
A. French. February 2017
e/m_e via the Fine Beam Tube

Calculating the electron charge to mass ratio using a Fine Beam Tube

This experiment refers to the Fine Beam Tube model number 555 58 Baur 2 made by Leybold.

https://www.leybold-shop.co.uk/

'Actual' values

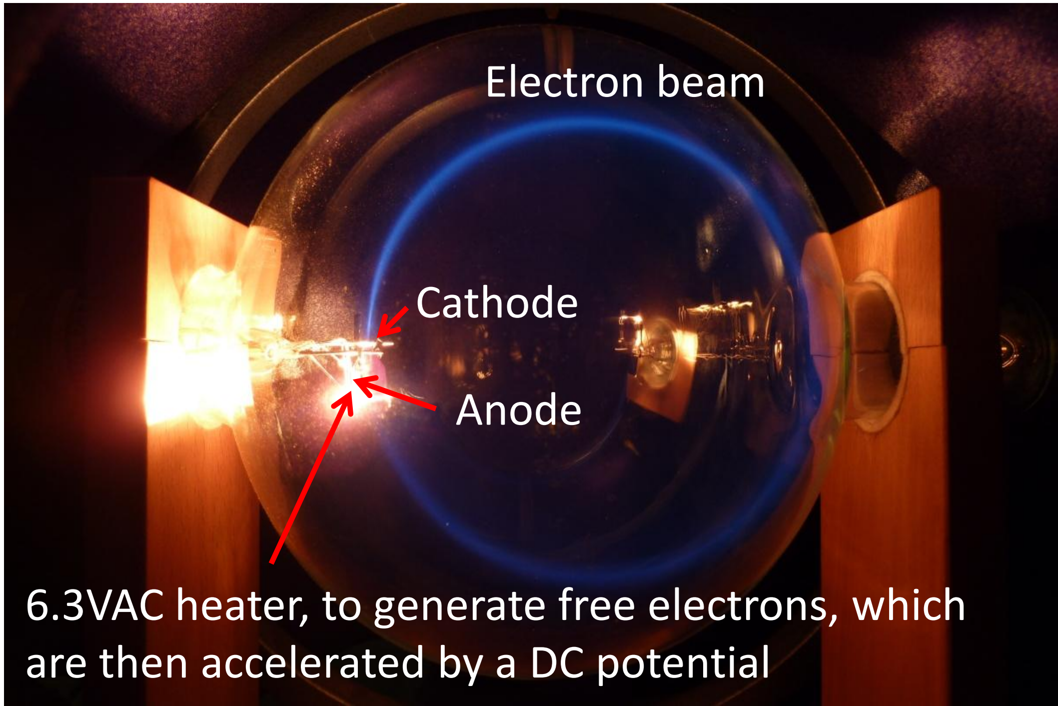
$$m_e = 9.109 \times 10^{-31}\,\mathrm{kg}$$

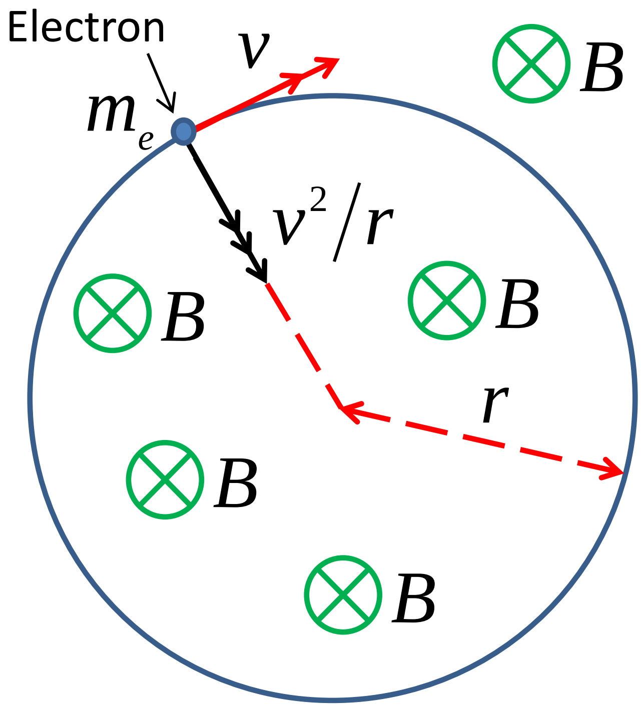
$$e = 1.602 \times 10^{-19}\,\mathrm{C}$$

$$\frac{e}{m_e} = 1.76 \times 10^{11}\,\mathrm{Ckg}^{-1}$$

Low pressure hydrogen gas inside a spherical tube is ionized by a beam of electrons, which are accelerated via a voltage of approximately 100V. A pair of Helmholtz coils (solenoids) produce a highly uniform magnetic field which bends the beam into a circle.

If the accelerating voltage, the coil current and the beam radius are measured, it is possible to calculate from these parameters the electron charge to mass ratio e/m_e





Assume *uniform* magnetic field of strength B between the Helmholtz coils.

The **force** on an electron (beyond cathode and deflection plates) is $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$

i.e. a purely *centripetal* force if the beam is initially vertical and perpendicular to the uniform magnetic field.

Newton II (+ve in radially inward direction):

$$\frac{m_e v^2}{r} = Bev \Rightarrow v = \frac{Ber}{m_e}$$

Assume electron **kinetic energy** is solely from the **accelerating potential**, and velocities are low enough such that relativistic effects can be ignored.*

$$\tfrac{1}{2} m_e v^2 = eV \therefore v = \sqrt{\frac{2eV}{m_e}}$$

Hence: $$\sqrt{\frac{2eV}{m_e}} = \frac{Ber}{m_e} \therefore \frac{2eV}{m_e} = \frac{B^2 e^2 r^2}{m_e^2}$$

The charge to mass ratio for an electron can therefore be determined in terms of readily measurable quantities via the Fine Beam Tube!

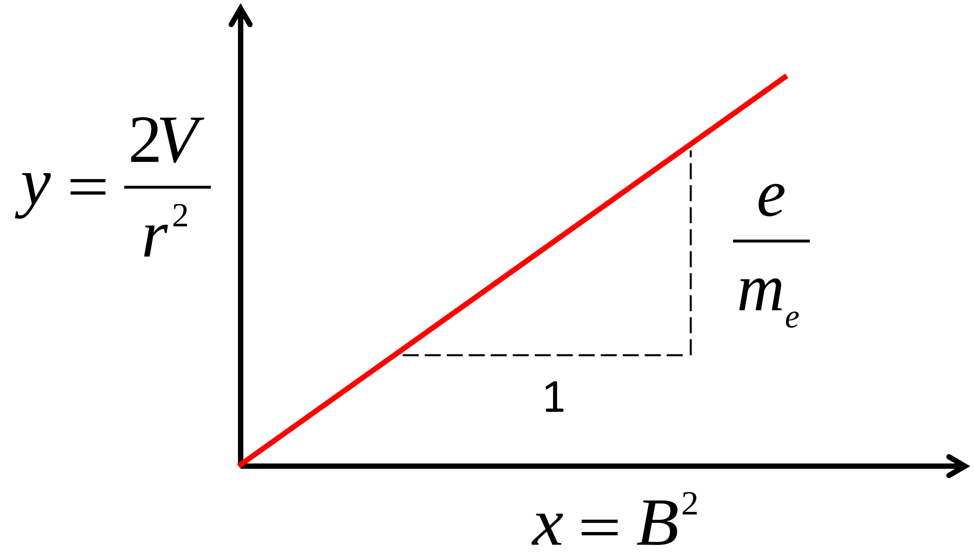
$$\therefore \boxed{\frac{e}{m_e} = \frac{2V}{B^2 r^2}}$$

* See next page!

Classical result: $$\frac{e}{m_e} = \frac{2V}{B^2 r^2}$$

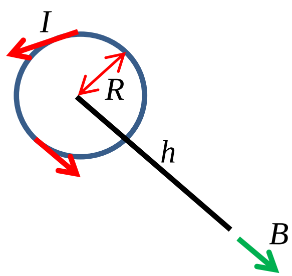
So the Fine Beam tube can be used to measure the **electron charge to mass ratio** by plotting a graph of y vs x and finding the gradient.

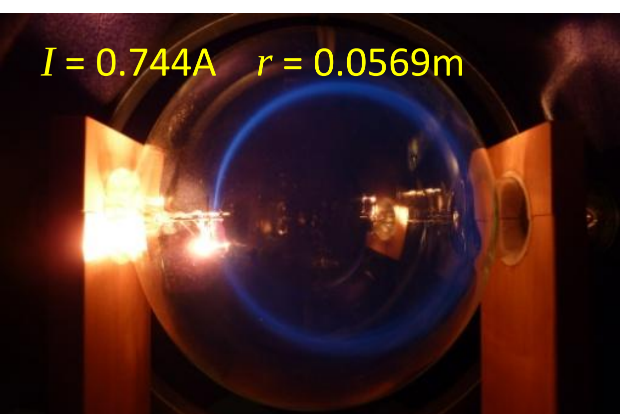
$$x = B^2, \quad y = \frac{2V}{r^2}$$



$$B = \frac{\frac{1}{2}\mu_0 NIR^2}{\left(R^2 + h^2\right)^{\frac{3}{2}}}$$

Magnetic field on axis from a current loop of N turns





For a *pair* of Helmholtz coils with N turns and radius R separated by distance $2h$, the magnetic field strength along the coil centre line, half way between the coils, is:

$$B = \frac{\mu_0 NIR^2}{\left(R^2 + h^2\right)^{\frac{3}{2}}} = \frac{\mu_0 NI}{R}\left(1 + \left(\frac{h}{R}\right)^2\right)^{-\frac{3}{2}}$$

$$R = 0.15\text{m}, \quad h = 0.75\text{m}$$

$$\therefore 1 + \left(\frac{h}{R}\right)^2 = \tfrac{5}{4} \Rightarrow \boxed{B = \frac{\mu_0 NI}{R}\left(\tfrac{4}{5}\right)^{\frac{3}{2}}}$$

Permeability of free space $\mu_0 = 4\pi \times 10^{-7}\,\text{Hm}^{-1}$

Do we need to account for relativistic effects?

$$\tfrac{1}{2} m_e v^2 = eV$$ ← Classical energy formula

$$\therefore \tfrac{1}{2} m_e \left(\frac{v}{c}\right)^2 = \frac{eV}{c^2}$$

$$\therefore V = \tfrac{1}{2}\frac{m_e c^2}{e}\left(\frac{v}{c}\right)^2$$

$$\therefore \boxed{V \approx 256{,}000\text{V} \times \left(\frac{v}{c}\right)^2}$$

Define 'Relativistic' if:

$$\frac{v}{c} > 0.1$$

$$\therefore V > 256{,}000\text{V} \times (0.1)^2$$

$$\therefore \boxed{V > 2{,}560\text{V}}$$

So since our accelerating potentials are about 180V, we might find the effects are small.

