Water Rocket

Dr Andrew French
Andrew Chesters
Jeremy Douglas
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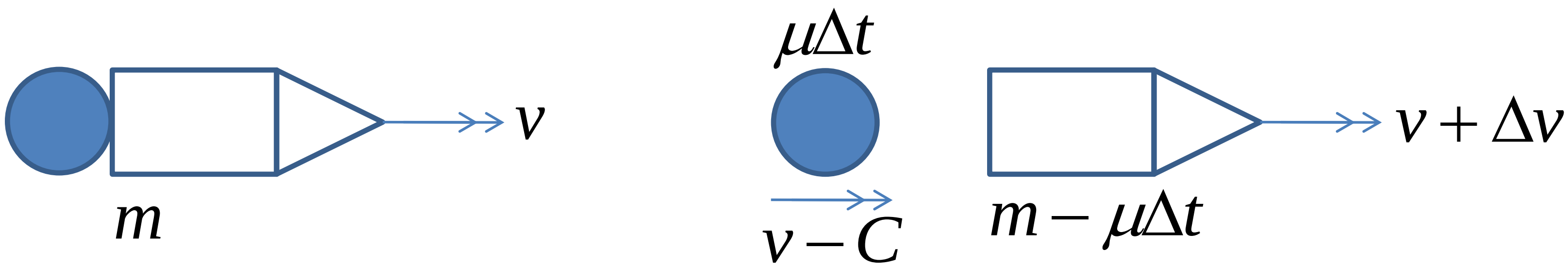
1. Water Rocket experiments
2. A simple model: *The Rocket Equation*
3. Measuring thrust
4. Modelling rocket dynamics
5. Measuring the rocket trajectory
6. Ideas for further work
7. The Rocket Scientists!



The Rocket Equation

In the first instance, only consider **thrust** due to the ejection of mass from a rocket. Assume mass is ejected at velocity *C* relative to the rocket and at mass rate μ



At time t At time $t + \Delta t$

By conservation of momentum:

$$mv = \mu\Delta t\left(v - C\right) + \left(m - \mu\Delta t\right)\left(v + \Delta v\right)$$

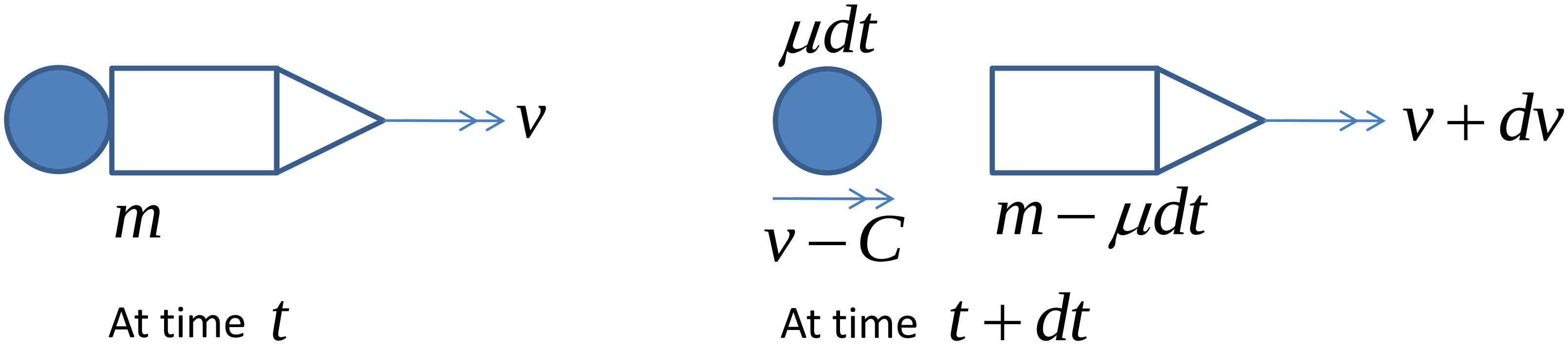
$$mv = \mu\Delta t v - \mu\Delta t C + mv - \mu\Delta t v + m\Delta v - \mu\Delta t\Delta v$$

$$m\Delta v - \mu\Delta t\Delta v = \mu\Delta t C$$

$$\left(m - \mu\Delta t\right)\frac{\Delta v}{\Delta t} = \mu C$$

Taking the limit $\Delta t \to 0$ $\quad m\dfrac{dv}{dt} = \mu C$

i.e. *Newton's Second Law.* Hence thrust $\boxed{T = \mu C}$



Now if the mass ejection rate and ejection velocity are both constant:

$$\left(m_0 - \mu t\right)\frac{dv}{dt} = \mu C$$

$$\int_0^v dv = \int_0^t \frac{\mu C}{m_0 - \mu t}dt = -C\int_0^t \frac{-\mu}{m_0 - \mu t}dt$$

$$v = \left[-C\ln\left|m_0 - \mu t\right|\right]_0^t$$

$$v = C\ln m_0 - C\ln\left(m_0 - \mu t\right)$$

$$\boxed{v = C\ln\left(\frac{m_0}{m_0 - \mu t}\right)}$$

This is the **Rocket Equation**

m_0 is the initial mass i.e. the mass of the rocket plus the total mass of the fuel.

