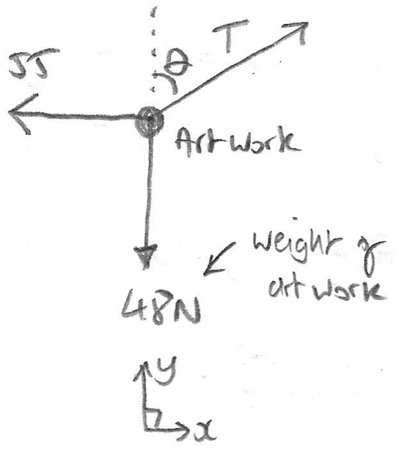
FORCES & MOMENTS

1/ (i)



System in equilibrium.

Newton II:

$//x: \quad 0 = T\sin\theta - 55$

$//y: \quad 0 = T\cos\theta - 48$

$\therefore \quad T\sin\theta = 55$

$T\cos\theta = 48$

$\therefore \quad \tan\theta = \frac{55}{48}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{55}{48}\right)$

$= \boxed{48.9^\circ}$

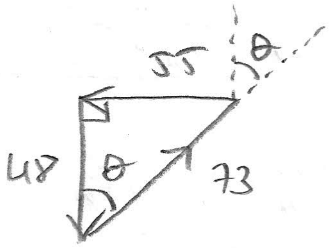
$T^2\sin^2\theta + T^2\cos^2\theta = 55^2 + 48^2$

$\therefore \quad T^2(\underset{\overset{\uparrow}{=1}}{\sin^2\theta + \cos^2\theta}) = 5329$

$\therefore \quad T = \sqrt{5329}$

$\therefore \quad \boxed{T = 73\text{ N}}$

Note:

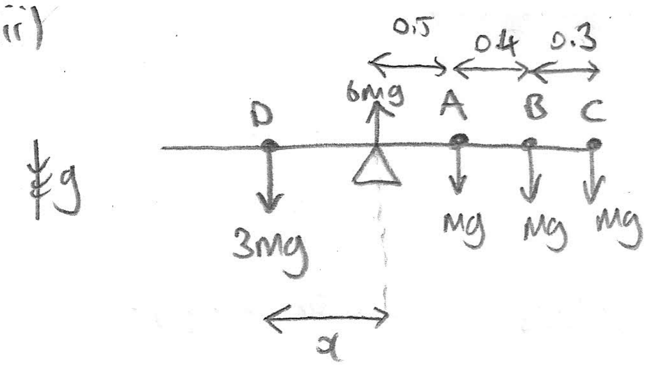


ie vector sum of forces is a closed triangle since system is in equilibrium

Also $73^2 = 55^2 + 48^2 \Rightarrow 73, 55, 48$ is a Pythagorean Triple.

And since the artwork is in Samos, so is this question!

(ii)



Net turning moment about pivot = 0

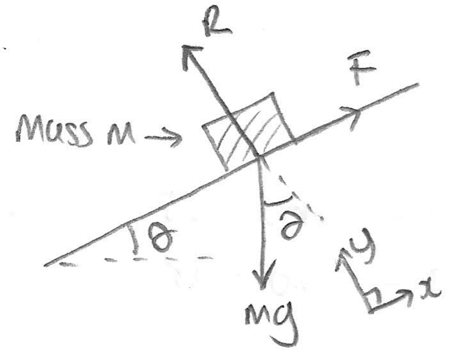
$\therefore \quad 3mgx = 0.5mg + 0.9mg + 1.2mg$

$3x = 2.6$

Hence:

$\boxed{x = 0.87\text{m}}$

(iii)



$\downarrow g$

Newton II, assuming equilibrium

$//x: \quad 0 = F - Mg\sin\theta$

$//y: \quad 0 = R - Mg\cos\theta$

so $F = Mg\sin\theta \qquad R = Mg\cos\theta$

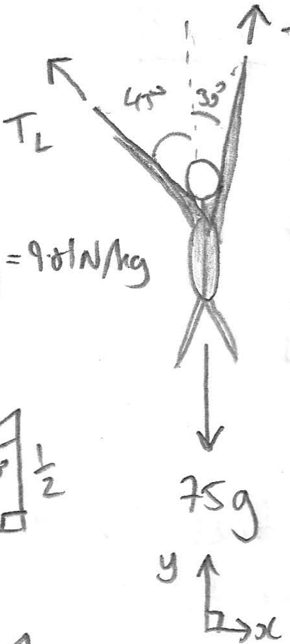
Now mass won't slide if $\boxed{F < \mu R}$.