However, if v/c = 0.01, the 'relativistic voltage' becomes **25.6V.**

So we might expect relativistic effects to be *small but perhaps not insignificant.*

'Actual' values

$$m_e = 9.109 \times 10^{-31}\,\text{kg}$$

$$e = 1.602 \times 10^{-19}\,\text{C}$$

$$c = 2.998 \times 10^{8}\,\text{ms}^{-1}$$

$$\frac{e}{m_e} = 1.76 \times 10^{11}\,\text{Ckg}^{-1}$$

Relativistic effects

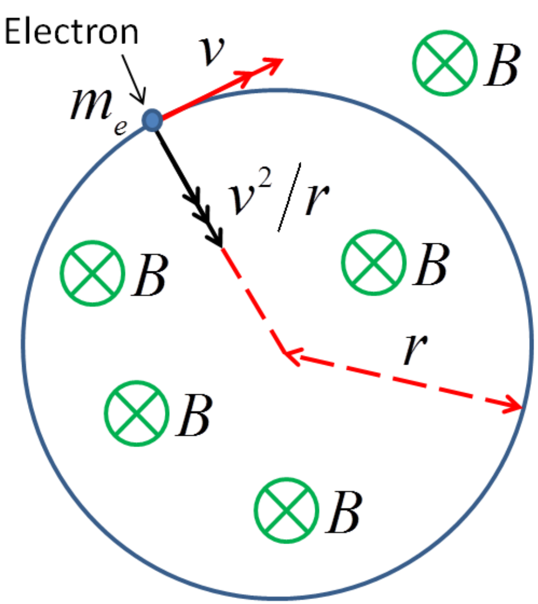
Newton II: $$\mathbf{f} = m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{v}}{c^2}\right)\mathbf{v}$$

Assume centripetal acceleration $\therefore \mathbf{a}\cdot\mathbf{v} = 0$

Newton II applied radially:

$$\therefore Bev = m_e\gamma\frac{v^2}{r}$$

$$\therefore \frac{Ber}{m_e} = \gamma v$$



$$eV = (\gamma - 1)m_e c^2$$

Relativistic Kinetic Energy equated to accelerating potential multiplied by the electron charge

$$\therefore \gamma = 1 + \frac{eV}{m_e c^2}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\therefore \gamma^{-2} = 1 - \frac{v^2}{c^2} \therefore v = c\sqrt{1-\gamma^{-2}}$$

$$\therefore \frac{Ber}{m_e} = c\left(1+\frac{eV}{m_e c^2}\right)\sqrt{1-\left(1+\frac{eV}{m_e c^2}\right)^{-2}}$$

$$\therefore r = \frac{m_e c}{Be}\left(1+\frac{eV}{m_e c^2}\right)\sqrt{1-\left(1+\frac{eV}{m_e c^2}\right)^{-2}}$$

Classical result

$$\frac{e}{m_e} = \frac{2V}{B^2 r^2}$$

$$\therefore r_C = \sqrt{\frac{2Vm_e}{B^2 e}}$$

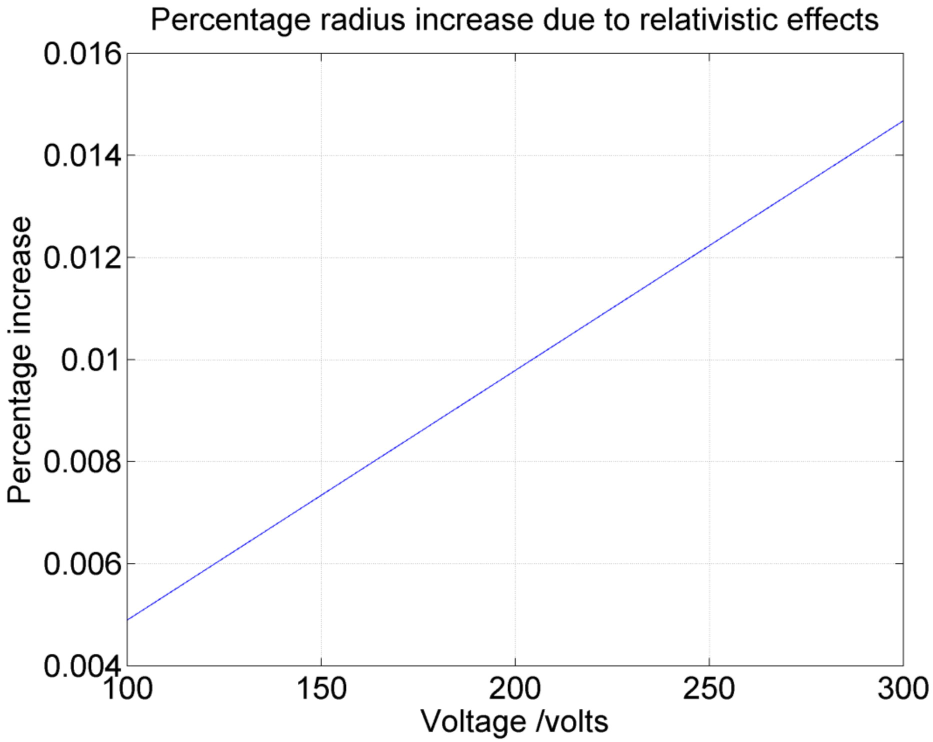
Relativistic result

$$r_R = \frac{m_e c}{Be}\left(1 + \frac{eV}{m_e c^2}\right)\sqrt{1 - \left(1 + \frac{eV}{m_e c^2}\right)^{-2}}$$

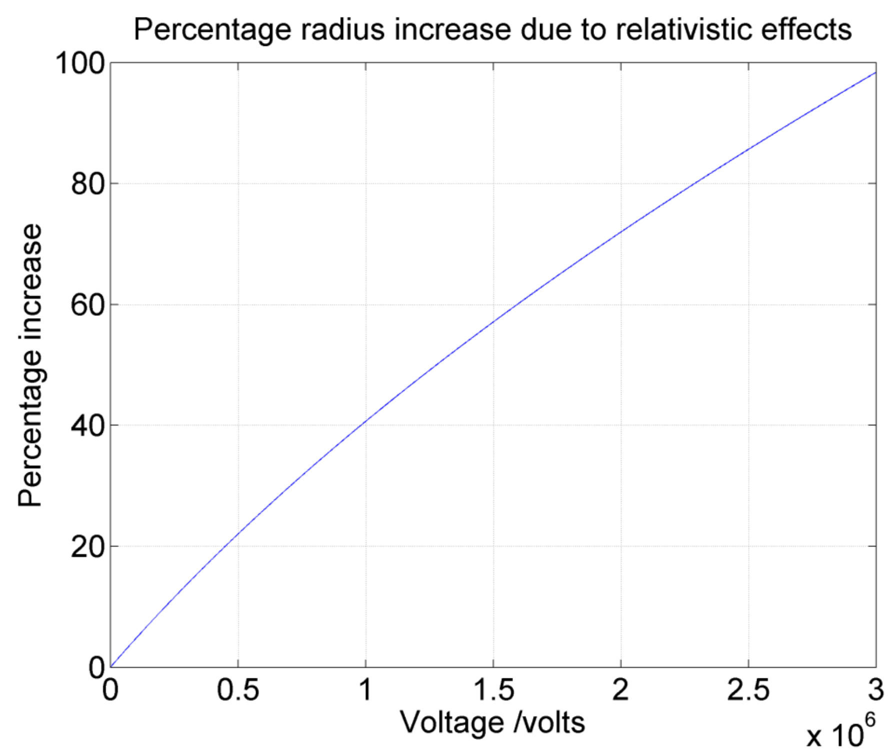
$$\therefore \frac{r_R}{r_C} = \sqrt{\frac{B^2 e}{2Vm_e}}\,\frac{m_e c}{Be}\left(1 + \frac{eV}{m_e c^2}\right)\sqrt{1 - \left(1 + \frac{eV}{m_e c^2}\right)^{-2}}$$

$$\frac{r_R}{r_C} = \sqrt{\frac{\frac{1}{2} m_e c^2}{eV}}\left(1 + \frac{eV}{m_e c^2}\right)\sqrt{1 - \left(1 + \frac{eV}{m_e c^2}\right)^{-2}}$$

Define percentage increase due to relativistic effects: $\dfrac{r_R - r_C}{r_C} \times 100 = 100\left(\dfrac{r_R}{r_C} - 1\right)$

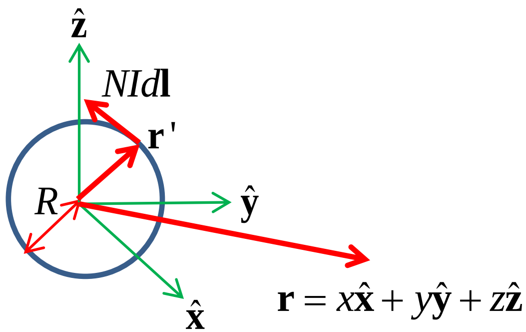


For the range of voltages we will be using, we anticipate the % change in beam radius to be **negligible** i.e. of the order of about 0.01%



If on the other hand we use an accelerating potential of a few million volts (or much higher, as they do in particle accelerator experiments) we would notice a *significant difference* between classical and relativistic results.

What about the effect of the electron beam being at radius r from the central axis between the Helmholtz coils? Let's firstly consider a 'single loop' of N tight coils of radius R:

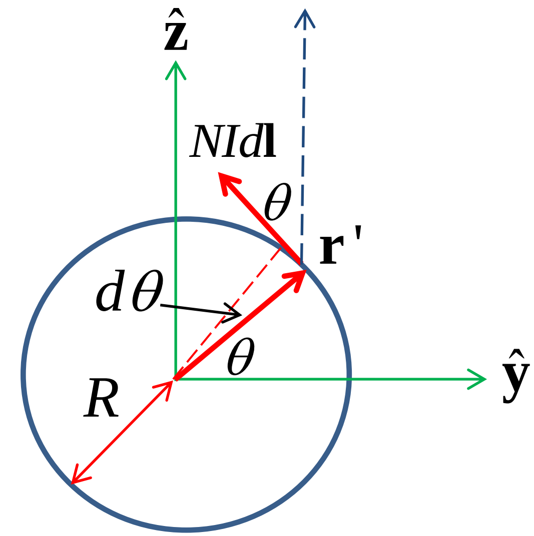


Biot-Savart law

$$\mathbf{B} = \frac{\mu_0 IN}{4\pi}\int_{loop}\frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$d\mathbf{l} = (-\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})Rd\theta$$

$$\mathbf{r}' = R\cos\theta\hat{\mathbf{y}} + R\sin\theta\hat{\mathbf{z}}$$



Define a *numerical method* to compute the x,y,z components of the magnetic field. Evaluate at $\mathbf{r} = h\hat{\mathbf{x}} + r\hat{\mathbf{z}}$

Given the 'cylindrical symmetry' of the system, this is a quite general result

Since the second solenoid is identical and equidistant we can simply multiply the magnetic field by two.