The mass of fuel gives an upper limit on the time that the rocket can be in a thrust phase.

$$\boxed{m_0 = m_R + m_F \qquad t_{\max} = \frac{m_F}{\mu}}$$

Measuring the thrust of a water rocket. The assumptions of constant thrust may be true for a short burn time for a space-borne rocket booster, but are unlikely to be valid for a terrestrial water rocket, which will rapidly depressurize as water is ejected.
To investigate how thurst varies with time, a test rig was designed such that upwards motion of the rocket was constrained by a force-meter attached to a data logger.

Screw-on fin attachment with pressure value

Pressure cable & bicycle pump

'Force plate' attachment for Pasco forcemeter

Pasco data-logger, set to record thrust (Newtons) vs time (seconds) at a maximum data rate of 5000 measurements per second

“I think we should use some guides to prevent any lateral movement”
“Nah, it will be fine...”

Oh dear! Perhaps the guides were a good idea then....

(The rocket ‘escaped’ Science School quad and was eventually found 60m away in Mr Munn’s garden!)

Stills from a 30 frame-per-second HD video recording using a Panasonic Lumix TZ8 mounted on a tripod.

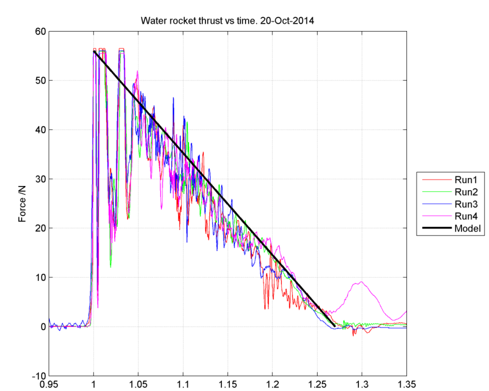
Thrust data analysis

Excel sheet
Run1
Run2
Run3
Run4

→ MATLAB Excel file importer → MATLAB processing

- Adjust for non-zero Force offset
- ↓ Time align peak thrust for all four runs
- ↓ Cut data around launch

