos\theta + \mathbf{e_y} r\sin\theta$ and $\dot{\mathbf{r}} = \mathbf{e_x}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right) + \mathbf{e_y}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)$. For a circular orbit $\dot{r} = 0$ so this simplifies to $\dot{\mathbf{r}} = \mathbf{e_x}\left(-r\dot{\theta}\sin\theta\right) + \mathbf{e_y}\left(r\dot{\theta}\cos\theta\right)$. Substituting for (16) we therefore find the correct initial velocity for a circular orbit (in terms of r and θ) to be

$$\dot{\mathbf{r}} = \mathbf{e_x}\left(\mp\frac{1}{\sqrt{r}}\sin\theta\right) + \mathbf{e_y}\left(\pm\frac{1}{\sqrt{r}}\cos\theta\right) \qquad (17)$$

The clockwise and anticlockwise orientations will of course depend on the pair of signs chosen.

(17) will yield the correct initial velocities for the central test masses. For the perturbation test masses we must add the initial velocity vector of the perturbation mass itself.

For bound orbits $E'_{TOT} < 0$. Now if $v_c = 0$ initially and $\mathbf{r_p} - \mathbf{r_c} = d\mathbf{e_x}$ using equation (9) we find our condition becomes:

$$\tfrac{1}{2}\frac{m_p}{m_c}v_p'^2 - \frac{m_p}{m_c}\frac{1}{d} < 0 \quad \Rightarrow \quad v_p < \sqrt{\frac{2}{d}} \qquad (18)$$

So for a parabolic orbit $E'_{TOT} = 0 \Rightarrow v_p = \sqrt{\frac{2}{d}}$.

Now for mutually circular orbits of the central and perturbation masses the magnitude of the separation vector $\mathbf{d} = \mathbf{r_p} - \mathbf{r_c}$ must be time independent. To find the correct initial condition to satisfy this let us consider the centre of mass vector $\mathbf{R}$, defined by the total linear momentum of the system.

$$\dot{\mathbf{R}}(m_p + m_c) = m_c\dot{\mathbf{r}}_\mathbf{c} + m_p\dot{\mathbf{r}}_\mathbf{p} \qquad (19)$$

Now the force acting *on* the perturbation mass due to the gravitation of the central mass $\mathbf{F_{pc}}$ is by Newton's 2nd law equal to $m_p\ddot{\mathbf{r}}_\mathbf{p}$. Similarly $\mathbf{F_{cp}} = m_c\ddot{\mathbf{r}}_\mathbf{c}$. Now by Newton's law of gravity $\mathbf{F_{cp}} = -\mathbf{F_{pc}}$. Hence $m_c\ddot{\mathbf{r}}_\mathbf{c} + m_p\ddot{\mathbf{r}}_\mathbf{p} = 0$. Since (19) is the time integral of this equation we deduce $\dot{\mathbf{R}} = \mathit{constant}$ regardless of the initial conditions. This is of course a statement of momentum conservation. Hence if the total momentum is zero, the centre of mass vector is also a constant.

We can express $\mathbf{r_p}$ and $\mathbf{r_c}$ in terms of $\mathbf{d}$ and $\mathbf{R}$ to exploit the above results. Let us define $\frac{m_p}{m_c} = \mu$ for clarity.

$$\mathbf{r_c} = \mathbf{R} - \frac{\mu}{1+\mu}\mathbf{d} \qquad (20) \qquad\qquad \mathbf{r}_p = \mathbf{R} + \frac{1}{1+\mu}\mathbf{d} \qquad (21)$$

Substituting the above into (6) and (7) and letting $\ddot{\mathbf{R}} = \dot{\mathbf{R}} = 0$ we arrive at the *same* equation in $\mathbf{d}$. i.e. we have reduced our problem to a single variable.

$$\ddot{\mathbf{d}} = -(1+\mu)\frac{\mathbf{d}}{|\mathbf{d}|^3} \qquad (22)$$

Now our condition for mutually circular orbits was that $|\mathbf{d}|$ = constant. Since (22) is practically identical to the equation used to calculate the initial velocities of the test masses for circular orbits (where $|\mathbf{r_i}|$ constant) we can write down the solution for $\dot{\mathbf{d}}$ which results in mutually circular orbits.

$$\dot{\mathbf{d}} = \mathbf{e_x}\left(\pm\sqrt{\frac{1+\mu}{d}}\sin\theta\right) + \mathbf{e_y}\left(\mp\sqrt{\frac{1+\mu}{d}}\cos\theta\right) \qquad (23)$$

So if initial conditions are $\mathbf{d} = d\mathbf{e_x}$ then our initial velocities are

$$\dot{\mathbf{r}}_{\mathbf{p}} = \frac{1}{\sqrt{(1+\mu)d}}\mathbf{e}_{\mathbf{y}} \qquad (24) \qquad\qquad \dot{\mathbf{r}}_{c} = \frac{-\mu}{\sqrt{(1+\mu)d}}\mathbf{e}_{\mathbf{y}} \qquad (25)$$

This corresponds to the anticlockwise configuration of mutually circular orbits about common centre $\mathbf{R}$ from the origin of our axis.

Note $E'_{TOT} = \frac{1}{d}\left(\frac{1+\mu^2}{2(1+\mu)} - \mu\right)$ i.e. ≤ 0 since $\mu \geq 0$. Hence system is in a bound state as expected.

4. Computational solution of problem.

4.1 Choice of software.

The essence of this problem is to compute the positions and velocities of all particles in our system using the VIM scheme illustrated above. This could be done rapidly by a Fortran or similar program that can store and modify elements of an array of numbers according to iterative rules. (In our case the VIM scheme). Although computational speed is fast with this method, the extra work required to deliver live plotting of results is beyond my current expertise and time allowed for this project. In this particular problem live plotting of particle trajectories is quite essential, in my view, to forming an understanding of the overall dynamics of the system. Hence I have resorted to a more standard package that can provide live update, Microsoft Excel.

As well as providing a direct on screen environment for simultaneously editing and plotting data, a 'macro' language is provided, based around Microsoft Visual Basic. This facility allows one to perform iterative tasks on arrays of cells in an equivalent manner to that required of a Fortran program. Since all the plotted data is linked to a particular array of cells, every time the macro updates them a new graph is generated. The moderate speed of Excel (about two iterations per second on most PC's for this problem) causes this to behave as a live animation of the system.