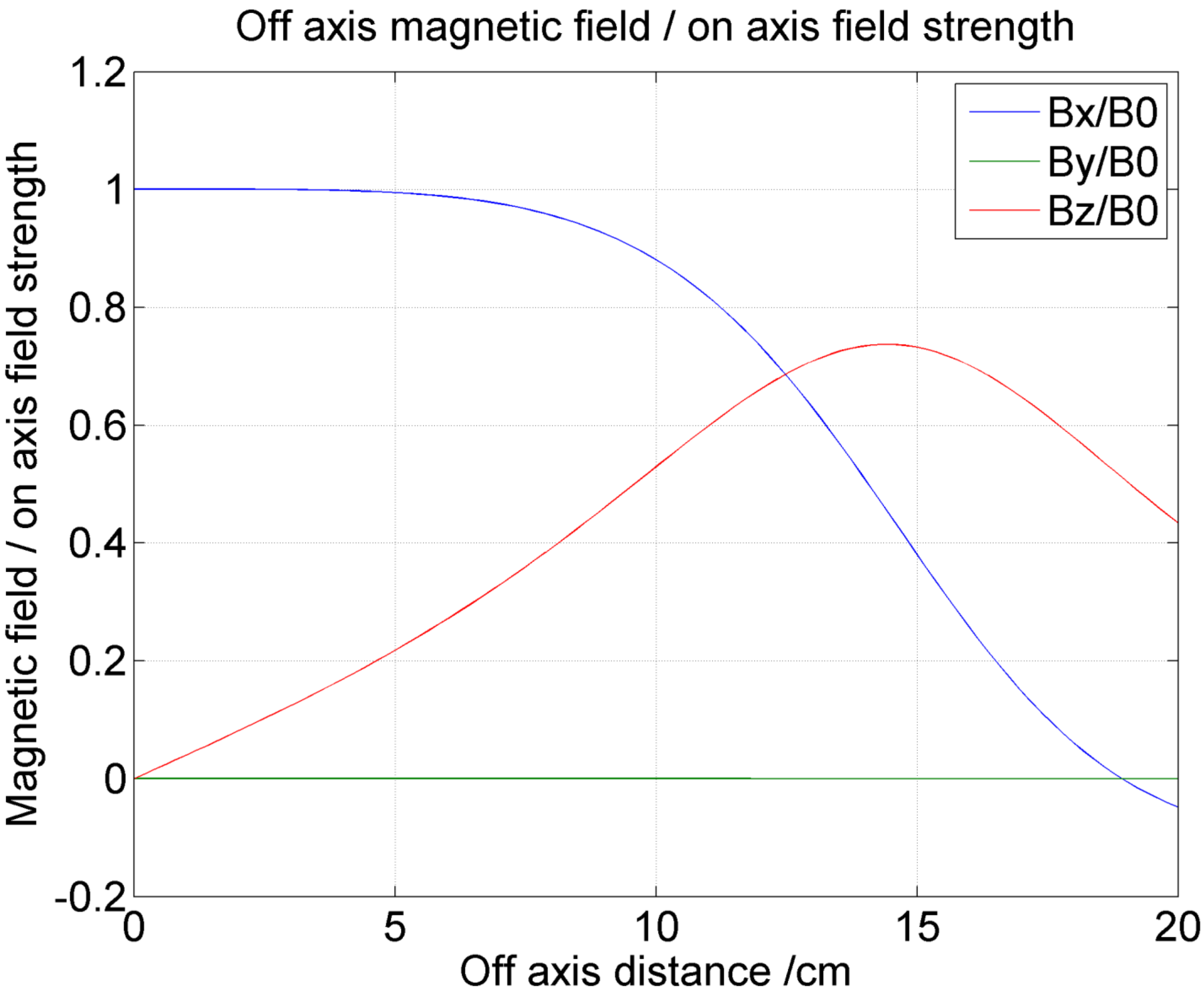
$$\Delta\theta = \frac{2\pi}{P-1} \qquad \text{e.g. } P = 1000$$

$$\theta_i = (i-1)\Delta\theta$$

$$\mathbf{B} = 2\times\frac{\mu_0 INR\Delta\theta}{4\pi}\sum_{i=1}^{P}\frac{(-\sin\theta_i\hat{\mathbf{y}} + \cos\theta_i\hat{\mathbf{z}})\times\Delta\mathbf{r}_i}{|\Delta\mathbf{r}_i|^3}$$

$$\Delta\mathbf{r}_i = h\hat{\mathbf{x}} + r\hat{\mathbf{z}} - R\cos\theta_i\hat{\mathbf{y}} + R\sin\theta_i\hat{\mathbf{z}}$$

The off-axis magnetic field strength is evaluated as a function of radius r from the centre of the Fine Beam Tube. For the range of measured r values, the degradation from on-axis field strength is a maximum of about 2%.

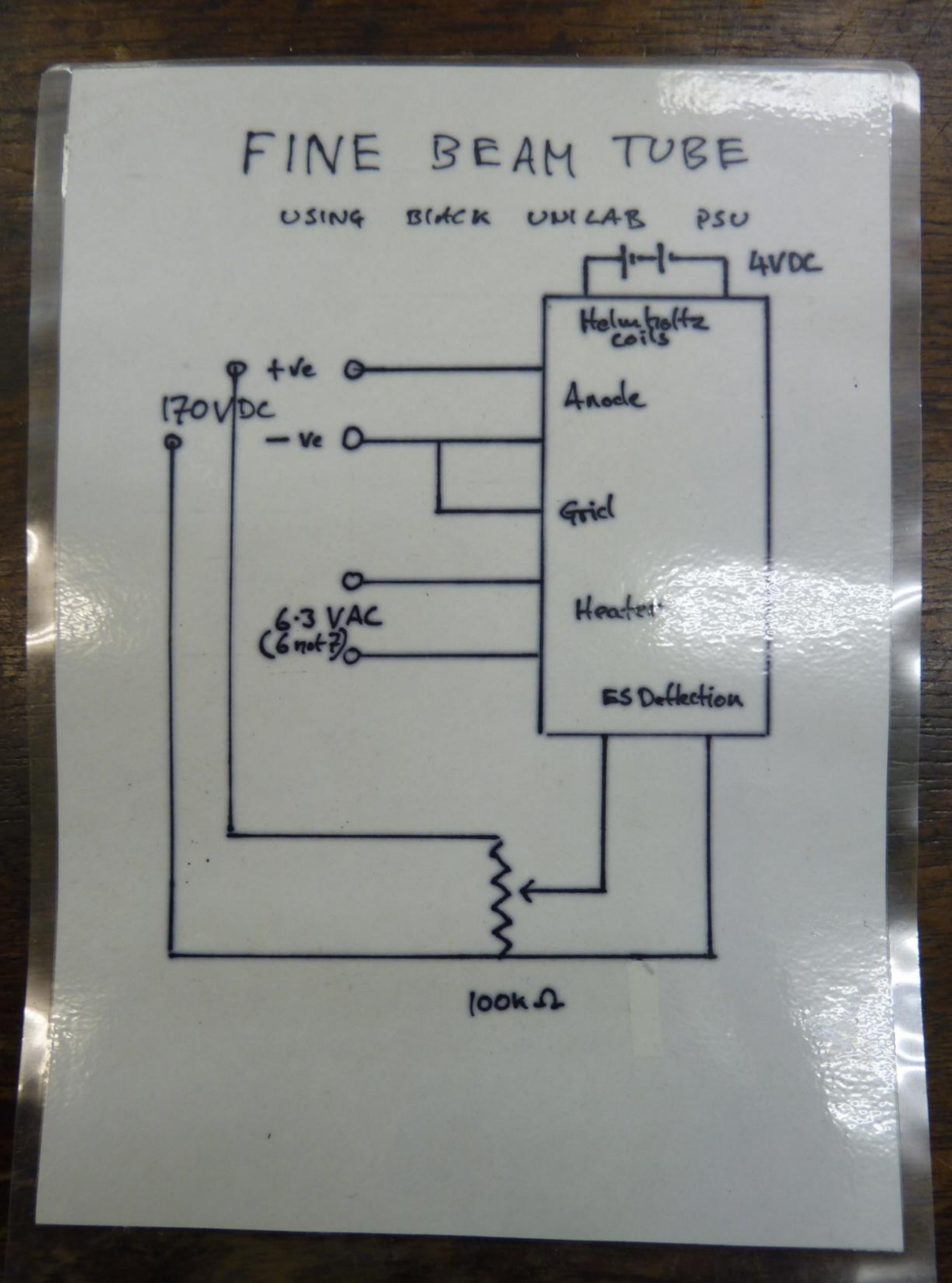


On axis field strength at $r = 0$ is:

$$B = \frac{\mu_0 NI}{R}\left(\frac{4}{5}\right)^{\frac{3}{2}}$$

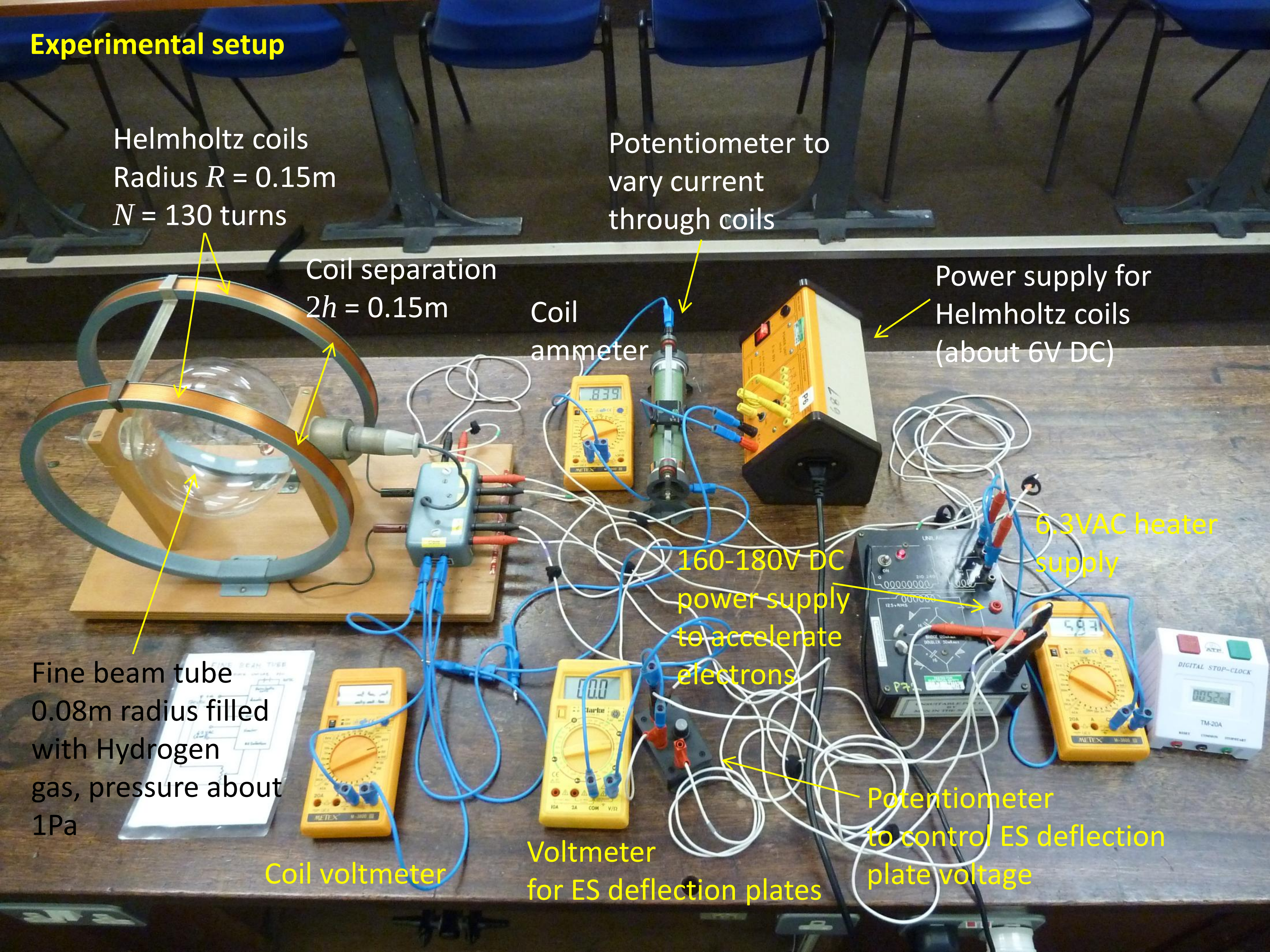
The field strength only significantly drops when the radius is greater than about 8cm.