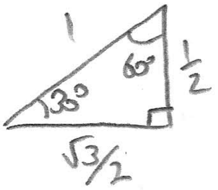
$\therefore \quad Mg\sin\theta < \mu Mg\cos\theta$

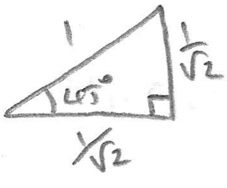
$$\boxed{\tan\theta < \mu}$$

(iv)

$\downarrow g = 9.81 N/kg$







Climber is in equilibrium

∴ Newton II

$//x: \quad 0 = T_R \sin 30^\circ - T_L \sin 45^\circ$

$//y: \quad 0 = T_R \cos 30^\circ + T_L \cos 45^\circ - 75g$

$\therefore \quad T_R/2 = T_L/\sqrt{2} \Rightarrow \boxed{T_R = T_L\sqrt{2}}$

$\therefore \quad 75g = T_L\sqrt{2}\,\frac{\sqrt{3}}{2} + T_L/\sqrt{2}$

$75g = \frac{T_L\sqrt{3}}{\sqrt{2}} + \frac{T_L}{\sqrt{2}}$

$75\sqrt{2}g = T_L(\sqrt{3}+1)$

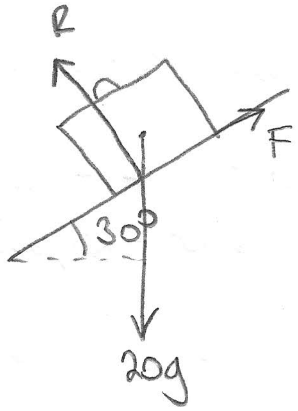
$$\boxed{\frac{75\sqrt{2}g}{\sqrt{3}+1} = T_L}$$

$\therefore \quad T_R = \sqrt{2}T_R = \boxed{\frac{150g}{\sqrt{3}+1}}$

$\therefore \quad T_R = \boxed{381N} \qquad T_L = \boxed{537N}$ (to 3s.f.).

(v)

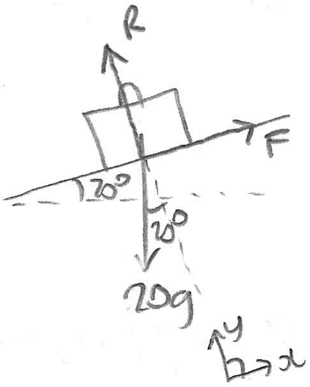
↓g



on the point of sliding so $F = \mu R$

and $\mu = \tan 30°$ $= \frac{1}{\sqrt{3}} \approx \boxed{0.577}$

↓g



Since elevation of ramp is < 30°, assume static equilibrium.

∴ by Newton II

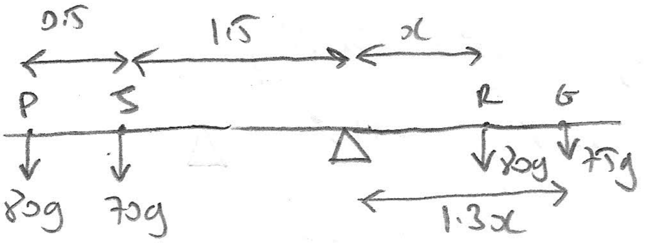
$/\!/x: \quad 0 = F - 20g\sin 20°$

$/\!/y: \quad 0 = R - 20g\cos 20°$

$\Rightarrow \quad F = 20g\sin 20° = \boxed{67.1\,N}$

$R = 20g\cos 20° = \boxed{184.4\,N}$

(vi)



↓g

Balancing turning moments about the see-saw pivot

$2 \times 80 + 1.5 \times 70 = 80x + 75 \times 1.3x$

$265 = 177.5x$

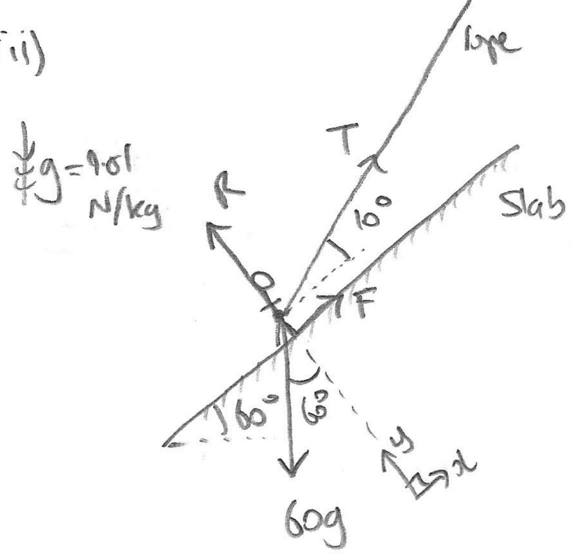
$\boxed{1.49 = x}$

∴ separation between Paul and George is :