4.2 General structure of Excel worksheet

Perturbation mass	1.000	Central mass radius 0.500	Perturbation mass radius 0.500	Time interval	0.010	Time	0.000	New time	0.010
Particle	**Index**	**x coordinate**	**x new**	**y coordinate**	**new y**	**vx component**	**new vx**	**vy component**	**new vy**
Central mass	0.000	0.000	0.000	0.000	-0.001	0.000	0.000	-0.139	-0.139
Perturbing mass	0.000	13.000	13.000	0.000	0.001	0.000	0.000	0.139	0.139
1st ring	1.000	2.000	2.000	0.000	0.006	0.000	-0.002	0.568	0.568
	2.000	1.732	1.729	1.000	1.005	-0.354	-0.355	0.474	0.473
	3.000	1.000	0.994	1.732	1.734	-0.612	-0.613	0.215	0.213
	4.000	0.000	-0.007	2.000	1.999	-0.707	-0.707	-0.139	-0.141
	5.000	-1.000	-1.006	1.732	1.727	-0.612	-0.611	-0.492	-0.494
	6.000	-1.732	-1.736	1.000	0.992	-0.354	-0.352	-0.751	-0.752
	7.000	-2.000	-2.000	0.000	-0.008	0.000	0.002	-0.846	-0.846
	8.000	-1.732	-1.729	-1.000	-1.008	0.354	0.356	-0.751	-0.750
	9.000	-1.000	-0.994	-1.732	-1.737	0.612	0.614	-0.492	-0.490
	10.000	0.000	0.007	-2.000	-2.001	0.707	0.707	-0.139	-0.136
	11.000	1.000	1.006	-1.732	-1.730	0.612	0.611	0.215	0.217
	12.000	1.732	1.736	-1.000	-0.995	0.354	0.352	0.474	0.475
2nd ring	13.000	3.000	3.000	0.000	0.004	0.000	-0.001	0.439	0.439

A section of the Excel spreadsheet created to solve this problem is illustrated above.

The positions and velocities of all the particles are given in two columns. The first of these specifies the initial conditions/start of iteration and the second gives the values after one time step. The Excel macro copies the second column and uses the "paste as values" function to update the original column with the new data while preserving the column linking equations that implement the VIM algorithm. This function is also used to update the 'clock' cell at the top right of the spreadsheet.

A graph of the first column positions of all the particles is plotted using separate data series for each type of particle. A different colour is used to distinguish each particle type. The graph of course updates automatically every iteration.

The number of iterations is controllable from within the Excel macro code[2] and is linked to a 'hot key' *(Ctrl +G)* which starts the process and the *Escape* key which causes the macro to be interrupted. This allows snapshots of the system plot to be saved.

Using the \${row}\${column} function cells at the top of the spreadsheet are used to contain fixed constants such as the time step, mass ratio (given as "perturbation mass") and 'softening' radius. The last constant is used to prevent the unlikely scenario of two interacting particles effectively colliding i.e. having a separation less than their real radii. In a computational sense, 'collision' results in often unrealistic accelerations since the inverse square law of interaction becomes infinite as the inter particle acceleration tends to zero. To avoid this problem a small, constant radius is added to the particle radii in the force law expression *for the test masses* to ensure a non singular acceleration when the particle positions coincide. The force laws of the central-perturbation mass interaction will be unaltered since it is unlikely that these particles will collide using our initial conditions and also since the softening radii will alter our condition for circular/parabolic orbits.

i.e. only equation (8) is altered.

$$\ddot{\mathbf{r}}_{\mathbf{i}}' = \frac{m_p}{m_c}\frac{(\mathbf{r}_p' - \mathbf{r}_i')}{\left(|\mathbf{r}_p' - \mathbf{r}_i'|^2 + \delta_p^2\right)^{3/2}} + \frac{(\mathbf{r}_c' - \mathbf{r}_i')}{\left(|\mathbf{r}_c' - \mathbf{r}_i'|^2 + \delta_c^2\right)^{3/2}}$$

where δ_c and δ_p are the softening radii of the central and perturbation masses respectively. (Default values are 0.5 units each).

[2] Listed in the Appendix

The ${row}${column} function is also used to set up the initial condition that all the perturbation test masses rotate about the perturbation mass AND move with it's initial velocity. i.e. one can specify the initial velocity of perturbation galaxy by just editing the cells corresponding to the perturbation mass. The same property also sets the position of the perturbation test masses.

4.3 Tests of the Program

The initial condition of circular test mass orbits is easily seen by running the program for the first few iterations. Setting the perturbation mass to zero and turning off the display of perturbation test masses allows us to check the stability of the circular orbits by observing over a much longer time scale. It was found a time step of 0.01 seconds caused inner orbits to deviate from their initial values of 2 units by 4% after 15.4 seconds. (1540 iterations). A time step of 0.1 seconds caused the test mass rings to collapse at an increasing rate after ~13 seconds. Hence a time step of 0.01 seconds was deemed appropriate for this simulation.

The correct trajectories of the central and perturbation masses (i.e. mutually circular and parabolic) were easily checked by setting d to be a relatively small amount (say 4 units) and turning off the test mass display. Since the angular speed of rotation varies as $1/d^{3/2}$ (16) for circular orbits, reducing d allows us to view a complete rotation within a relatively short time. This check proved very helpful as in the fist instance my I had incorrectly derived the initial velocities to result in mutually circular orbits. Setting d = 2 units I discovered an elliptic orbit was performed – indicating that the error was in my mathmatics and not integration errors. (For which a notable change in d after one closed orbit would be expected).

For circular orbits, period T is given by $T = 2\pi d^{3/2}$. So for $d = 13$ units we expect a complete cycle to occur after 295 seconds. Now for a time step of 0.01 seconds we expect a complete cycle to occur after ~4 hours. (1 iteration takes about ½ second of real time).

5.1 Results

The time evolution of our system is illustrated below for three distinct initial conditions.

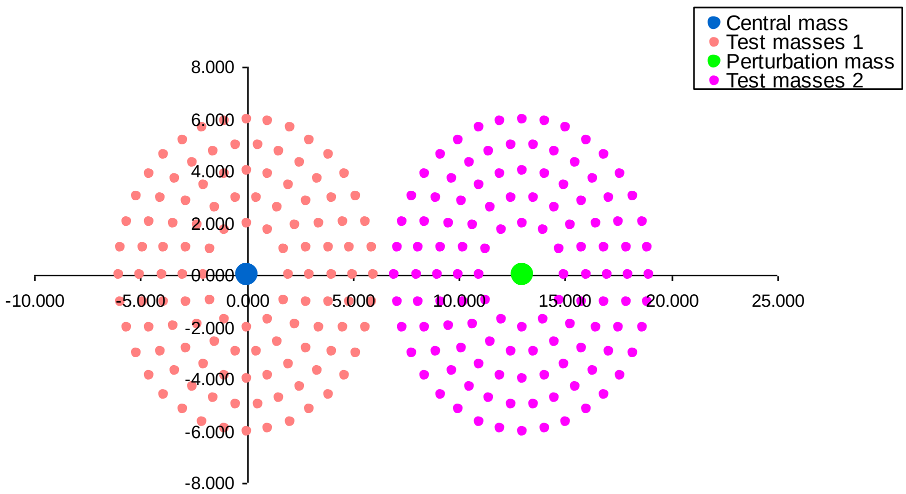
(i) Central and perturbation masses are identical. Initial separation of galaxies is $\mathbf{r_p} - \mathbf{r_c} = d\mathbf{e_x}$, where d = 13 units. Initial velocities are such to yield a parabolic orbit. i.e. $E_{TOT} = 0$. To achieve this $v'_c = 0$ and $v'_p = \sqrt{\frac{2}{d}}$. (c.f. equation (18)).