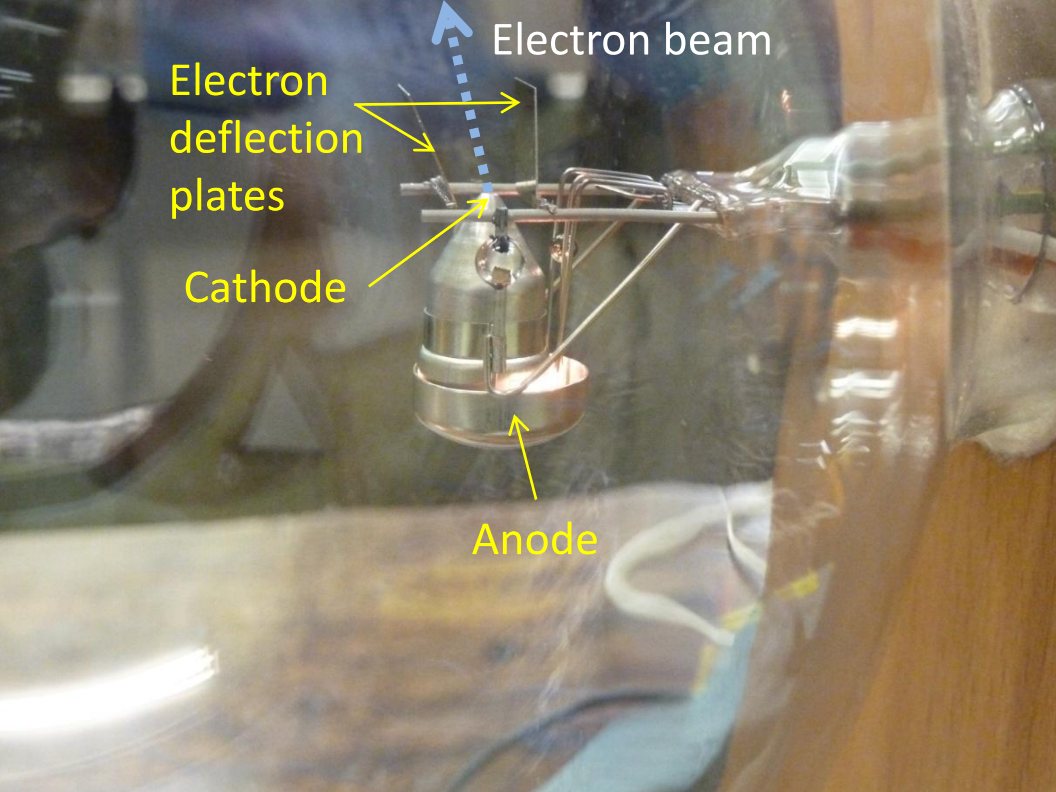
So since the tube has an 8cm radius we expect the magnetic field to be essentially uniform.



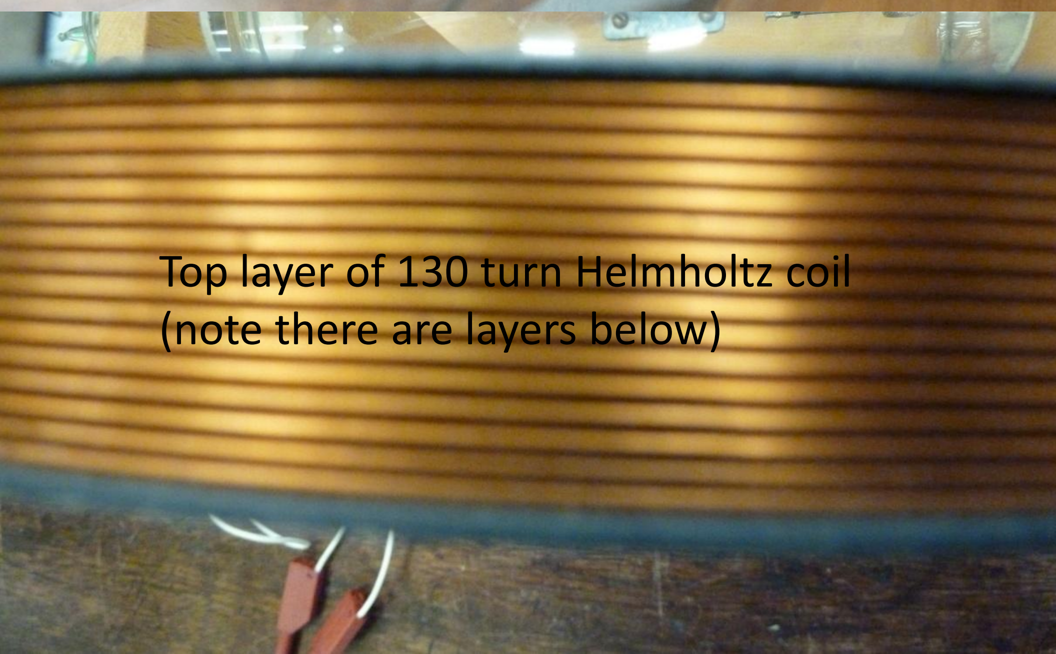
Assembly & storage

Fine beam tube + Helmholtz coils


Experimental setup
Helmholtz coils
Radius R = 0.15m
N = 130 turns
Coil separation
$2h$ = 0.15m
Coil
ammeter
Potentiometer to
vary current
through coils
Power supply for
Helmholtz coils
(about 6V DC)
6.3VAC heater
supply
160-180V DC
power supply
to accelerate
electrons
Fine beam tube
0.08m radius filled
with Hydrogen
gas, pressure about
1Pa
Coil voltmeter
Voltmeter
for ES deflection plates
Potentiometer
to control ES deflection
plate voltage

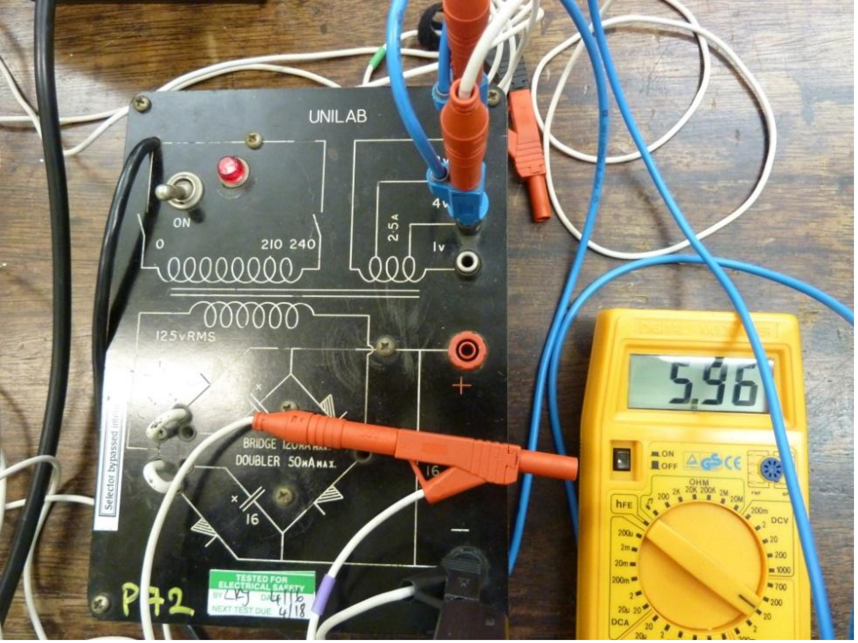


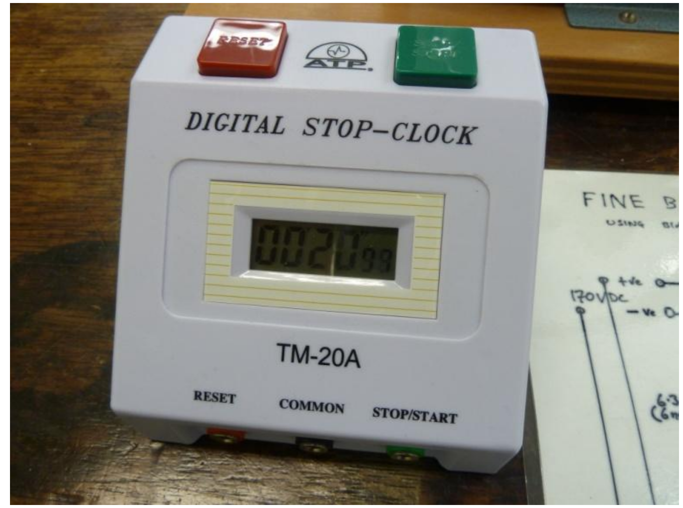
Helmholtz coil power supply & potentiometer. Current should be between 0.6 and 1.5 amps



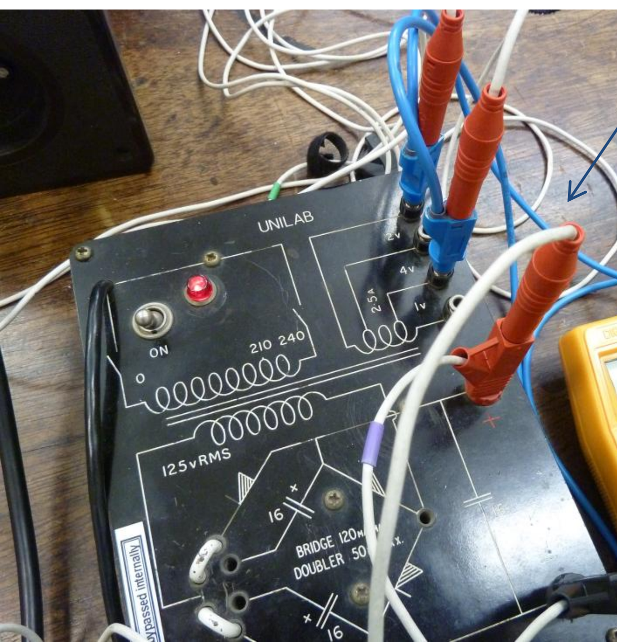
Top layer of 130 turn Helmholtz coil (note there are layers below)

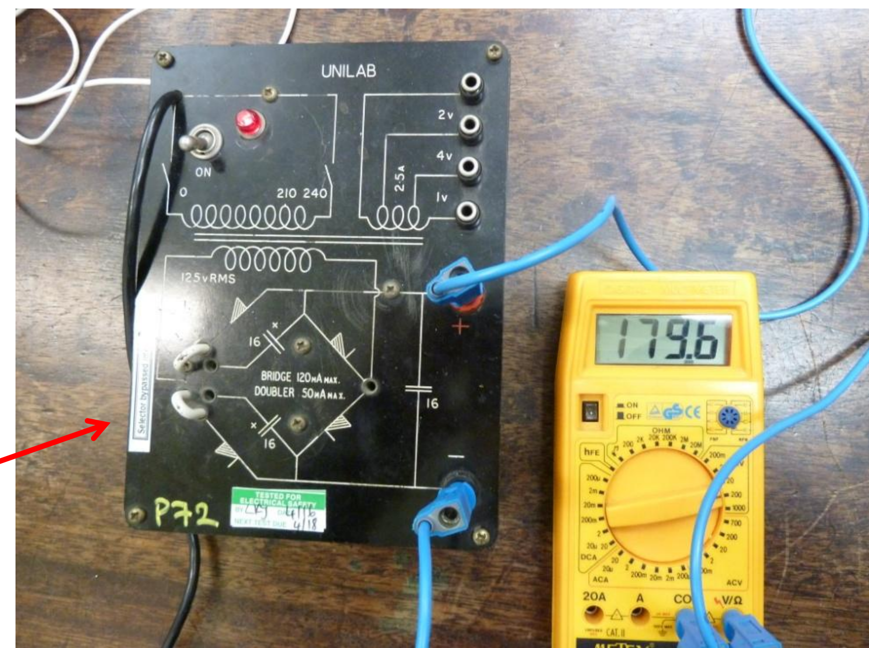
Fine beam tube power connectors





Switch on the heater circuit (max 6.3VAC) for about a minute *before* connecting up the 180V DC accelerating voltage