↓ Plot overlay of (time,force) graphs

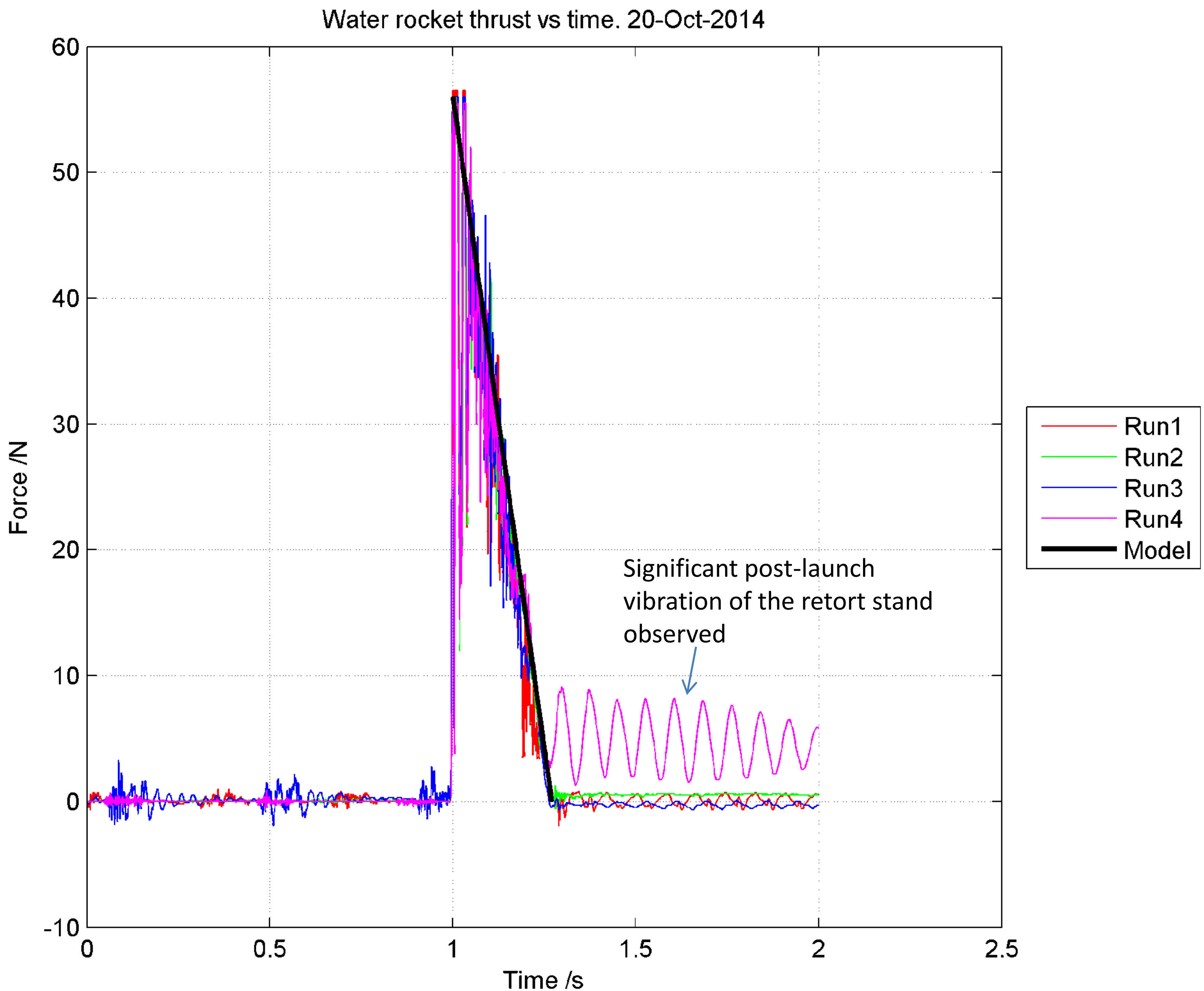


↑ .csv file export

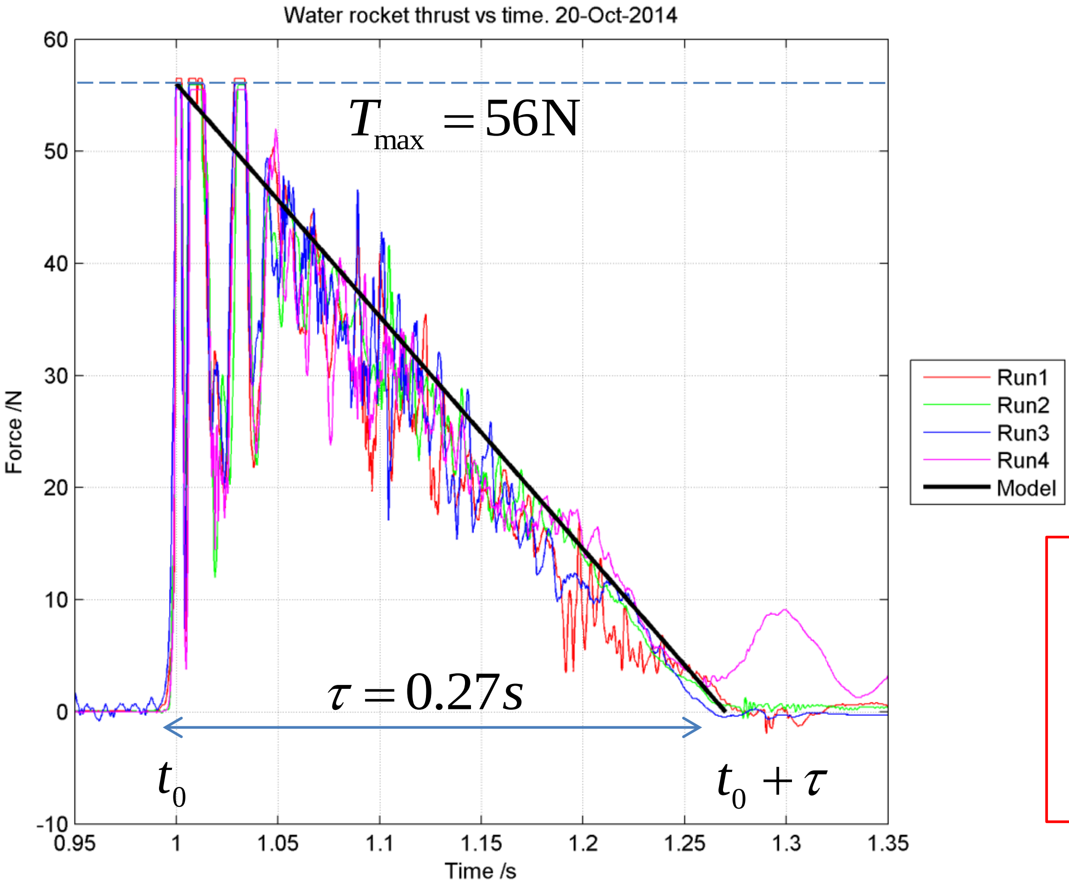
↑ Pasco Data Studio

↑ USB connection to PC

↑


Water rocket thrust vs time. 20-Oct-2014
Force /N
Time /s
60
50
40
30
20
10
0
-10
0
0.5
1
1.5
2
2.5
Run1
Run2
Run3
Run4
Model
Significant post-launch vibration of the retort stand observed

Modelling thrust



Experimental data appears to indicate a linearly decaying thrust T from a maximum T_{max} = 56N to zero in τ = 0.27s is a reasonable approximation.

$$\boxed{T = \begin{cases} T_{\max}\left(1 - \dfrac{t - t_0}{\tau}\right) & t_0 \le t \le t_0 + \tau \\ 0 & \text{otherwise} \end{cases}}$$

The 'rocket equation' states thrust T is the product of mass ejection rate μ and water ejection velocity C $\boxed{T = \mu C}$

Since water is approximately incompressible and the nozzle area remains the same during launch, we may assume the mass ejection rate is proportional to the ejection velocity $\boxed{\mu = kC}$

Putting this together, we can find an expression for C and μ in terms of time t.
If m_w is the mass of water to be ejected, this allows us to find k and hence C and μ.

$$T = \mu C$$

$$\mu = kC$$

$$\therefore T_{\max}\left(1 - \frac{t - t_0}{\tau}\right) = kC^2$$

$$\therefore \boxed{C = \sqrt{\frac{T_{\max}}{k}\left(1 - \frac{t - t_0}{\tau}\right)}}$$