$0.5 + 1.5 + 1.3 \times x = \boxed{3.94\,M}$

③

(vii)



Newton II (assume equilibrium)

$/\!/x: \; 0 = T\cos 10^\circ + F - 60g\sin 60^\circ$

$/\!/y: \; 0 = R + T\sin 10^\circ - 60g\cos 60^\circ$

So $F = 60g\sin 60^\circ - T\cos 10^\circ$

$R = 60g\cos 60^\circ - T\sin 6^\circ$

(The rope is not to drag the climber up the slab, so use the situation of 'fall protection' ie where friction acts <u>up</u> the slope)

For no slip $F < \mu R$

and $\mu = 1.23$.

$\therefore \; 60g\sin 60^\circ - T\cos 10^\circ < \mu\left(60g\cos 60^\circ - T\sin 10^\circ\right)$

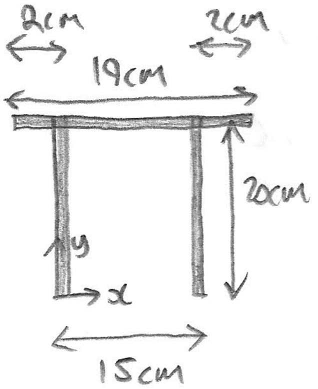
$60g\sin 60^\circ - \mu \times 60g\cos 60^\circ < T\left(\cos 10^\circ - \mu\sin 10^\circ\right)$

$$\therefore T > \frac{60g\left(\sin 60^\circ - \mu\cos 60^\circ\right)}{\cos 10^\circ - \mu\sin 10^\circ}$$

$\therefore \boxed{T > 192\text{ N}}$ to 3.s.f.

(viii)

a)



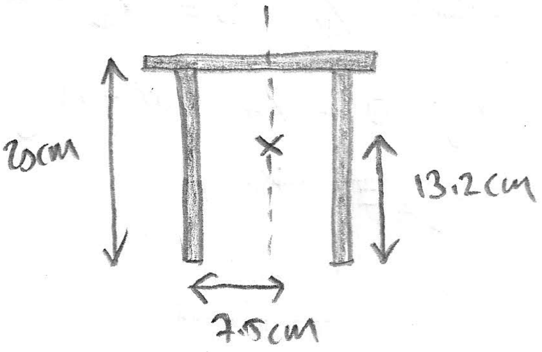
Mass / unit length is 31.4 kg/m $= \rho$ (of iron rods)

Centre of mass is:

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[\begin{pmatrix}0\\0.1\end{pmatrix}\times 0.2\rho + \begin{pmatrix}0.075\\0.2\end{pmatrix}\times 0.19\rho + \begin{pmatrix}0.15\\0.1\end{pmatrix}\times 0.2\rho\right]}{(0.2+0.2+0.19)\rho}$$

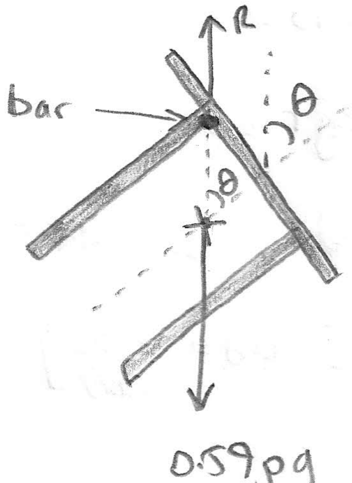
$$= \begin{pmatrix}0.04425\\0.078\end{pmatrix}\frac{1}{0.59} = \begin{pmatrix}0.075\\0.132\end{pmatrix}$$

1e/

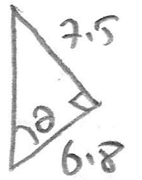


× is centre of mass.

b)



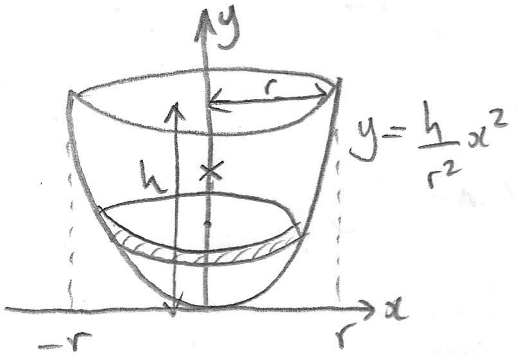
g = 9.81 N/kg



so $\theta = \tan^{-1}\left(\frac{7.5}{6.8}\right)$

$= \boxed{47.8^\circ}$

(ix)