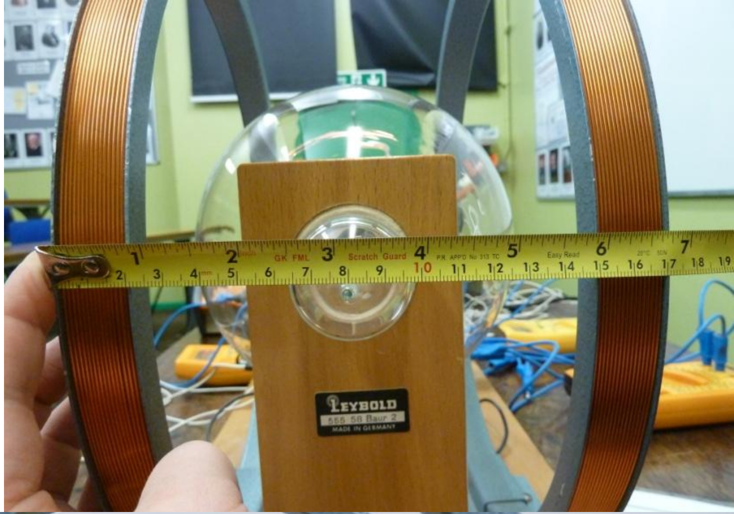


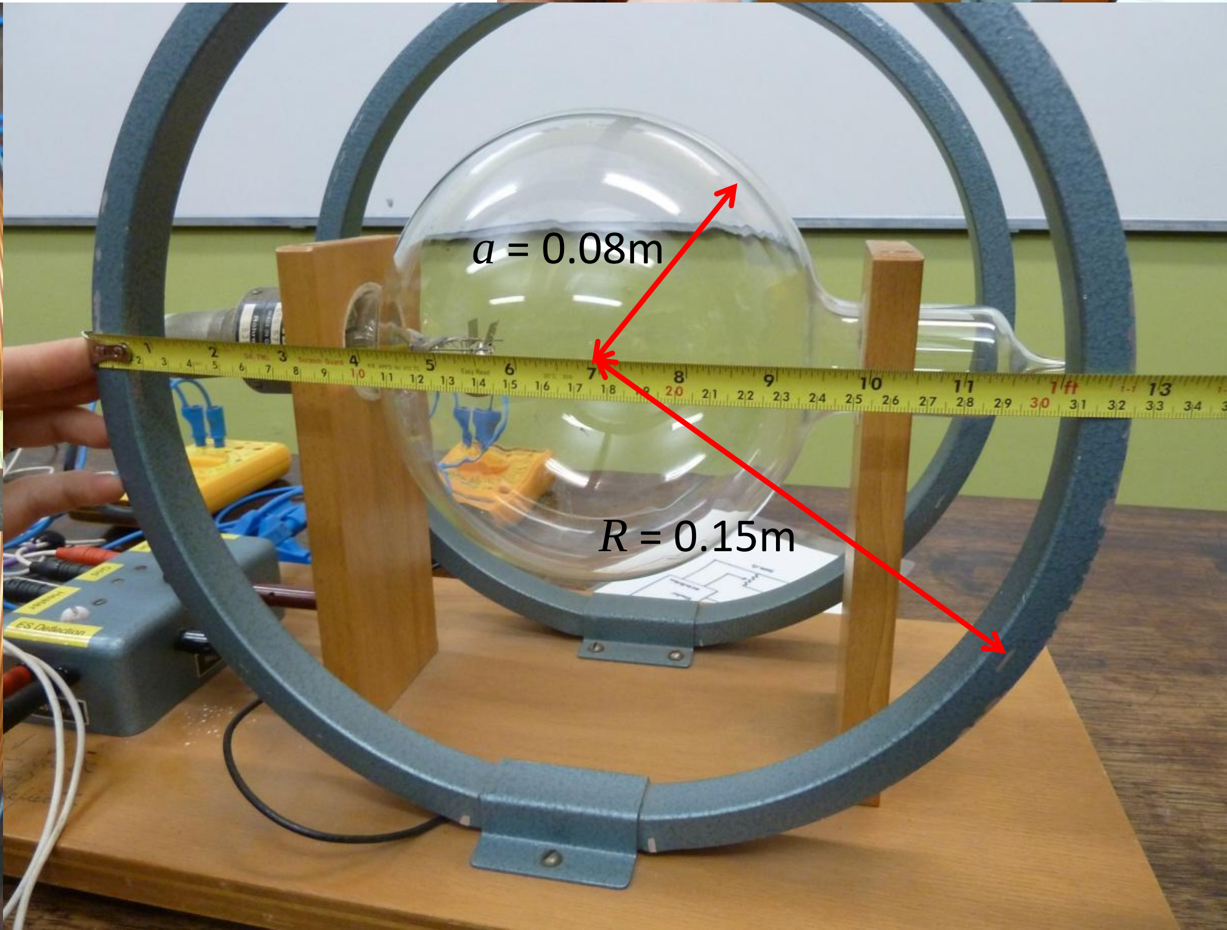


Checking the supply voltage before wiring up the Fine Beam Tube

Fine beam tube dimensions

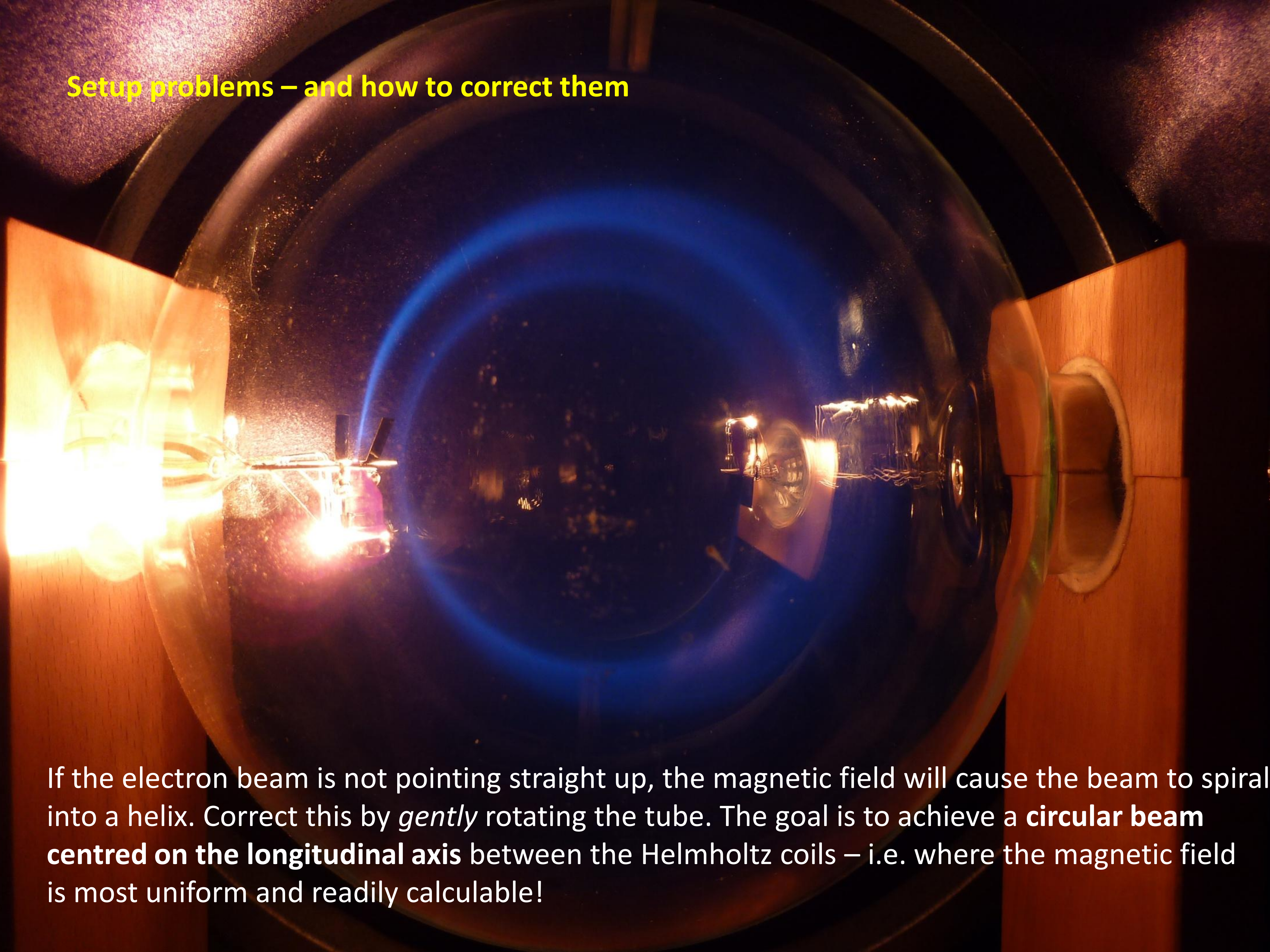
(Spherical) tube radius a = 0.08m
Helmholtz coil radius R = 0.15m
Coil separation $2h$= 0.15m
Number of turns per coil N = 130





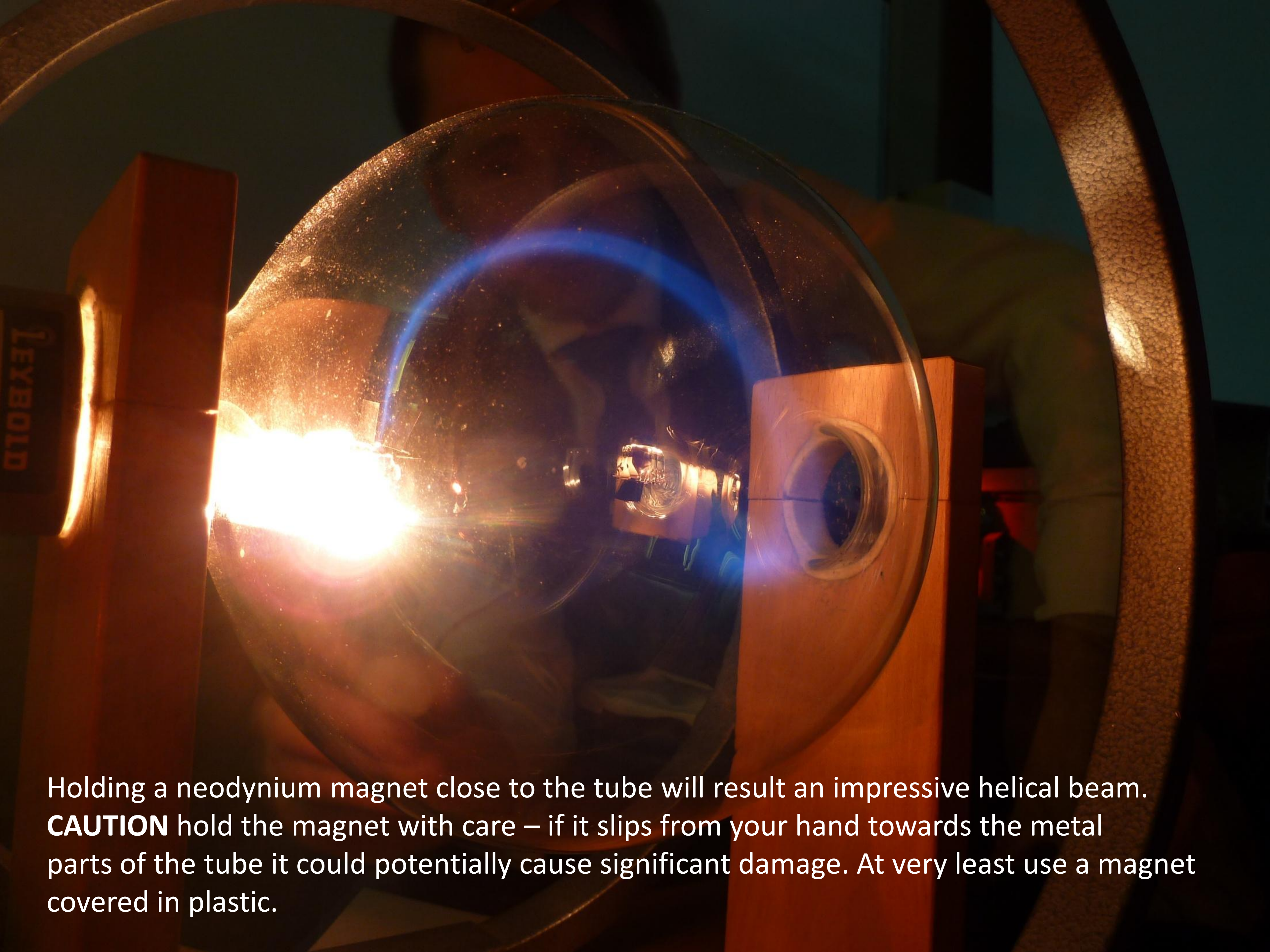
Setup problems – and how to correct them