$$\therefore \boxed{\mu = \sqrt{kT_{\max}\left(1 - \frac{t - t_0}{\tau}\right)}}$$

$$m_w = \int_{t_0}^{t_0 + \tau} \mu \, dt$$

$$m_w = \sqrt{kT_{\max}} \int_{t_0}^{t_0 + \tau} \sqrt{1 - \frac{t - t_0}{\tau}} \, dt$$

$$m_w = \sqrt{kT_{\max}} \left[-\tfrac{2}{3}\tau \left(1 - \frac{t - t_0}{\tau}\right)^{\frac{3}{2}} \right]_0^{t_0 + \tau}$$

$$m_w = \sqrt{kT_{\max}} \ \tfrac{2}{3}\tau$$

$$\therefore \boxed{k = \frac{9m_w^2}{4\tau^2 T_{\max}}}$$

An *iterative* dynamic model for the water rocket

Thrust phase

$$m_0 = m_R + m_F$$

$$t \le \frac{m_F}{\mu}$$

ρ is the density of air (around 1kgm^{-3})

A is the cross sectional area of the forward section of the rocket (assume 3.5 cm radius)

c_D is the drag coefficient. Assume about 0.5 since bottle has no nosecone.

$$a = \frac{\overbrace{T_{\max}\left(1 - \frac{t}{\tau}\right)}^{\text{Thrust}} - \overbrace{m_t g}^{\text{Weight}} - \overbrace{\tfrac{1}{2} c_D \rho A v_t^2}^{\text{Air resistance}}}{m_t}$$

$$\Delta v = a \Delta t$$

$$v_{t+\Delta t} = v_t + \Delta v$$

$$x_{t+\Delta t} = v_{t+\Delta t} \Delta t$$

i.e. a *very simple* (first order) differential equation solving routine. Use a small timestep Δt = 0.001s

$$\mu = \sqrt{\frac{9m_w^2}{4\tau^2 T_{\max}} \times T_{\max}\left(1 - \frac{t}{\tau}\right)} = \frac{3m_w}{2\tau}\sqrt{\left(1 - \frac{t}{\tau}\right)}$$