(ii) Central and perturbation masses are identical and undergo mutually circular orbits. Separation is therefore constant ($d = 13$ as above) though of course the separation vector $\mathbf{d}$ will rotate about the common centre of mass from the initial case of $\mathbf{d} = d\mathbf{e_x}$. Initial velocities are given by equations (24) and (25) and since in this case $\mu = 1$ they reduce to $\dot{\mathbf{r}}_\mathbf{c} = -\frac{1}{\sqrt{2d}}\mathbf{e_y}$, $\dot{\mathbf{r}}_\mathbf{p} = \frac{1}{\sqrt{2d}}\mathbf{e_y}$.

(iii) Perturbation galaxy contains 'supermassive' object. In this case $\mu = 500$. Initial conditions for position and velocity are identical to that of (i). i.e. we expect the system to be unbound by the μ independence of equation (18).

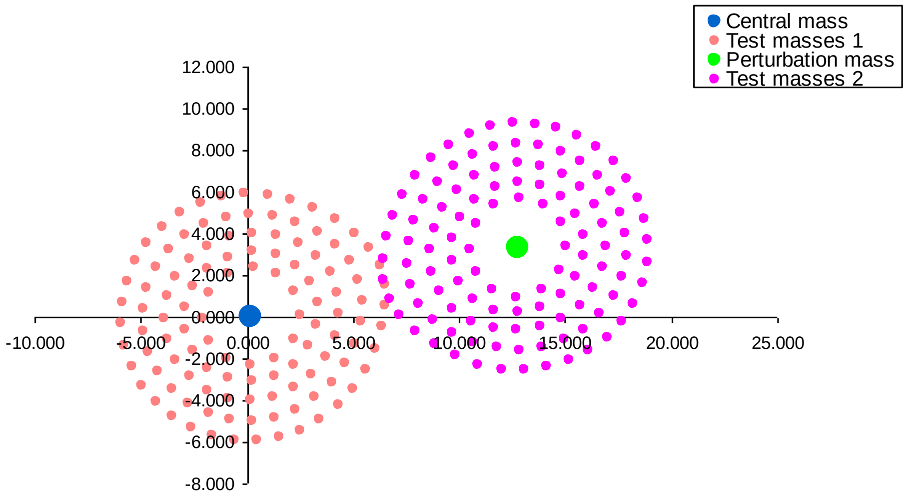
Case (i) : Parabolic Orbits of identical galaxies[3].

t=0

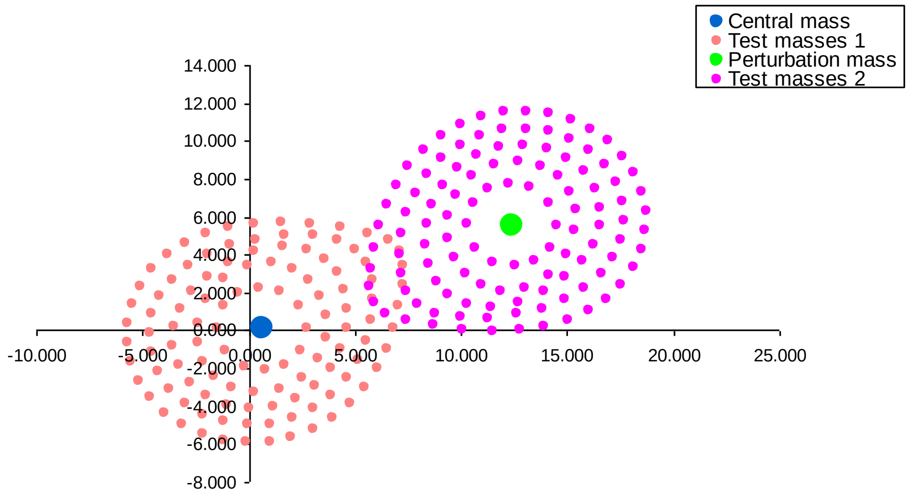


[3] Note mutual anticlockwise rotation.