Too much ES deflection will divert the electron beam off centre



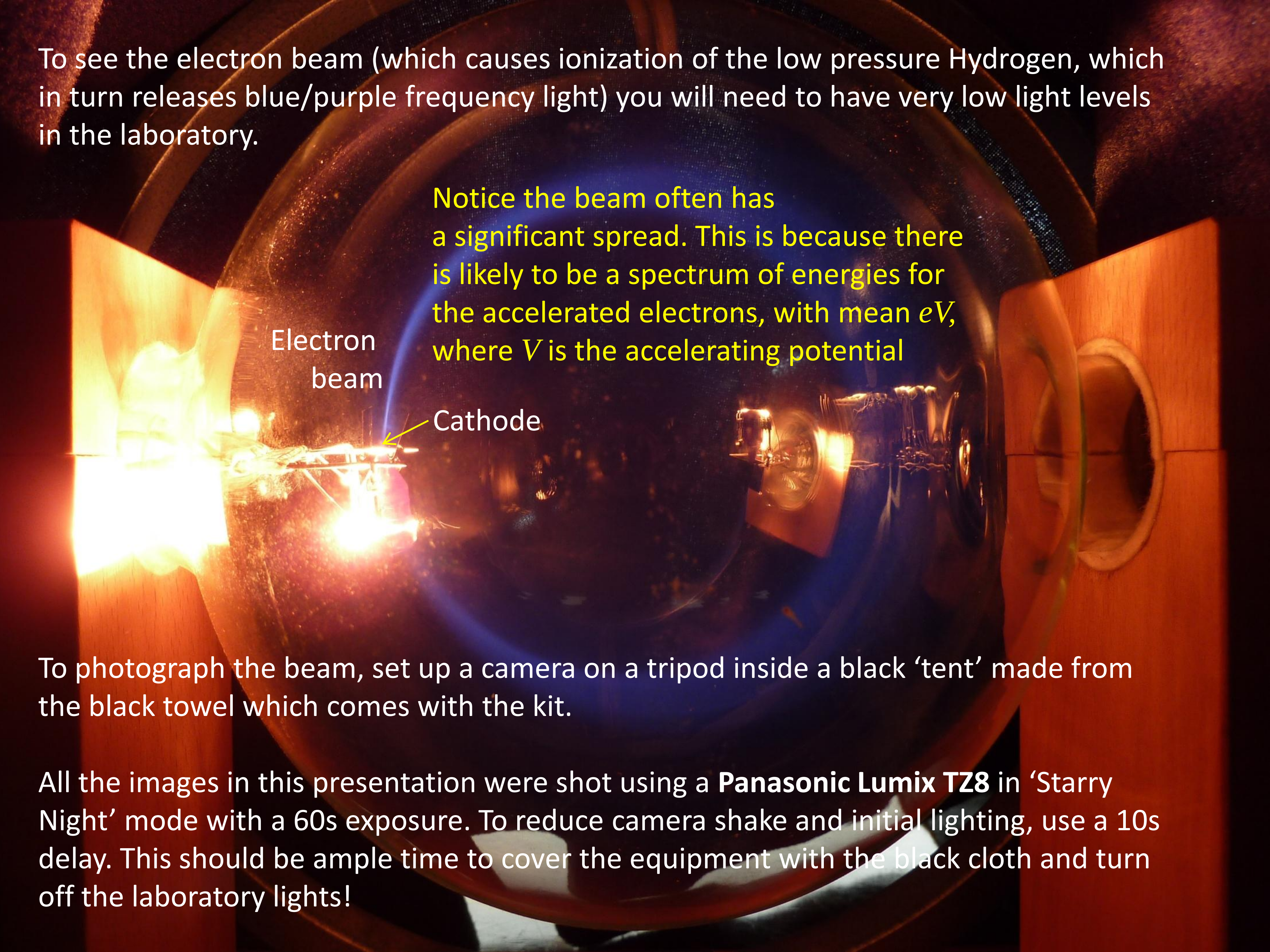
Setup problems – and how to correct them

If the electron beam is not pointing straight up, the magnetic field will cause the beam to spiral into a helix. Correct this by *gently* rotating the tube. The goal is to achieve a **circular beam centred on the longitudinal axis** between the Helmholtz coils – i.e. where the magnetic field is most uniform and readily calculable!



Holding a neodymium magnet close to the tube will result an impressive helical beam.
CAUTION hold the magnet with care – if it slips from your hand towards the metal parts of the tube it could potentially cause significant damage. At very least use a magnet covered in plastic.

To see the electron beam (which causes ionization of the low pressure Hydrogen, which in turn releases blue/purple frequency light) you will need to have very low light levels in the laboratory.



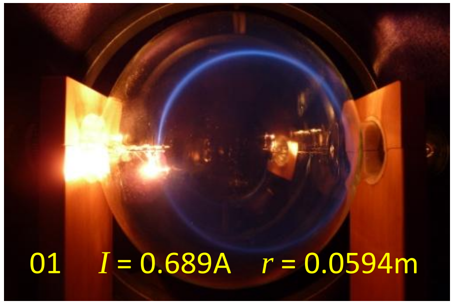
Notice the beam often has a significant spread. This is because there is likely to be a spectrum of energies for the accelerated electrons, with mean eV, where V is the accelerating potential

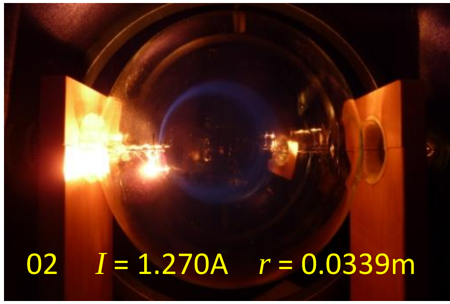
To photograph the beam, set up a camera on a tripod inside a black ‘tent’ made from the black towel which comes with the kit.

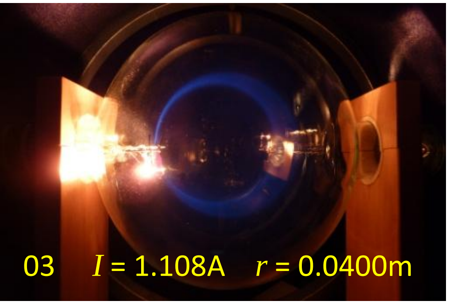
All the images in this presentation were shot using a **Panasonic Lumix TZ8** in ‘Starry Night’ mode with a 60s exposure. To reduce camera shake and initial lighting, use a 10s delay. This should be ample time to cover the equipment with the black cloth and turn off the laboratory lights!

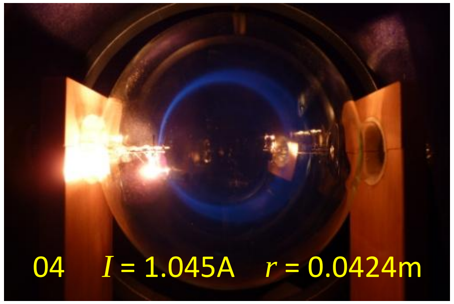
Using a potentiometer to vary the Helmholtz coil current, and hence magnetic field strength, and therefore the beam radius

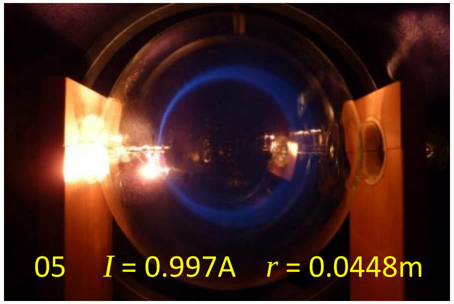
NOTE: Power supply and variable resistor settings for ES deflection were unmodified. However, the voltage across the power supply was observed to change slightly (between 169 and 176V DC). The power supply voltage was recorded for each of the fourteen experiments, and this variable result was used in the calculations.