For greater accuracy use a higher order method such as *Verlet* or *Runge-Kutta*

$$m_{t+\Delta t} = m_t - \mu \Delta t$$

An *iterative* dynamic model for the water rocket

Projectile phase: Going up

$$v_t > 0$$

Weight

Air resistance

$$a = \frac{-m_t g - \frac{1}{2} c_D \rho A v_t^{\,2}}{m_t}$$

$$\Delta v = a \Delta t$$

$$v_{t+\Delta t} = v_t + \Delta v$$

$$x_{t+\Delta t} = v_{t+\Delta t} \Delta t$$

Projectile phase: Going down

$$v_t < 0,\ x_t \geq 0$$

Weight

Air resistance

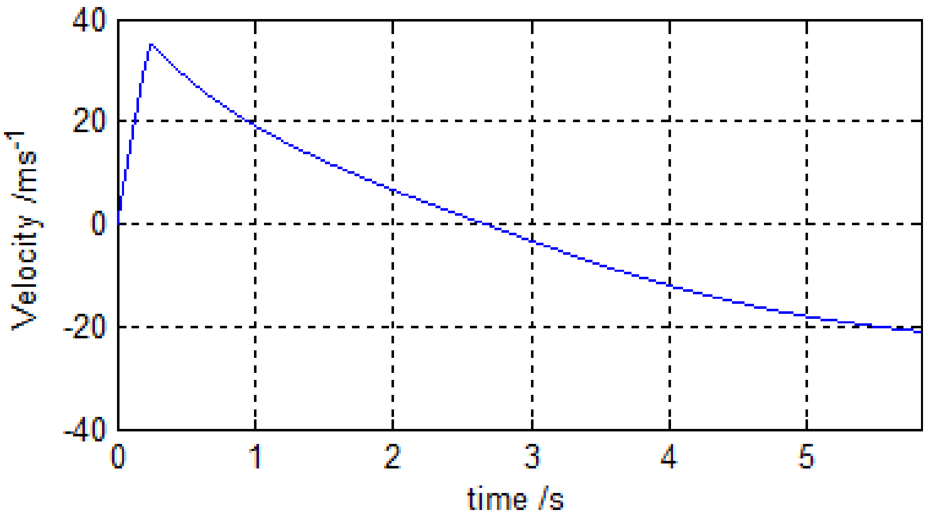
$$a = \frac{-m_t g + \frac{1}{2} c_D \rho A v_t^{\,2}}{m_t}$$