By considering the sum of cylindrical masses of mass $dm = \rho\pi x^2 dy$

centre of mass of paraboloid of density ρ is at $y = \bar{y}$

(note by symmetry it must be on the y axis)

$$\bar{y} = \frac{\int y\, dm}{\int \rho\pi x^2 dy}$$

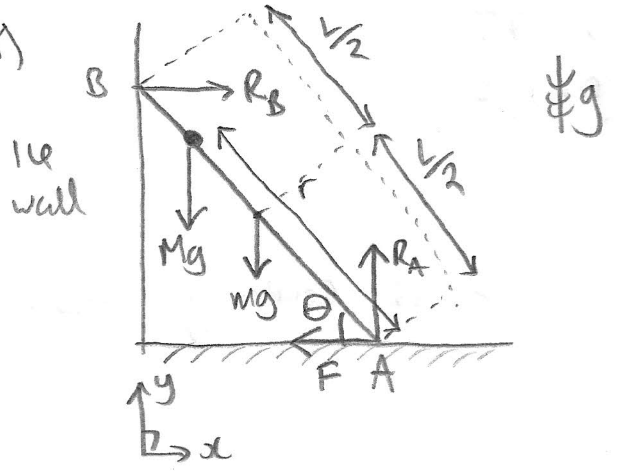
↑ mass of paraboloid

Makes sense to integrate over y not x ∴ note $x^2 = yr^2/h$

$$\therefore \bar{y} = \frac{\int_0^h y \times \rho\pi y r^2/h \, dy}{\int_0^h \rho\pi y r^2/h \, dy}$$

$$\Rightarrow \bar{y} = \frac{\int_0^h y^2 dy}{\int_0^h y\, dy} = \frac{\frac{1}{3}h^3}{\frac{1}{2}h^2} = \boxed{\frac{2}{3}h}$$

(x)



Assume equilibrium:

NII: $//x: \quad 0 = R_B - F$

$//y: \quad 0 = R_A - (m+M)g$

$$\therefore \boxed{\begin{array}{l} R_B = F \\ R_A = (m+M)g \end{array}}$$

Assume net turning moment at A = zero.

$$\circlearrowleft + : \quad 0 = R_B L \sin\theta - mg \tfrac{L}{2}\cos\theta - Mg r \cos\theta$$

[Climber on ladder is r up the ladder ie max $r = L$]

$$\text{So} \quad \boxed{R_B = \left(\frac{m}{2} + \frac{Mr}{L}\right)\frac{1}{\tan\theta}} \qquad \left[\tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

Now if no slip at A: $\quad F < \mu R_A$

so using $R_B = F$

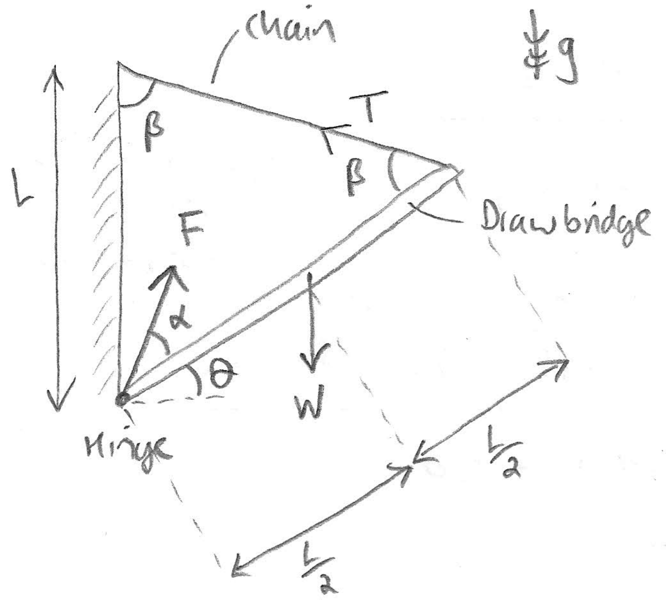
$$\Rightarrow \left(\frac{m}{2} + \frac{Mr}{L}\right)\frac{1}{\tan\theta} < \mu(m+M)g$$

$$\therefore \quad \mu > \frac{\left(\frac{m}{2} + \frac{Mr}{L}\right)\frac{1}{\tan\theta}}{m+M}$$