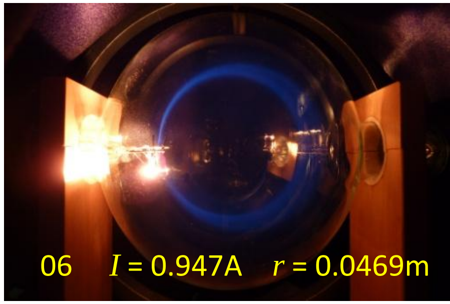


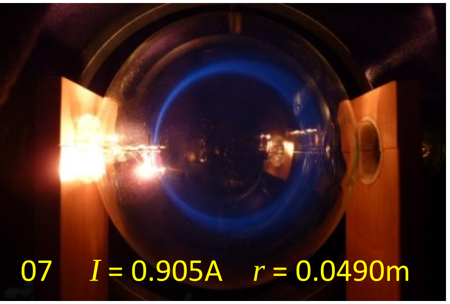


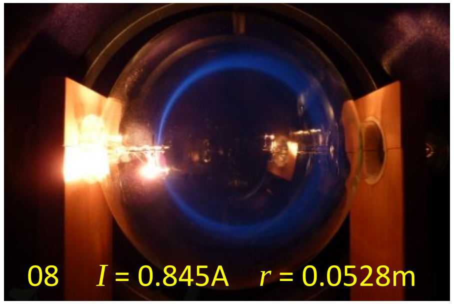


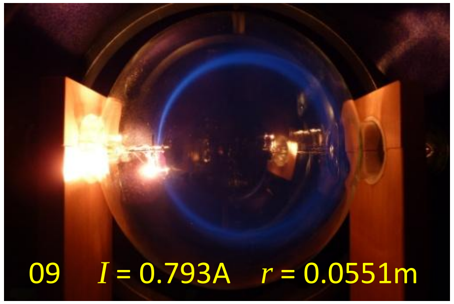


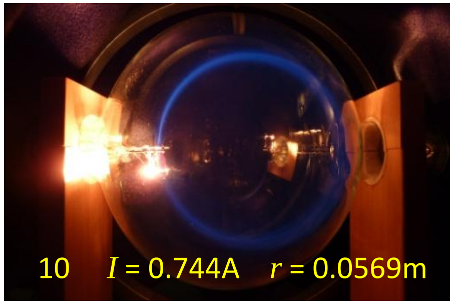


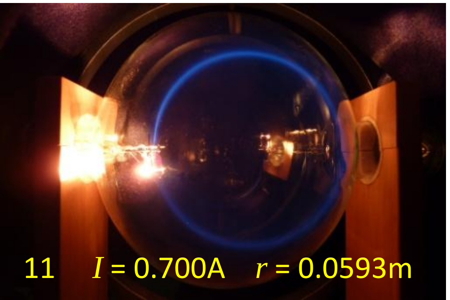


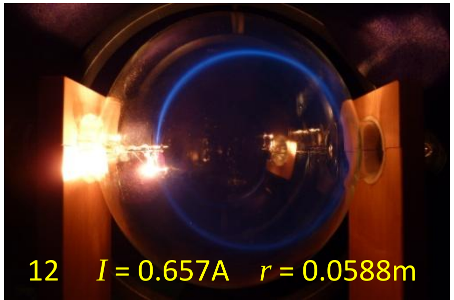


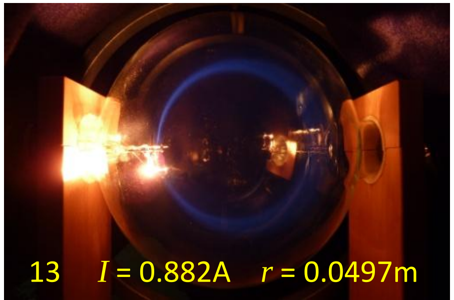


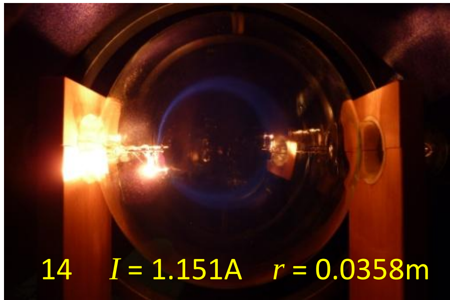


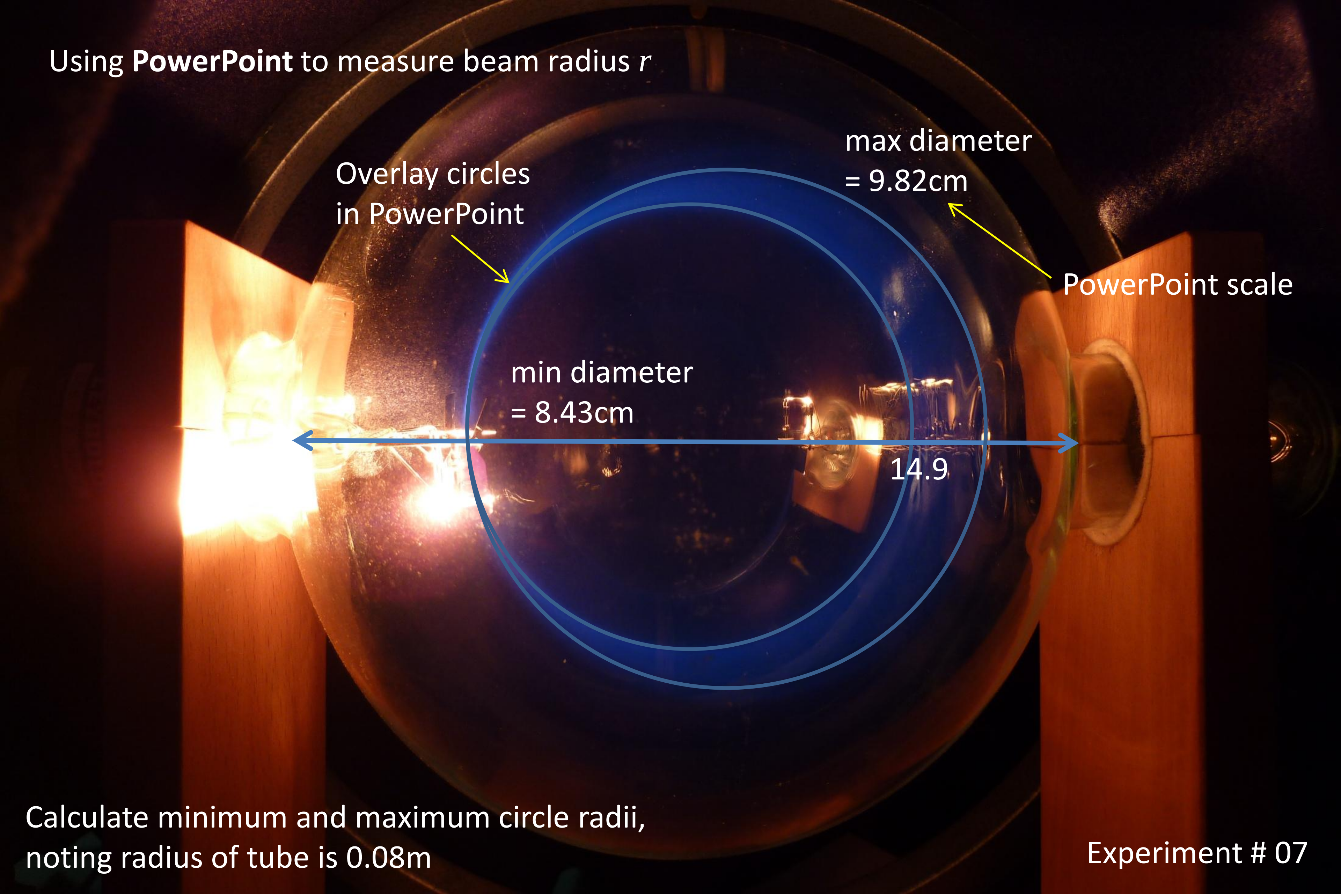












$$r_{\text{min}} = \frac{8.43}{14.9} \times 0.08\text{m} = 0.0453\text{m} \qquad r_{\text{max}} = \frac{9.82}{14.9} \times 0.08\text{m} = 0.0527\text{m} \qquad I = 0.905\text{A}$$

Helmholtz coils

#	Current I /Amps	error /amps
1	0.689	0.005
2	1.270	
3	1.108	
4	1.045	
5	0.997	
6	0.947	
7	0.905	
8	0.845	
9	0.793	
10	0.744	
11	0.700	
12	0.657	
13	0.882	
14	1.151	

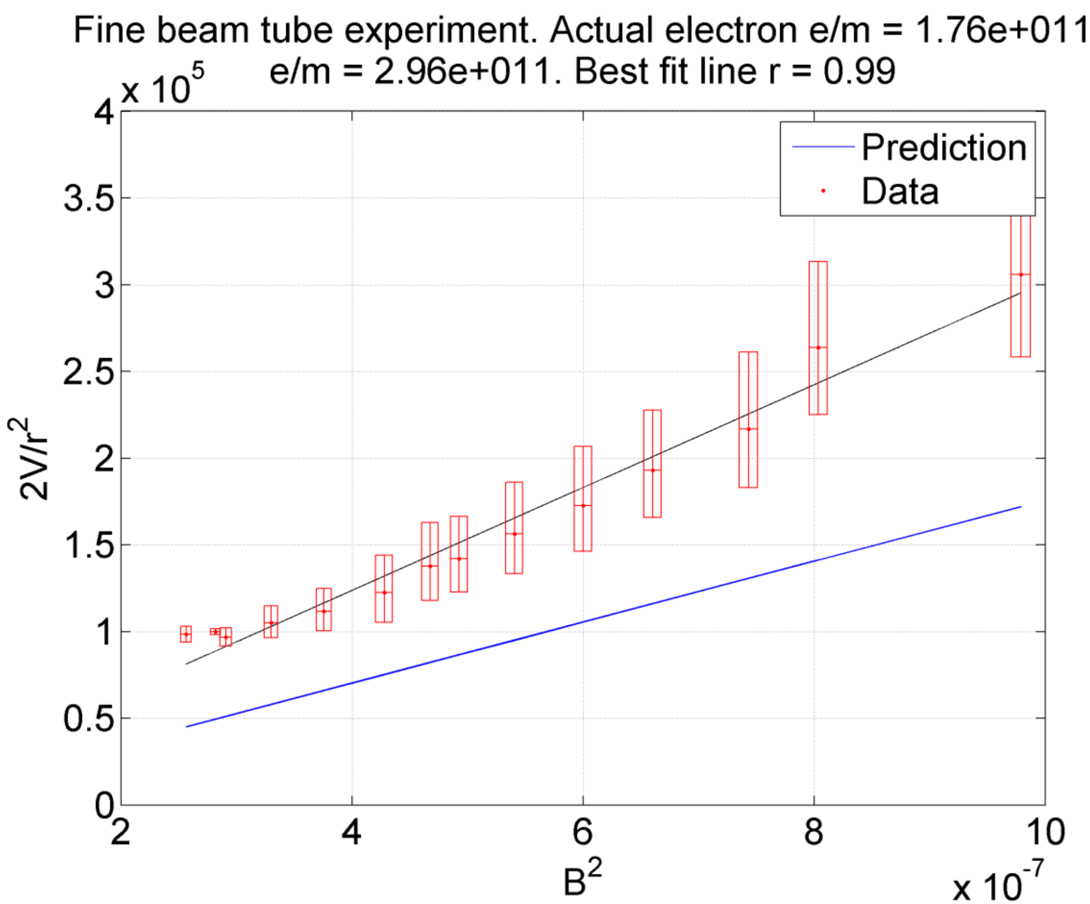
Voltage /Volts	error /volts
2.96	0.02
5.59	
4.81	
4.55	
4.36	
4.16	
4	
3.75	
3.49	
3.3	
3.06	
2.88	
3.91	
5.13	

Accelerating PD /volts	error /volts
176	0.5
175.8	
173.5	
173.5	
173.2	
172	
170.6	
170.7	
169.3	
170	
169.9	
170	
170.2	
169	

From PowerPoint slides

Beam ring radius /m	error /m	%error
0.0594	0.0005	0.8
0.0339	0.0030	8.8
0.0400	0.0036	8.9
0.0424	0.0034	7.9
0.0448	0.0039	8.6
0.0469	0.0039	8.3
0.0490	0.0037	7.6
0.0528	0.0041	7.8
0.0551	0.0030	5.4
0.0569	0.0025	4.3
0.0593	0.0017	2.8
0.0588	0.0014	2.3
0.0497	0.0040	8.0
0.0358	0.0030	8.2

Off axis factor	B /Tesla	2*V/r^2	B^2	(e/m)*B^2
0.9802	5.263E-04	9.976E+04	2.770E-07	4.930E+04
0.9887	9.785E-04	3.059E+05	9.575E-07	1.704E+05
0.98667	8.519E-04	2.169E+05	7.258E-07	1.292E+05
0.98587	8.028E-04	1.930E+05	6.446E-07	1.147E+05
0.98507	7.653E-04	1.726E+05	5.858E-07	1.043E+05
0.98437	7.264E-04	1.564E+05	5.277E-07	9.394E+04
0.98367	6.937E-04	1.421E+05	4.813E-07	8.567E+04
0.9824	6.469E-04	1.225E+05	4.185E-07	7.449E+04
0.98163	6.066E-04	1.115E+05	3.680E-07	6.550E+04
0.98103	5.688E-04	1.050E+05	3.235E-07	5.759E+04
0.98023	5.347E-04	9.663E+04	2.859E-07	5.089E+04
0.9804	5.020E-04	9.834E+04	2.520E-07	4.485E+04
0.98343	6.759E-04	1.378E+05	4.569E-07	8.133E+04
0.98807	8.863E-04	2.637E+05	7.854E-07	1.398E+05



Beam radius is taken as the mean average of the max and min measured radii. The error is defined as half the difference between these radii.

Errors in voltage and current measurements were inferred by eye from the fluctuations in the multimeter readings.

Error boxes correspond to the full range of possible values (i.e. upper and lower bounds) given the stated errors.