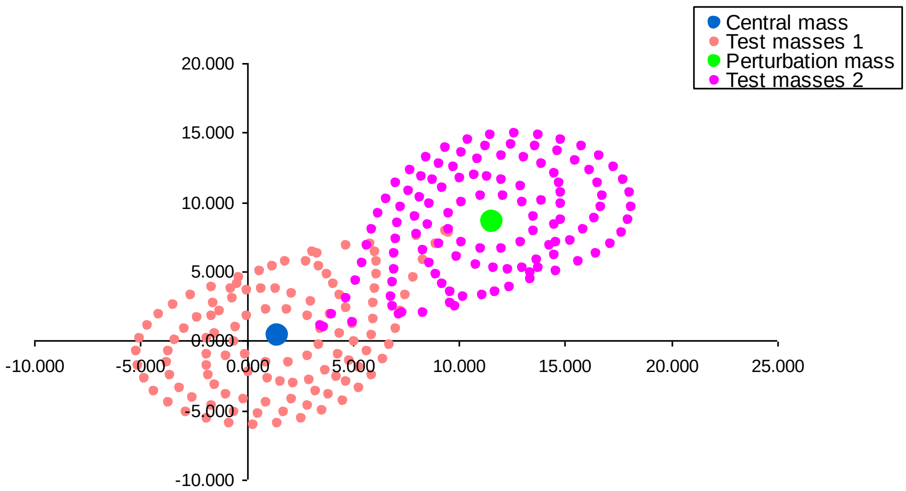
t=8.520



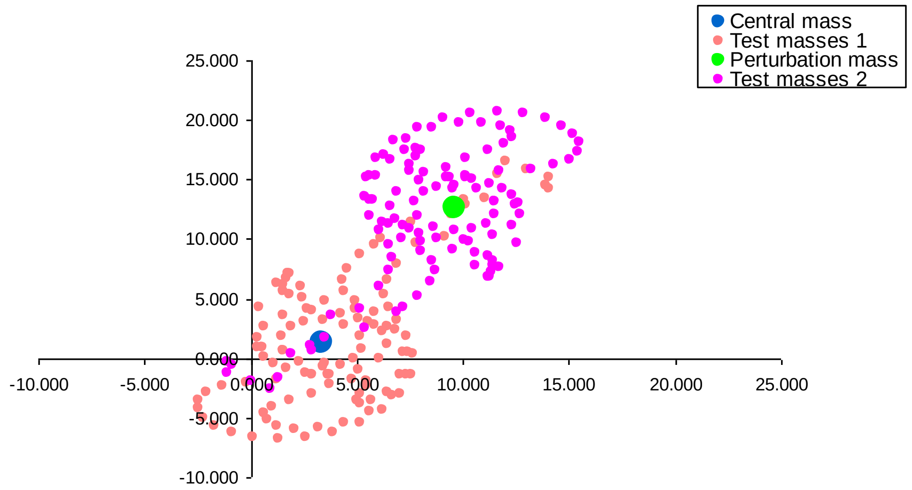
t=14.430



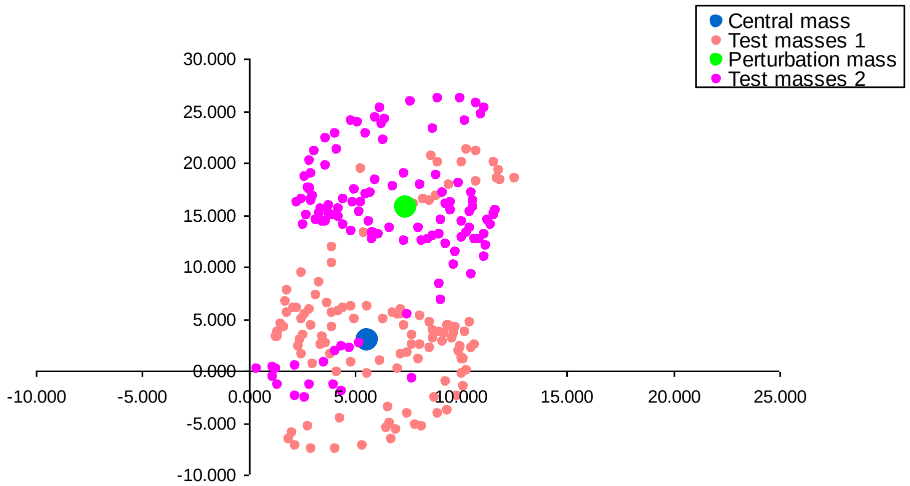
t=22.570



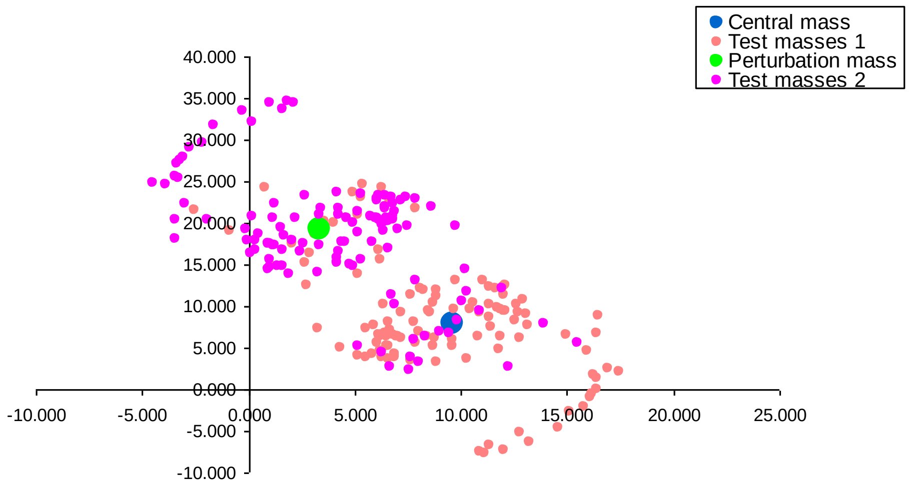
t=35.320



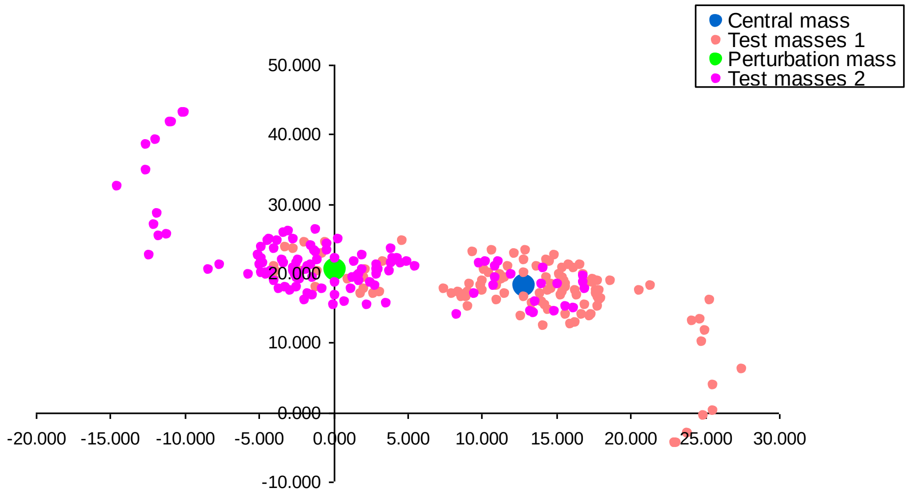
t=47.390



t=68.710

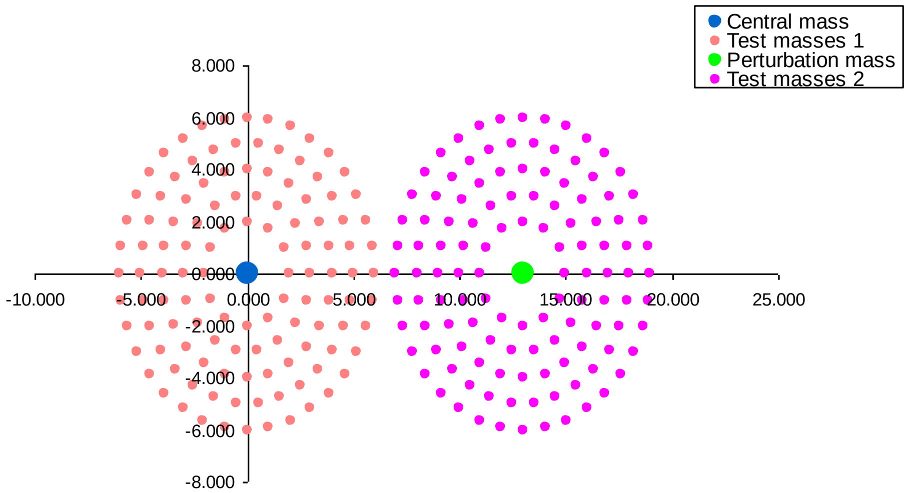


t=98.520

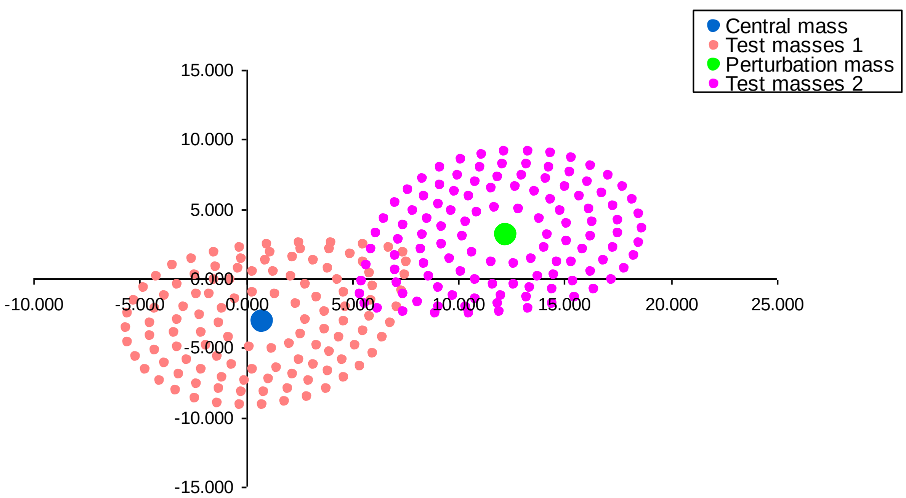


Case (ii) : Mutual circular orbits of identical galaxies

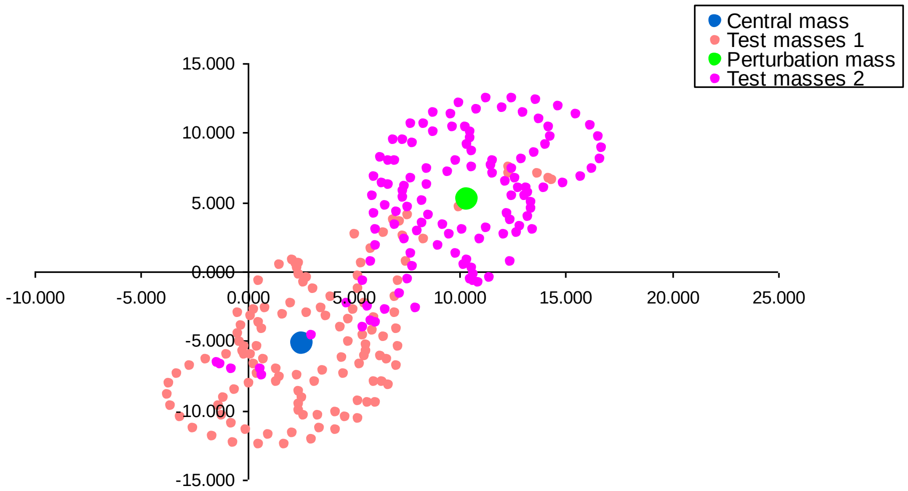
t=0



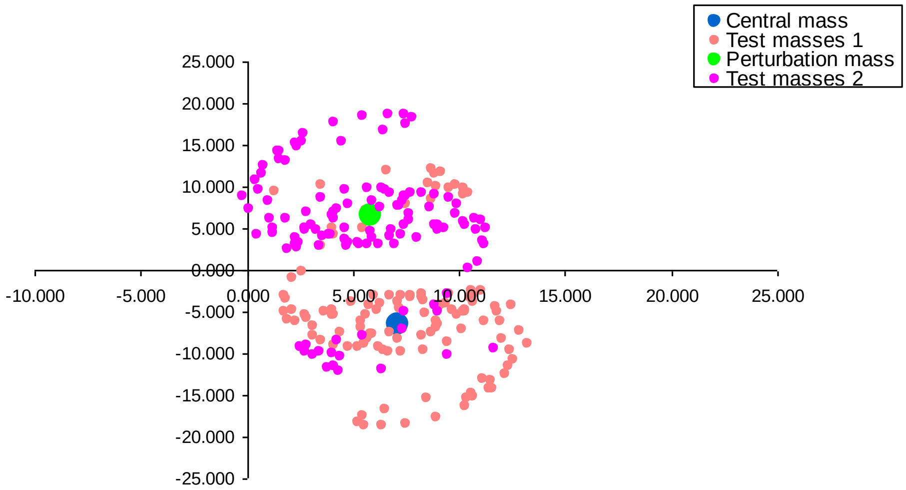
t=16.390



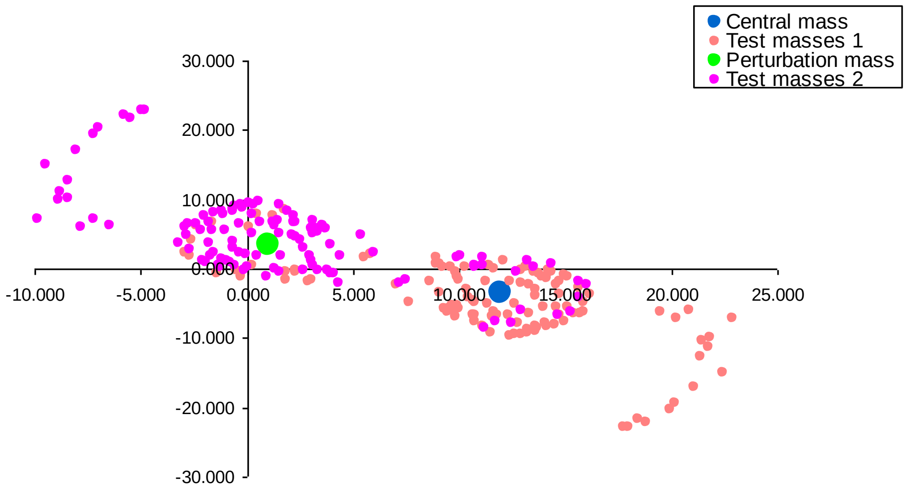
t=30.770



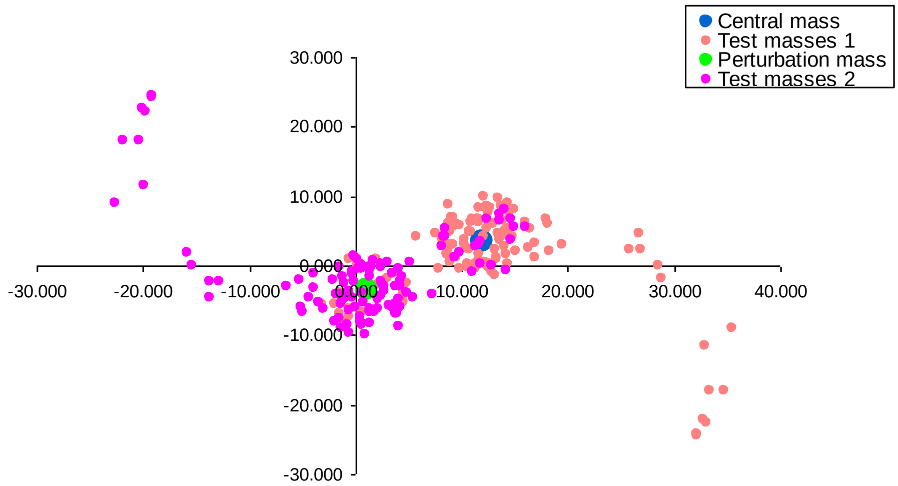
t=55.330



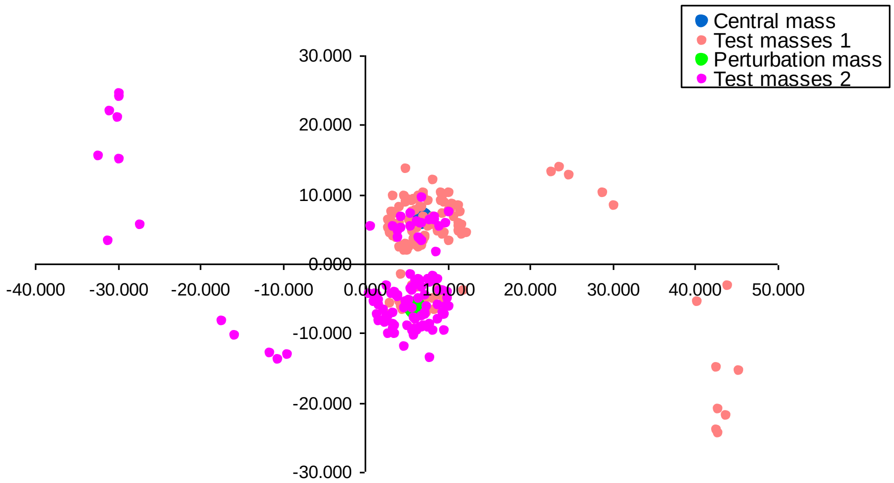
t=85.340



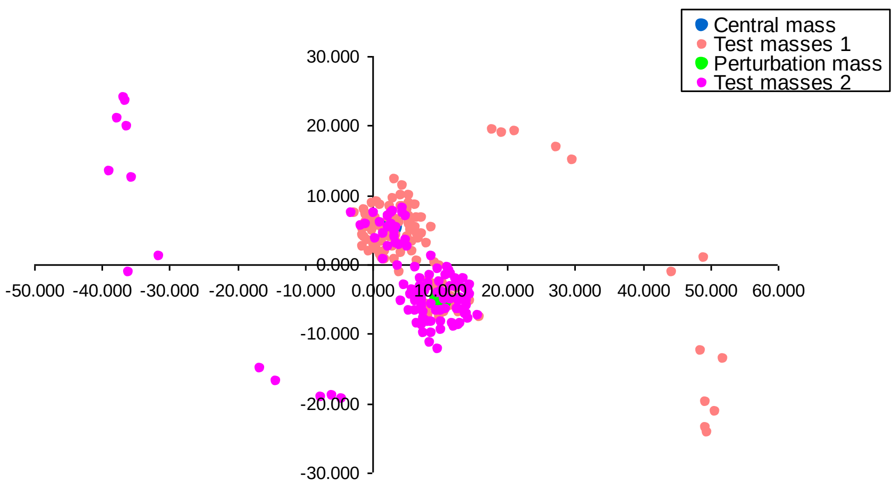
t=122.910



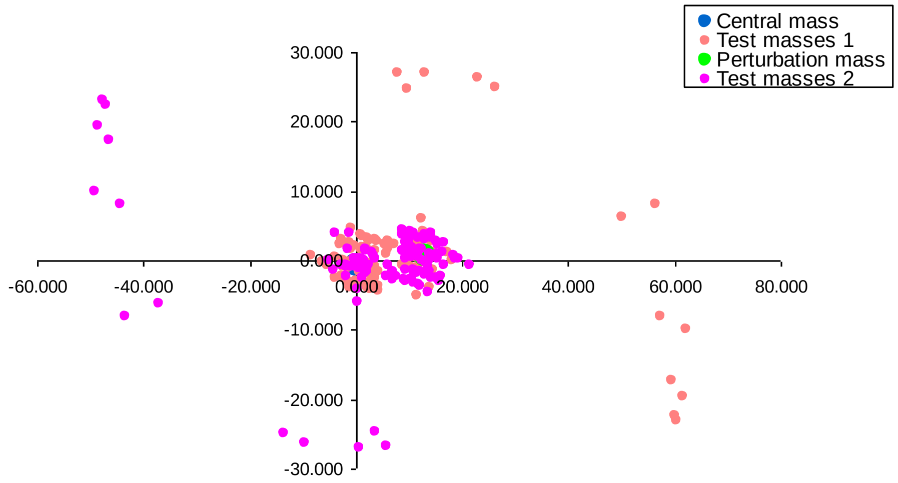
t=154.010



t=175.520

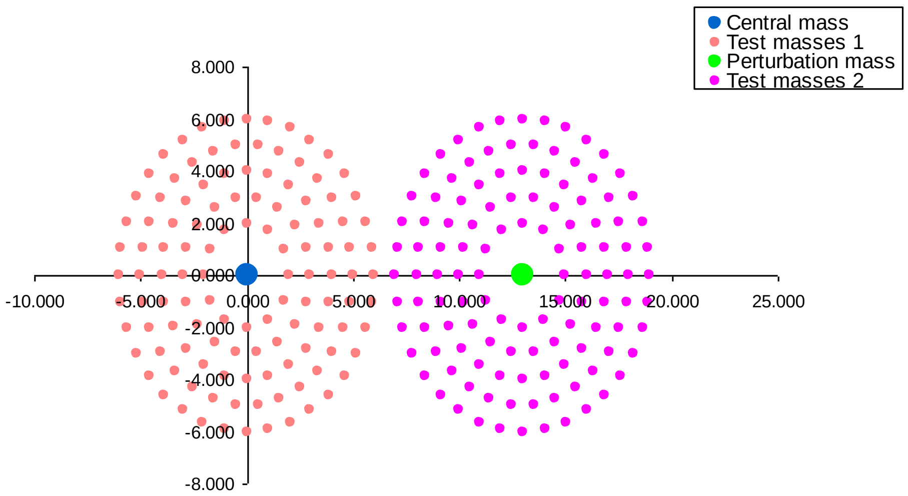


t=210.700

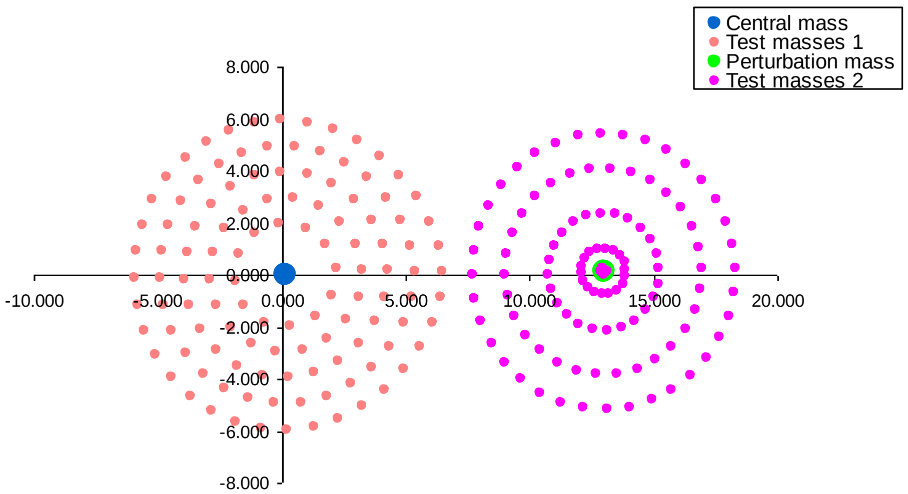