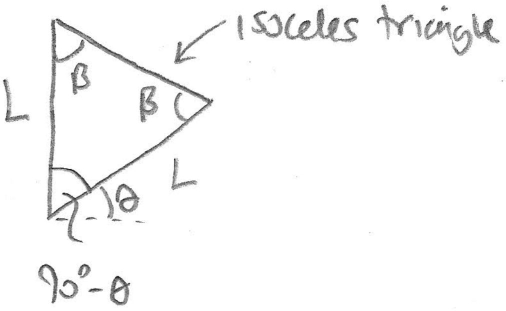
For this to be true for $0 \leq r \leq L$
(ie the whole ascent of the ladder)

$$\Rightarrow \boxed{\mu > \frac{\left(\frac{m}{2} + M\right)\frac{1}{\tan\theta}}{m+M}} \quad \text{as required.}$$

2/



a)



$$180° = 2\beta + 90° - \theta$$

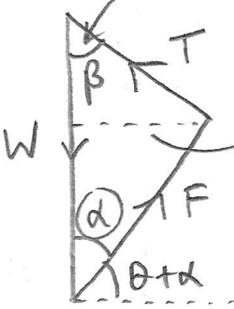
$$90° = 2\beta - \theta$$

$$\therefore \quad \boxed{45° + \tfrac{1}{2}\theta = \beta}$$

b) Taking moments ↺+ from centre of drawbridge

$$0 = F\sin\alpha \times \frac{L}{2} - T\sin\beta \times \frac{L}{2} \quad \Rightarrow \quad \boxed{F\sin\alpha = T\sin\beta}$$

Triangle of forces:
(since in equilibrium)



This must be β from above diagram (translate W...)

This length is $T\sin\beta$

Since this $= F\sin\alpha$

⇒ ○ circled angle must be α.

$$\therefore \quad 90° = 2\alpha + \theta \quad \Rightarrow \quad \boxed{\alpha = 45° - \tfrac{1}{2}\theta}$$

c) Taking moments ↺+ about the hinge

$$0 = W \times \frac{L}{2}\cos\theta \quad - T\sin\beta \times L$$

$$\Rightarrow \quad \boxed{T = \tfrac{1}{2}W\cos\theta / \sin\beta}$$

d) $\cos\theta = \cos(\theta/2 + \theta/2) = \cos^2\theta/2 - \sin^2\theta/2$

$\sin\beta = \sin(45° + \tfrac{1}{2}\theta) = \sin 45°\cos\tfrac{1}{2}\theta + \cos 45°\sin\tfrac{1}{2}\theta = \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta)$

$\sin\alpha = \sin(45° - \tfrac{1}{2}\theta) = \sin 45°\cos\tfrac{1}{2}\theta - \cos 45°\sin\tfrac{1}{2}\theta = \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)$

$$\therefore \quad T = \frac{\frac{1}{2}W\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}{\frac{1}{\sqrt{2}}\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)}$$

Difference of two Squares
$(a^2-b^2) = (a+b)(a-b)$

$\left[\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\right]$

$$T = \frac{\frac{W}{\sqrt{2}}\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)\left(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\right)}{\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta}$$

$$\therefore \quad \boxed{T = \frac{W}{\sqrt{2}}\left(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\right)}$$

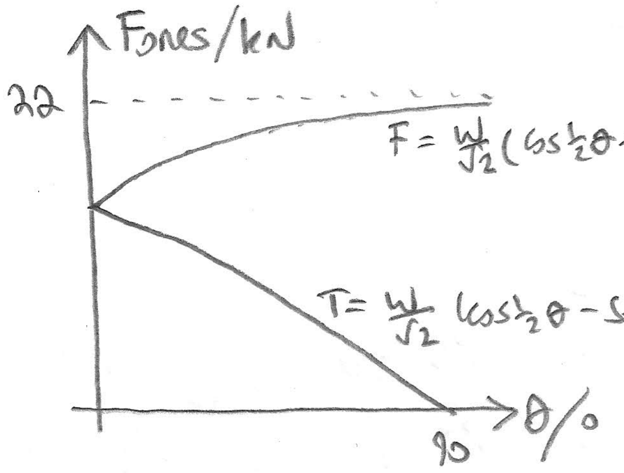
From (b) $F = \frac{T\sin\beta}{\sin\alpha}$