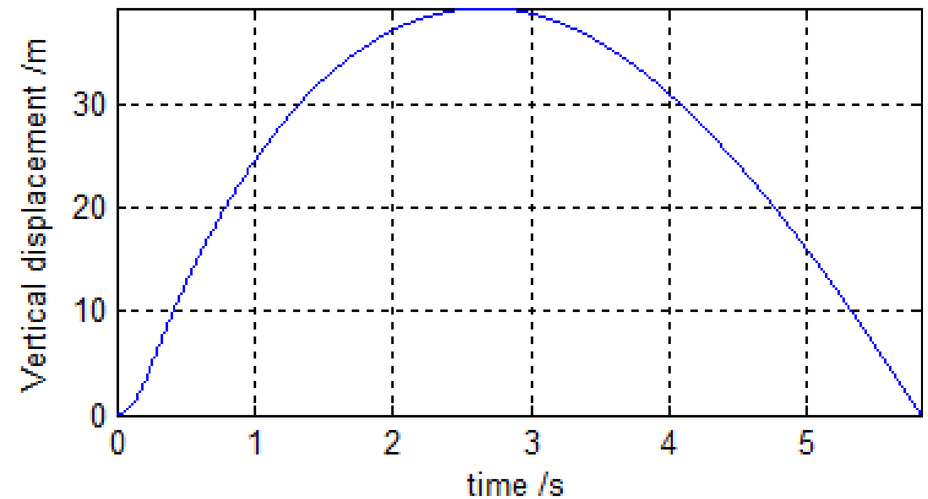
$$\Delta v = a \Delta t$$

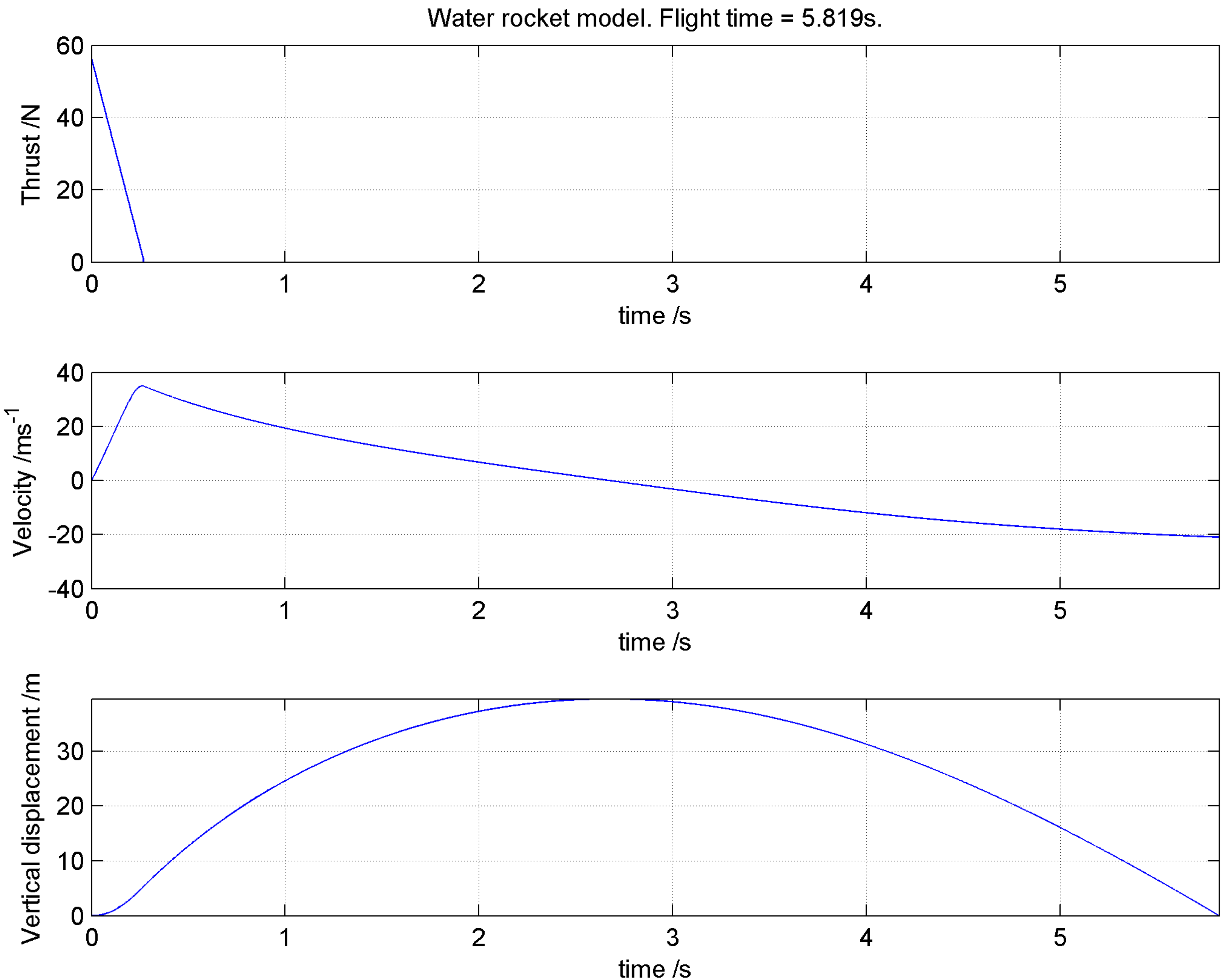
$$v_{t+\Delta t} = v_t + \Delta v$$

$$x_{t+\Delta t} = v_{t+\Delta t} \Delta t$$

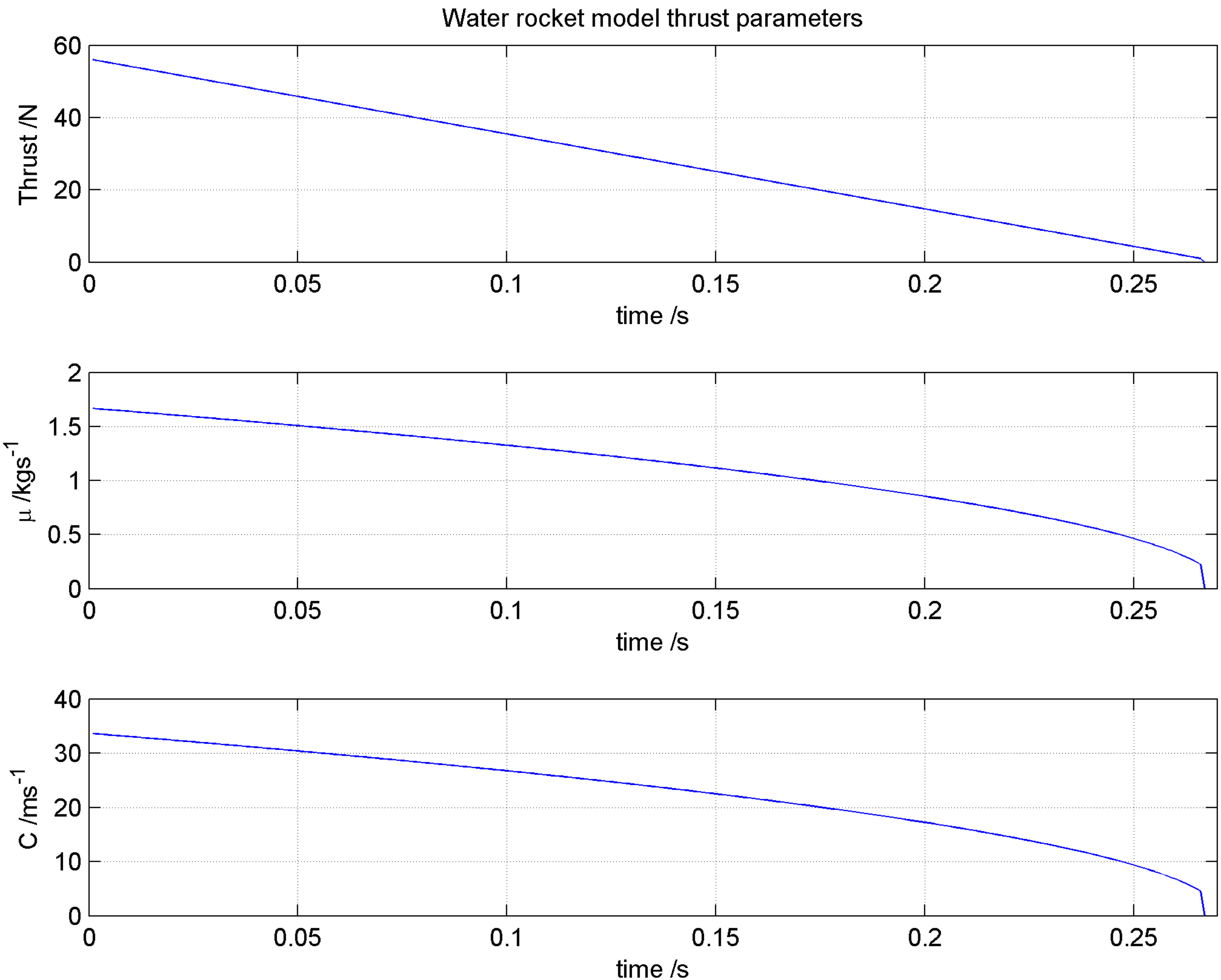
```
%Empty rocket mass /kg
m_r = 0.06;

%Mass of water added /kg
m_w = 0.3;

%Gravitational field strength /ms^-2
g = 9.81;

%Water ejection time /s
t_eject = 0.27;

%Max thrust /N
Tmax = 56;

%Cross sectional area of rocket /m^2
A = pi*(3.5/100)^2;

%Drag coefficient for vertical rocket
motion
cD = 0.5;

%Air density /kgm^-3
rho_air = 1;

%Timestep for differential equation
solver
dt = 0.001;
```







Water rocket model. Flight time = 5.819s.
