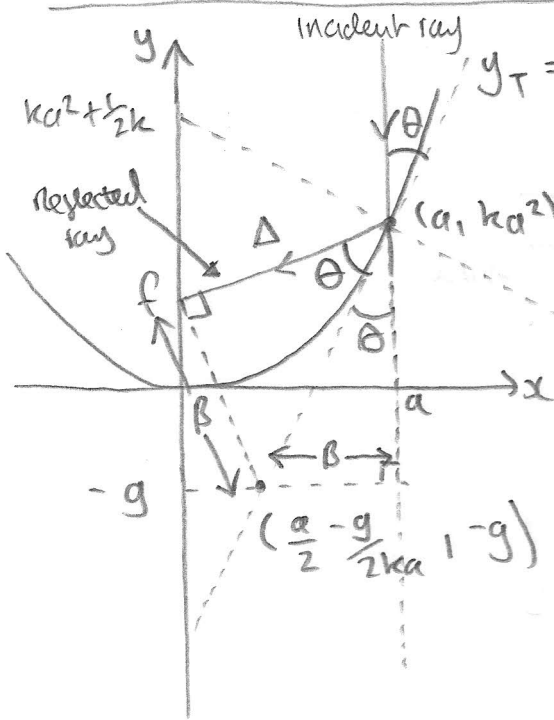


Proof that a parabolic reflector focuses vertical rays to the point $(0, \frac{1}{4k})$, if the equation of the parabola is $y = kx^2$, $k > 0$



$$\left[\begin{aligned} \text{if } y_T &= 2kax - ka^2 \\ y_N &= -\frac{1}{2ka}x + d \\ \text{using } (a, ka^2) \\ ka^2 &= -\frac{1}{2ka}a + d \\ \therefore y_N &= -\frac{1}{2ka}x + ka^2 + \frac{1}{2k} \end{aligned} \right]$$

$$y_N = -\frac{1}{2ka}x + ka^2 + \frac{1}{2k}$$

For a quadratic $y = kx^2$

$$\frac{dy}{dx} = 2kx$$

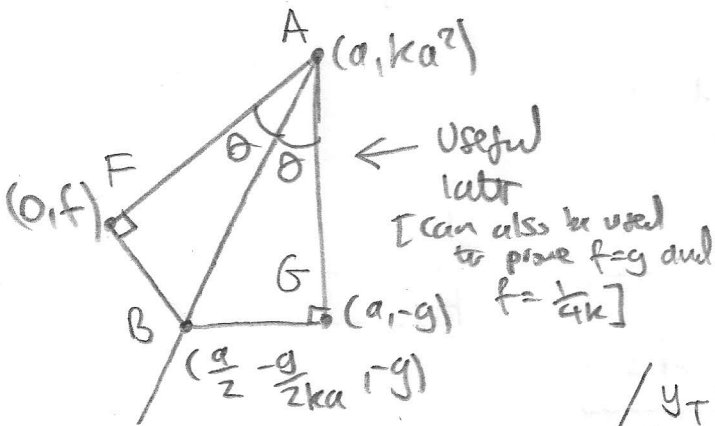
\therefore Gradient at (a, ka^2) is $2ka$

$$y_T = 2kax + c$$

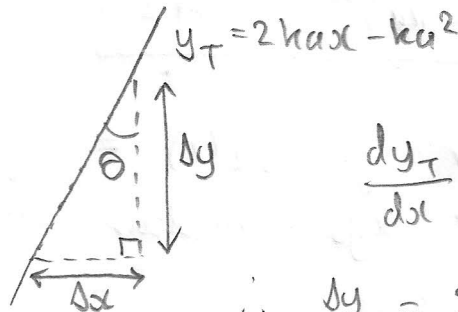
$$\text{using } (a, ka^2): ka^2 = 2ka^2 + c$$

$$\therefore c = -ka^2$$

$$\boxed{y_T = 2kax - ka^2}$$



From the diagram



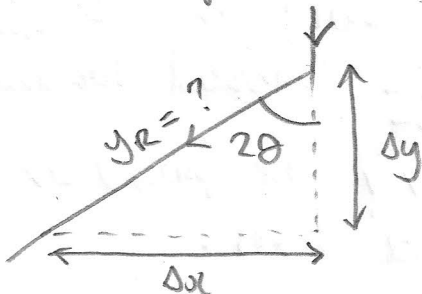
$$\frac{dy_T}{dx} = \frac{dy}{dx}$$

\leftarrow Since y_T is a straight line

$$\frac{dy}{dx} = 2ka$$

$$\text{Now } \tan \theta = \frac{\Delta x}{\Delta y} = \frac{1}{2ka}$$

For the reflected ray:



$$\tan 2\theta = \frac{\Delta x}{\Delta y}$$

so gradient of $y_R(x)$

$$\text{is } \frac{\Delta y}{\Delta x} = \frac{1}{\tan 2\theta} = \frac{dy_R}{dx}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