Helmholtz coils

#	Current I /Amps	error /amps
1	0.689	0.005
2	1.270	
3	1.108	
4	1.045	
5	0.997	
6	0.947	
7	0.905	
8	0.845	
9	0.793	
10	0.744	
11	0.700	
12	0.657	
13	0.882	
14	1.151	

Voltage /Volts	error /volts
2.96	0.02
5.59	
4.81	
4.55	
4.36	
4.16	
4	
3.75	
3.49	
3.3	
3.06	
2.88	
3.91	
5.13	

Accelerating PD /volts	error /volts
176	0.5
175.8	
173.5	
173.5	
173.2	
172	
170.6	
170.7	
169.3	
170	
169.9	
170	
170.2	
169	

From PowerPoint slides

Beam ring radius /m	error /m	%error
0.0594	0.0005	0.8
0.0339	0.0030	8.8
0.0400	0.0036	8.9
0.0424	0.0034	7.9
0.0448	0.0039	8.6
0.0469	0.0039	8.3
0.0490	0.0037	7.6
0.0528	0.0041	7.8
0.0551	0.0030	5.4
0.0569	0.0025	4.3
0.0593	0.0017	2.8
0.0588	0.0014	2.3
0.0497	0.0040	8.0
0.0358	0.0030	8.2

Off axis factor	B /Tesla	2*V/r^2	B^2	(e/m)*B^2
0.9802	5.263E-04	5.686E+04	2.770E-07	4.930E+04
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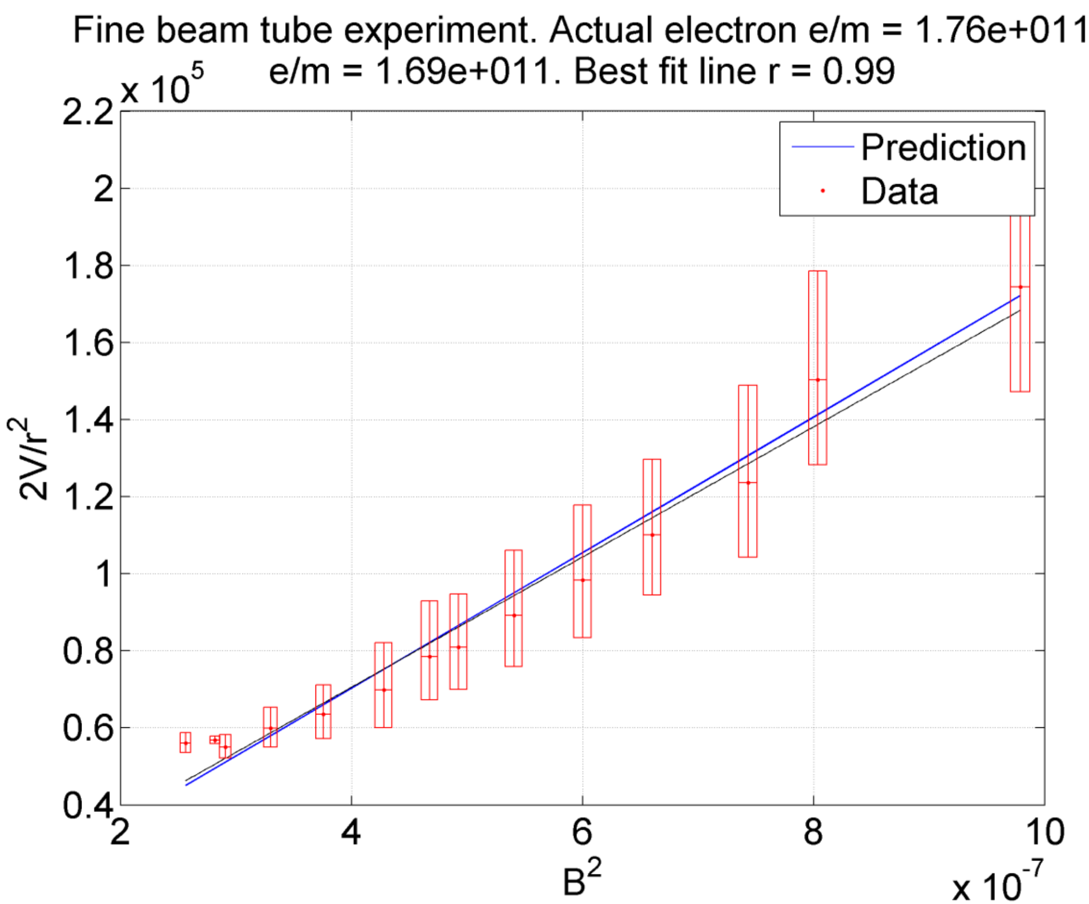
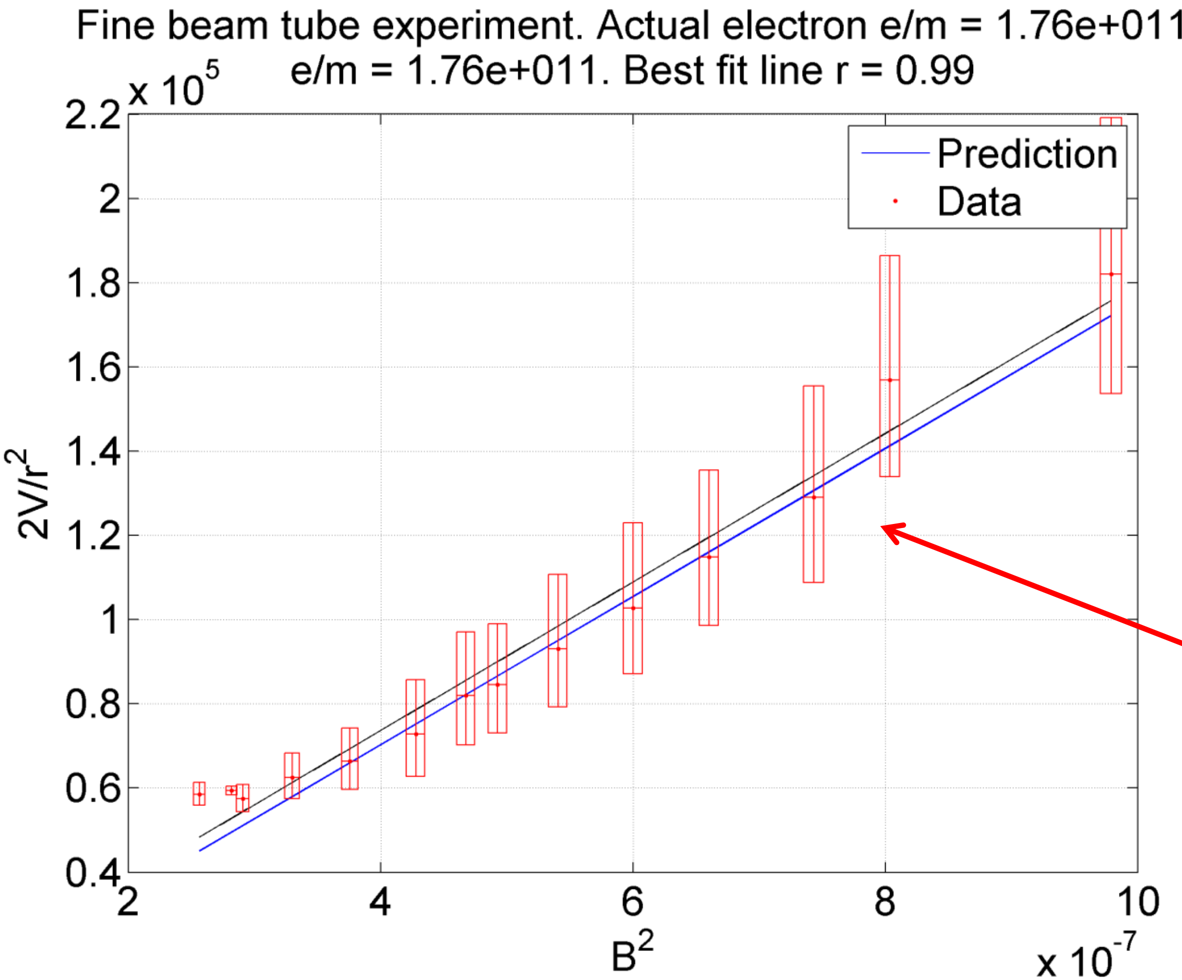
0.98667	8.519E-04	1.236E+05	7.258E-07	1.292E+05
0.98587	8.028E-04	1.100E+05	6.446E-07	1.147E+05
0.98507	7.653E-04	9.838E+04	5.858E-07	1.043E+05
0.98437	7.264E-04	8.914E+04	5.277E-07	9.394E+04
0.98367	6.937E-04	8.100E+04	4.813E-07	8.567E+04
0.9824	6.469E-04	6.980E+04	4.185E-07	7.449E+04
0.98163	6.066E-04	6.357E+04	3.680E-07	6.550E+04
0.98103	5.688E-04	5.986E+04	3.235E-07	5.759E+04
0.98023	5.347E-04	5.508E+04	2.859E-07	5.089E+04
0.9804	5.020E-04	5.605E+04	2.520E-07	4.485E+04
0.98343	6.759E-04	7.855E+04	4.569E-07	8.133E+04
0.98807	8.863E-04	1.503E+05	7.854E-07	1.398E+05

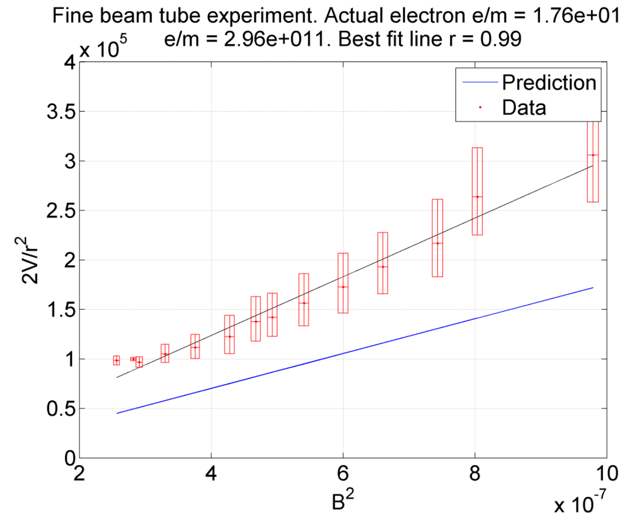


With '0.57 fudge factor'

i.e. assume the accelerating potential V is actually 57% of the voltage across the power supply.

This is proposed as the most obvious source of discrepancy between the measured and predicted lines.





With '0.595 fudge factor'

i.e. assume the accelerating potential V is actually 59.5% of the voltage across the power supply.

This is proposed as the most obvious source of discrepancy between the measured and predicted lines.

The ratio of the measured to actual *e/m* value is:
1.76/2.96 = 0.595

Summary

$x = B^2, \quad y = \dfrac{2V}{r^2}$ indeed yields a fairly straight line with significant correlation, although the measured gradient of 2.96 x 10^{11} C/kg differs quite significantly from the predicted value of 1.76 x 10^{11} C/kg.

The most likely source of error is that the actual accelerating voltage is *much less* than the measured voltage across the terminals. i.e. there is a significant internal resistance.

Off-axis magnetic field degradation or Relativistic effects are not thought to be as important.

Multiplying the measured e/m by 0.595 results in the actual value, i.e. an expected voltage loss of 40.5%. Note a loss factor ('fudge factor'!) of 0.57 actually gives a slightly better agreement of the measured data with the predicted line, although the gradient differs slightly from the predicted e/m value.

The beam was rarely thin, indicating a range of electron energies at the point of emission. This might augment the 'internal resistance' hypothesis for a modification of V, although this effects is reflected in the error bars.

Further investigation:

- Change the power supply, perhaps using a higher voltage (up to perhaps 250V). Do x,y values lie on the same line as in this experiment? Is the voltage loss systematic?
- Find some mechanism for directly measuring the accelerating potential. Can the anticipated 40.5% loss be calculated independently?
- Note Leybold have upgraded the equipment over the years. Perhaps the internal resistance / spectrum of electron energies issue has been fixed? It might be worth contacting Leybold directly.

To be continued!