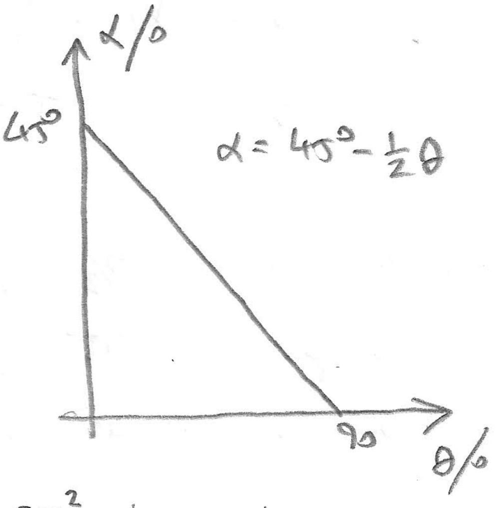
$$F = \underbrace{\frac{\frac{1}{2}W\cos\theta}{\sin\beta}}_{T}\ \frac{\sin\beta}{\sin\alpha} = \frac{\frac{1}{2}W\cos\theta}{\sin\alpha}$$

$$\therefore \quad F = \frac{\frac{1}{2}W\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)\left(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\right)}{\frac{1}{\sqrt{2}}\left(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\right)}$$

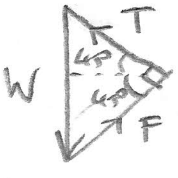
$$\therefore \quad \boxed{F = \frac{W}{\sqrt{2}}\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)}$$

let $L = 10.0\text{m}$
$W = 22\text{kN}$



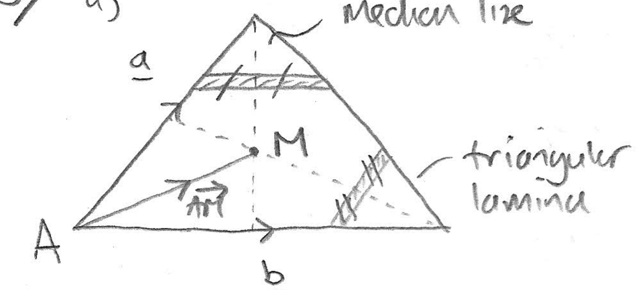


when $\theta = 0$: $F = T = \frac{W}{\sqrt{2}}$



$W^2 = 2T^2 \quad \therefore T = \frac{W}{\sqrt{2}}$

3/ a)



We could construct a triangular lamina from rectangular strips, with centre of each strip being on the MEDIAN LINE. Hence the centre of mass of the lamina must be on the median line.

To find out where, determine the intersection of two of the median lines.

ie/ $\vec{AM} = \frac{1}{2}\underline{a} + \lambda\left(-\frac{1}{2}\underline{a} + \underline{b}\right)$ [ie λ is fractional distance along $-\frac{1}{2}\underline{a}+\underline{b}$]

$\vec{AM} = \frac{1}{2}\underline{b} + \mu\left(-\frac{1}{2}\underline{b} + \underline{a}\right)$

$\therefore \quad \underline{a}\left(\frac{1}{2} - \frac{\lambda}{2} - \mu\right) + \underline{b}\left(\lambda - \frac{1}{2} + \mu/2\right) = \underline{0}$

So comparing coefficients of $\underline{a}$, $\underline{b}$ (ie since above equation is true for all $\underline{a}$, $\underline{b}$)

$\frac{1}{2} - \lambda/2 - \mu = 0$ ①

$\lambda - \frac{1}{2} + \mu/2 = 0$ ②

$1 - \lambda - 2\mu = 0$ ①

$1 - \frac{1}{2} - \frac{3}{2}\mu = 0$ ① + ②

$\therefore \frac{1}{2} = \frac{3}{2}\mu$

$\therefore \boxed{\mu = \frac{1}{3}}$

in ①: $\lambda = 1 - 2\mu$

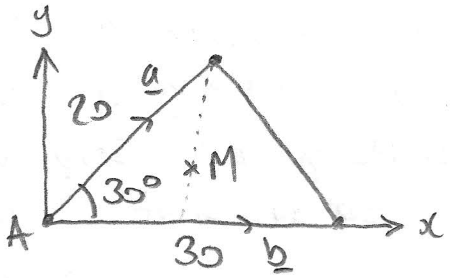
$\boxed{\lambda = \frac{1}{3}}$

$\therefore \quad \vec{AM} = \frac{1}{2}\underline{a} + \frac{1}{3}\left(-\frac{1}{2}\underline{a} + \underline{b}\right)$

$\vec{AM} = \frac{1}{3}\left(\frac{3}{2}a - \frac{1}{2}\underline{a} + \underline{b}\right)$

$\boxed{\vec{AM} = \frac{1}{3}(\underline{a} + \underline{b})}$ as required.

b)