So using $\tan\theta = \frac{1}{2ka}$

$$\tan 2\theta = \frac{\frac{1}{ka}}{1 - \frac{1}{4k^2a^2}} = \frac{1}{ka - \frac{1}{4ka}}$$

$$\therefore \frac{dy_R}{dx} = \frac{1}{\tan 2\theta} = ka - \frac{1}{4ka}$$

Hence $y_R = \left(ka - \frac{1}{4ka}\right)x + c$

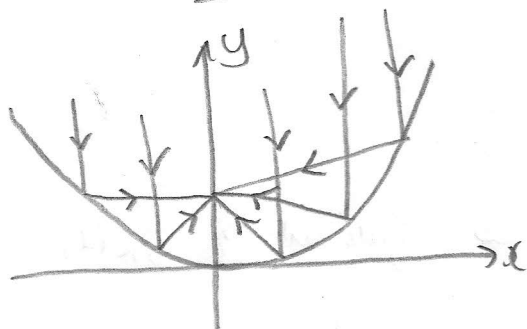
$y_R(x)$ passes through (a, ka^2)

So $ka^2 = \left(ka - \frac{1}{4ka}\right)a + c$

$$\Rightarrow \frac{1}{4k} = c$$

$$\therefore y_R = \left(ka - \frac{1}{4ka}\right)x + \frac{1}{4k}$$

Notice the y intercept of $y_R(x) = \frac{1}{4k}$ is independent of a . i.e. all \downarrow rays incident onto the parabola are focussed at $(0, \frac{1}{4k})$. So $f = \frac{1}{4k}$

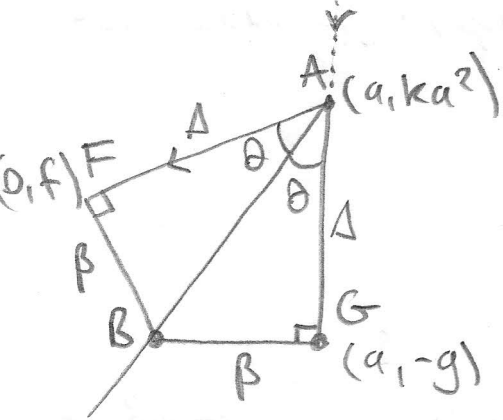


This is why a parabolic reflector is a useful antenna.

* An isotropic source at $(0, \frac{1}{4k})$ creates $\left(\begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}\right)$ collimated radiation

* incoming $\downarrow \downarrow \downarrow$ are focussed at $(0, \frac{1}{4k})$
eg Satellite dishes.

Geometrical remarks



$$y_T = 2ka^2 - ka^2$$

$$\text{Now } \Delta = ka^2 + g$$

$$\text{Also } \Delta = \sqrt{(ka^2 - f)^2 + a^2} \quad (\text{Pythagoras, using } \Delta = |\vec{AF}|)$$

$$\text{Now if } f = \frac{1}{4k}$$

$$\Delta = \sqrt{\left(ka^2 - \frac{1}{4k}\right)^2 + a^2}$$

$$= \sqrt{k^2 a^4 - \frac{1}{2} a^2 + \frac{1}{16k^2} + a^2}$$

$$= \sqrt{k^2 a^4 + \frac{1}{2} a^2 + \frac{1}{16k^2}}$$

$$= \sqrt{\left(ka^2 + \frac{1}{4k}\right)^2}$$

$$\therefore \Delta = ka^2 + \frac{1}{4k}$$

$$\therefore ka^2 + \frac{1}{4k} = ka^2 + g$$

$$\Rightarrow \boxed{\frac{1}{4k} = g} \quad \text{So } \boxed{g = f}$$

We can form a kite from the reflected ray vector \vec{AF} and its extension to the incident ray \vec{AG}

$$|\vec{AG}| \text{ is defined to be } = |\vec{AF}| = \Delta$$

$$\text{By symmetry } |\vec{FB}| = |\vec{BG}|$$