Thrust /N
60
40
20
0
0
1
2
3
4
5
time /s
Velocity /ms^-1
40
20
0
-20
-40
0
1
2
3
4
5
time /s
Vertical displacement /m
30
20
10
0
0
1
2
3
4
5
time /s


Water rocket model thrust parameters
Thrust /N
time /s
μ /kgs-1
time /s
C /ms-1
time /s

Measuring the maximum height attained by the rocket

A practical method for measuring the height of the rocket at apogee *is quite difficult to achieve.* Effects of windage and non-vertical takeoff due to uneven ground or rocket shake during pumping complicate the dynamics, making the scenario a two-or three dimensional problem.

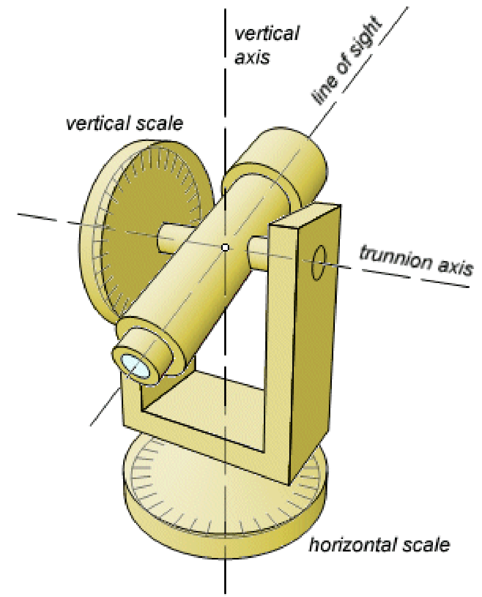
Potential techniques investigated have been (so far with limited success):