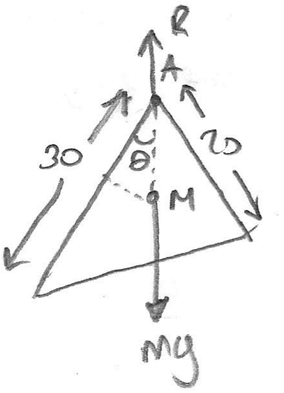


$$\underline{a} = \begin{pmatrix} 20\sqrt{3}/2 \\ 20\cdot\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 10\sqrt{3} \\ 10 \end{pmatrix}$$

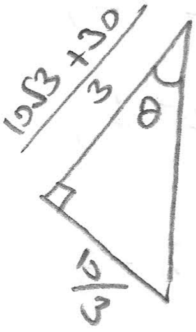
$$\underline{b} = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

$$\therefore \quad \vec{AM} = \frac{1}{3}\begin{pmatrix} 10\sqrt{3}+30 \\ 10 \end{pmatrix} \qquad \text{ie } \vec{AM} = \frac{1}{3}(\underline{a}+\underline{b})$$

#g



Hang lamina from A.

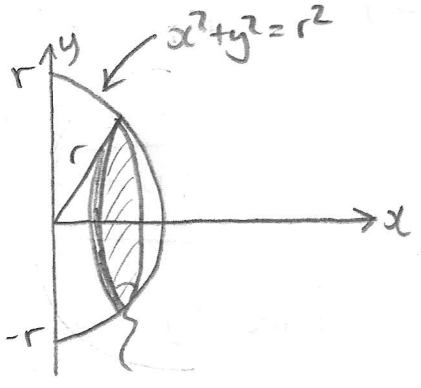


So angle of tilt of base from vertical is

$$\theta = \tan^{-1}\left(\frac{10}{10\sqrt{3}+30}\right)$$

$$\theta = \boxed{11.9°}$$

4/ a)



cylindr mass $dM = \rho\pi y^2 dx$

Centre of mass of a solid hemisphere of density ρ is at $\bar{x}$ on symmetry axis.

$$\bar{x} = \frac{\int \rho\pi y^2 dx \times x}{\frac{2}{3}\pi r^3 \rho} \nwarrow \text{mass of hemisphere}$$

Now $r^2 = x^2+y^2 \quad \therefore y^2 = r^2 - x^2$

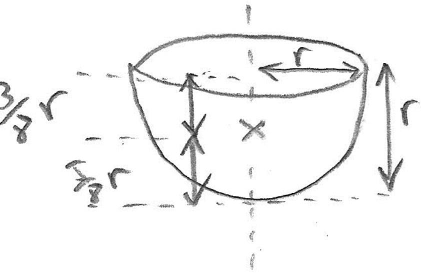
$$\therefore \quad \bar{x} = \frac{\int_0^r \rho\pi(r^2-x^2)x\,dx}{\frac{2}{3}\pi r^3 \rho}$$

$$= \frac{3}{2}\frac{1}{r^3}\int_0^r (xr^2 - x^3)dx$$

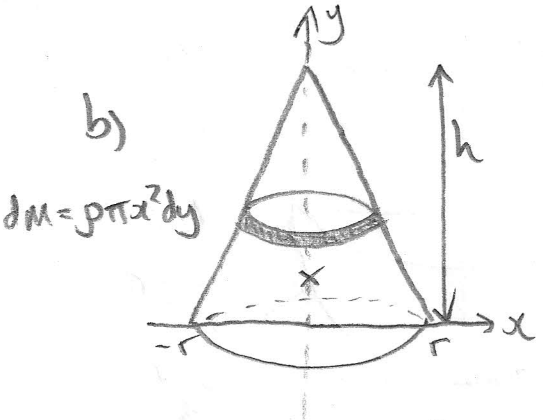
$$= \frac{3}{2r^3}\left[\frac{1}{2}x^2r^2 - \frac{1}{4}x^4\right]_0^r$$

$$= \frac{3}{2r^3}\left[\frac{1}{4}r^4\right] = \boxed{\frac{3}{8}r}$$

So in summary for hemisphere: (centre of mass at ×)



b)



For cone:

$$\boxed{y = h - \frac{h}{r}x}$$

centre of mass at $\bar{y} = \dfrac{\int y \times \rho\pi x^2 \, dy}{\frac{1}{3}\pi r^2 h\rho}$