Case (iii) Massive Perturbation Galaxy ($\mu = 500$) in parabolic orbit

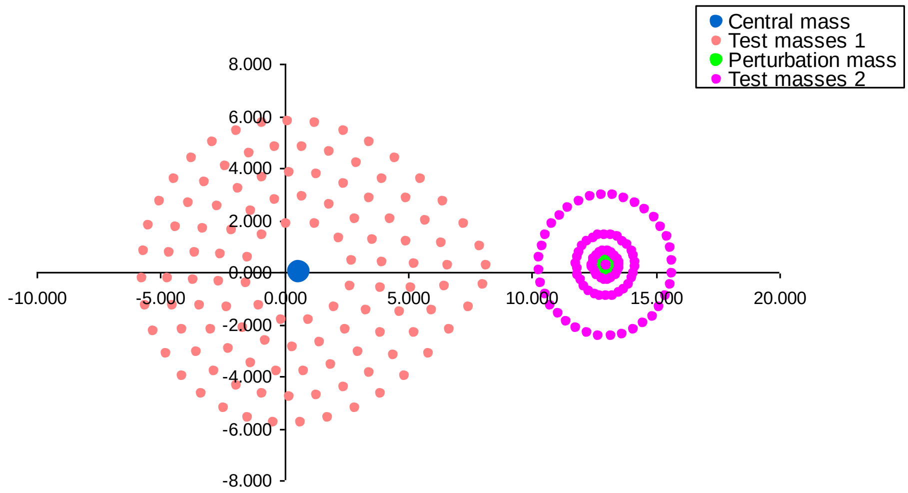
t=0



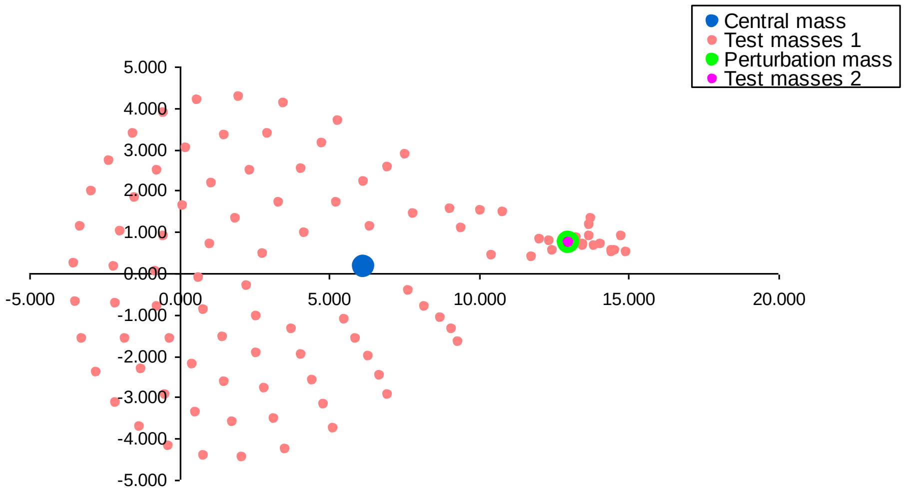
t=0.310



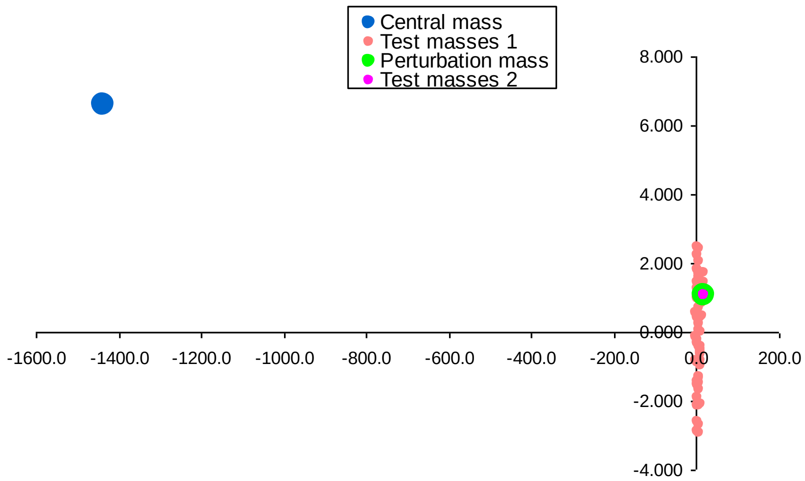
t=0.620



t=1.860



t= 2.790



6.1 Conclusion

The in-depth physical interpretation of the above results is beyond the scope of this report. However, there are some general comments I can make concerning the "success" of the computational procedure employed. i.e. does the method used yield physically reasonable results consistent with the above theoretical discussions of the initial conditions?

I will consider each of the three cases above in turn and then make some general comments.

For case (i) the above plots clearly demonstrate the right form of orbit. i.e. the separation of the galaxies after initial exchange of test masses continues to increase indicating an unbound system. (Or near to it – the parabolic case is