1. **Launch the rocket attached to a spool of light but strong fishing line.**

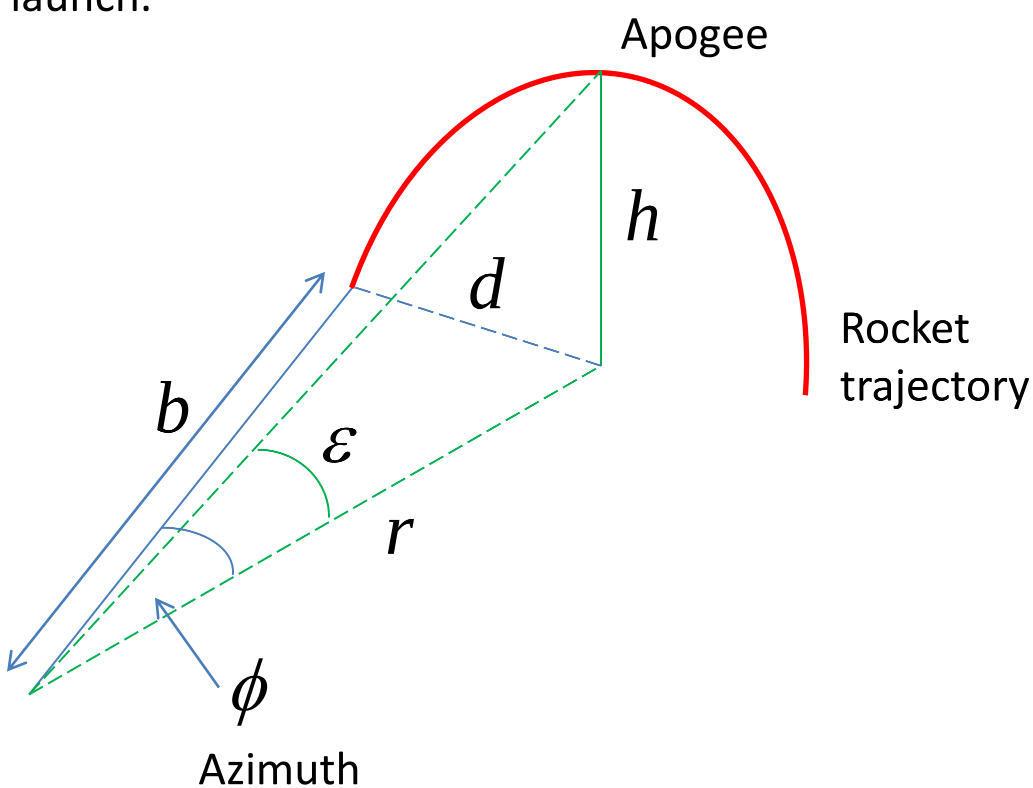
When the rocket lands, cut the line and weigh the line. The ratio of the mass of the line compared to the mass of 1m of line gives the length of the line and hence the height.

Problems: Spooling the wire without friction is not easy. If there is any friction the spool will shock-load and break. This will also arrest the launch of the rocket as the shock loaded line will temporarily act as a leash! You need a heavy vertically spooling cop of thread for this to work. Also, windage effects will increase the line length beyond the apogee.

2. Use an angle measuring setup to determine the apogee using trigonometry.



This can be difficult to achieve given the rapid motion of the rocket. An expensive solution (!) could be to track the motion using a tripod-mounted *theodolite* attached to a data-logger. i.e. record a time history of elevation and azimuth vs time. This could be co-mounted and synchronized with a video camera to enable the apogee to be accurately determined after the launch.



Drift from baseline at apogee

$$\boxed{d = b \tan \phi}$$

$$r \cos \phi = b$$

$$h = r \tan \varepsilon$$

Height at apogee

$$\boxed{h = \frac{b \tan \varepsilon}{\cos \phi}}$$

Further work:

1. Reduce vibration in static test rig. Create an exhaust flue to prevent splashback!

2. Investigate a practical apogee measuring device, or perhaps indeed an ‘entire trajectory’ tracking system!

3. Investigate the effect of varying the amount of water. In all experiments 300ml was used.

4. Investigate using a valve which fails at a higher pressure, i.e. generating more thrust.
   Note the Pasgo force-meter is rated at 50N. Therefore any extra thrust would take it well beyond its design spec.

Yes it is rocket science!

Andrew French Jeremy Douglas Andrew Chesters