at the boundary between unbound and bound states). Clearly visible are the 'tidal tails' created from the outlying test particles as the galaxies move past each other. An interesting observation (which also holds true for the mutually circular orbit) is that an sort of equilibrium test mass distribution is created by the system after initial penetration of one galaxy by its counterparts test masses. This *could* be an artificial property caused by an overly large softening radius (0.5 units for each large mass) in the sense that instead of being absorbed or scattered by a mass particle, test particles reside in an artificially shallow potential well. The *MathCAD* figure below demonstrates how the softening radius distorts the Gravitational potential. ($g(r)$ is the normal scaled gravitational potential, $f(r)$ is the softened version).

$i := 0..100$

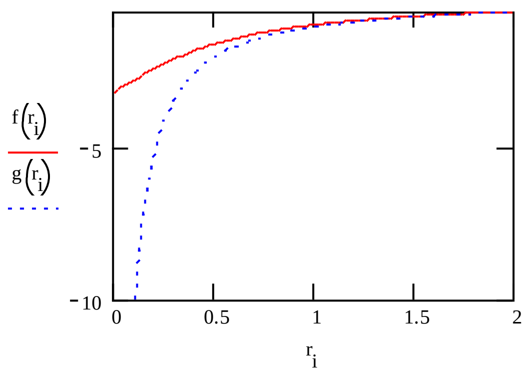
$r_i := \frac{i}{50}$

$\delta := 0.5$

$$\int_r^{\infty} \frac{-1}{\left(x^2+\delta^2\right)}\,dx \rightarrow \frac{-1}{2}\cdot \mathrm{csgn}\left(\frac{1}{\delta}\right)\cdot\frac{\pi}{\delta} + \frac{\mathrm{atan}\left(\frac{r}{\delta}\right)}{\delta}$$

$$f(r) := \frac{-1}{2}\cdot \mathrm{csgn}\left(\frac{1}{\delta}\right)\cdot\frac{\pi}{\delta} + \frac{\mathrm{atan}\left(\frac{r}{\delta}\right)}{\delta}$$

$$g(r) := \frac{-1}{r}$$



For case (ii) we observe very similar behaviour initially to that of the parabolic orbit. Quite beautiful structures result as the test mass distribution is perturbed such as the "shell-like" formation at around 31 seconds. (Units of scaled time!!) In the latter stages the circular orbits become apparent and, as before, large tidal tails of test masses are swept out as the galaxies rotate around each other. Note the lengths of these tails are much larger than the dimensions of the central clusters.

For case (iii) the expected symmetric collapse of test masses into the centre of the massive perturbation galaxy is observed followed by a "sucking" of the central mass galaxy into orbits around it. The last caption demonstrates the effect of not including a softening radius in the non test mass force law. The central mass acquires an enormous velocity as it approaches the perturbation mass. Unless the co-ordinate of the perturbation mass lies symmetrically within a step of position the central mass will be ejected from the system with a huge velocity as the acceleration suddenly changes sign. Hence explaining the x length scale of ~1000 units in the last figure. Although a rather dramatic confirmation of the prediction of an unbound system (the initial velocities were fixed for a parabolic orbit) the (probably) superluminal real ejection speed hardly represents a real event!

Overall my comments can be summarised as follows:

- The computer simulation of interacting galaxies (modelled by two interacting masses plus non self-interacting test particles) yields the well known tidal tail phenomenon observable in real galaxies.

- "Bridges" of test particles that penetrate their opposing galaxy are observed before a 'equilibrium' distribution results. (Running the program for many more iterations would confirm whether the latter statement is generally true).

- Overall the VIM solution of the particle trajectories seems to yield physically reasonable results except in the case when the interacting masses become very close. For the test particles, introduction of a softening radius removes much of the problems though one wonders whether this may lead to a fallacious equilibrium condition mentioned above.

- Obvious extensions to the project are to firstly compare the results with known observations and possibly attempt to predict current galactic structures from previous conditions as described by more sophisticated cosmological models. (i.e. as in the paper by *Toomre and Toomre* given in References). Another would be to reduce the time step and run the program on a faster computer (or a slower machine for longer). This would show how much of my results can be attributed to trajectory errors and how much is behaviour characteristic of the true solution of the equations of motion. Of course there is a limit to how much information can possibly be gained from my model since the use of Newtonian Gravity is itself an approximation. The behaviour of "supermassive" galaxies like that of case (iii) really need to be described by a more correct theory such as General Relativity.

7.1 References

1. A. TOOMRE, J. TOOMRE. "Galactic Bridges and Tails." *The Astrophysical Journal.* **178**. pp 623-666. Dec.15. 1972.

2. P. ALEXANDER. "Part II Computational Physics Lecture Handout." *Cambridge University.* 1999.

3. B. R. WEBBER. "Part IB Dynamics Lecture Handout." *Cambridge University.*1998.

8. Appendix

8.1 The Verlet Integration Method

Derivation of the VIM equations for numerical calculation of position and velocity given the known time dependence of acceleration.

Consider a small, finite time step of length h . Let us expand the position vector of a particle $\mathbf{r}(t)$ using a Taylor series involving h .

$$\mathbf{r}(t+h) = \mathbf{r}(t) + \dot{\mathbf{r}}(t)h + \tfrac{1}{2}\ddot{\mathbf{r}}(t)h^2 + \tfrac{1}{6}\dddot{\mathbf{r}}(t)h^3 + O(h^4) \qquad \text{(I)}$$

Similarly

$$\mathbf{r}(t-h) = \mathbf{r}(t) - \dot{\mathbf{r}}(t)h + \tfrac{1}{2}\ddot{\mathbf{r}}(t)h^2 - \tfrac{1}{6}\dddot{\mathbf{r}}(t)h^3 + O(h^4) \qquad \text{(II)}$$

Now (I) + (II) gives

$$\mathbf{r}(t+h) = 2\mathbf{r}(t) - \mathbf{r}(t-h) + \ddot{\mathbf{r}}(t)h^2 + O(h^4) \qquad \text{(III)}$$

Now the average velocity in region $-h < t < h$ is given by

$$\dot{\mathbf{r}}(t) = \frac{\mathbf{r}(t+h) - \mathbf{r}(t-h)}{2h} \qquad \text{(IV)}$$

We can generate two more equations from (III) and (IV) by change of variable $t \to t+h$.

$$\mathbf{r}(t+2h) = 2\mathbf{r}(t+h) - \mathbf{r}(t) + \ddot{\mathbf{r}}(t+h)h^2 + O(h^4) \qquad \text{(V)}$$

$$\dot{\mathbf{r}}(t+h) = \frac{\mathbf{r}(t+2h) - \mathbf{r}(t)}{2h} \qquad \text{(VI)}$$

Eliminating $\mathbf{r}(t+2h)$ and $\mathbf{r}(t-h)$ from (III) - (VI) we arrive at the VIM set of equations.

$$\mathbf{r}(t+h) = \mathbf{r}(t) + \dot{\mathbf{r}}(t)h + \tfrac{1}{2}\ddot{\mathbf{r}}(t)h^2 \qquad (10)$$

$$\dot{\mathbf{r}}(t+h) = \dot{\mathbf{r}}(t) + \tfrac{1}{2}h\left(\ddot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t+h)\right) \qquad (11)$$

8.2 Annotated listing of Excel Macro

```
Sub Dynamic1()                    Begins subroutine "Dynamic1"
'
' Dynamic1 Macro
' Macro recorded 01/11/99 by Andy French
'

'
For a = 0 To 30 Step 1            Specifies iteration range and step length.
   Range("D6:D247").Select        Selects cells containing the VIM generated
                                  co-ordinates.

   Selection.Copy                 Copies selection to the internal clipboard.
   Range("C6").Select             Selects destination column – the original set of
                                  co-ordinates
   Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone,
SkipBlanks:= _
       False, Transpose:=False    Utilisation of "Paste as values" command. This is
                                  a subset of the general "Paste special"
                                  command, hence the extra logical statements to
                                  define the operation.
   Range("F6:F247").Select
   Application.CutCopyMode = False

                                  {The next lines repeat the above for each of the
                                  columns of cells corresponding to x and y
                                  components of position and velocity}
   Selection.Copy
   Range("E6").Select
   Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone,
SkipBlanks:= _
       False, Transpose:=False
   Range("H6:H247").Select
   Application.CutCopyMode = False
   Selection.Copy
   Range("G6").Select
   Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone,
SkipBlanks:= _
       False, Transpose:=False
   Range("J6:J247").Select
   Application.CutCopyMode = False
   Selection.Copy
   Range("I6").Select
   Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone,
SkipBlanks:= _
       False, Transpose:=False
   Range("J3").Select
   Selection.Copy
   Range("H3").Select
```

```
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
```

Next a	*Perform iteration a +1.*
End Sub	*End subroutine.*