↑ mass of cone

so $\frac{h}{r}x = h - y$

$x = \frac{r}{h}(h-y)$

$x^2 = \frac{r^2}{h^2}(h-y)^2$

$x^2 = \frac{r^2}{h^2}(h^2 - 2hy + y^2)$

$\therefore \bar{y} = \dfrac{\int_0^h y\rho\pi \frac{r^2}{h^2}(h^2 - 2hy + y^2)dy}{\frac{1}{3}\pi r^2 h\rho}$

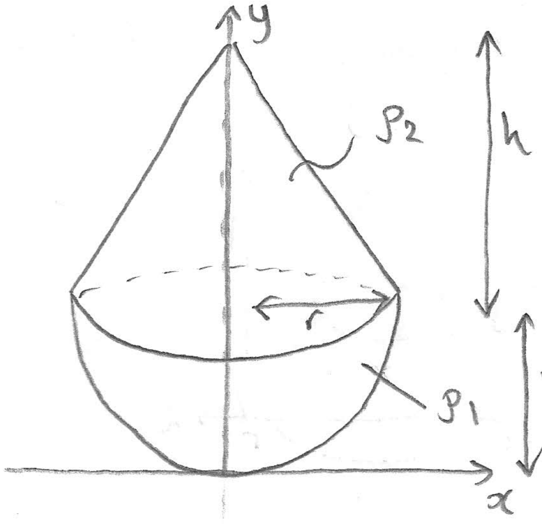
$\Rightarrow \bar{y} = \frac{3}{h^3}\int_0^h (yh^2 - 2hy^2 + y^3)\, dy$

$$\Rightarrow \bar{y} = \frac{3}{h^3}\left[\frac{1}{2}y^2h^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right]_0^h$$

$$\Rightarrow \bar{y} = \frac{3}{h^3}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]h^4$$

$$\Rightarrow \boxed{\bar{y} = \frac{1}{4}h}$$ as required.

c) Weeble

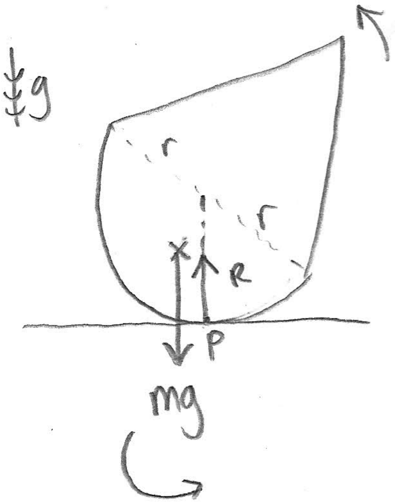


centre of mass of weeble is :

$$\bar{y} = \frac{\frac{5}{8}r \times \frac{2}{3}\pi r^3 \rho_1 + \left(r + \frac{h}{4}\right) \times \frac{1}{3}\pi r^2 h \rho_2}{\frac{2}{3}\pi r^3 \rho_1 + \frac{1}{3}\pi r^2 h \rho_2}$$

$$\boxed{\bar{y} = \frac{\frac{5}{8}r + \left(r + \frac{h}{4}\right)\frac{1}{2}\frac{h}{r}\frac{\rho_2}{\rho_1}}{1 + \frac{1}{2}\frac{h}{r}\frac{\rho_2}{\rho_1}}}$$

fig



Turning effect of weight about point of contact P

Now, to cause a weeble to 'wobble but not fall down", the centre of mass $\boxed{\bar{y} < r}$. If this is the case, the weight of the weeble will provide a turning moment which will cause the weeble to return to the upright position.

$$\therefore \frac{5}{8}r + \left(r + \frac{h}{4}\right)\frac{h}{2r}\frac{\rho_2}{\rho_1} < r + \frac{1}{2}h\frac{\rho_2}{\rho_1}$$

$$\frac{5}{8}r + \cancel{\frac{h}{2}\frac{\rho_2}{\rho_1}} + \frac{h^2}{8r}\frac{\rho_2}{\rho_1} < r + \cancel{\frac{1}{2}h\frac{\rho_2}{\rho_1}}$$

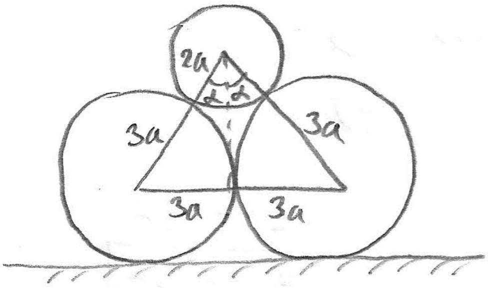
$$\therefore \quad \frac{h^2}{8r}\frac{\rho_2}{\rho_1} < \frac{3}{8}r$$

$$\therefore \quad h^2 < 3r^2 \rho_1/\rho_2$$

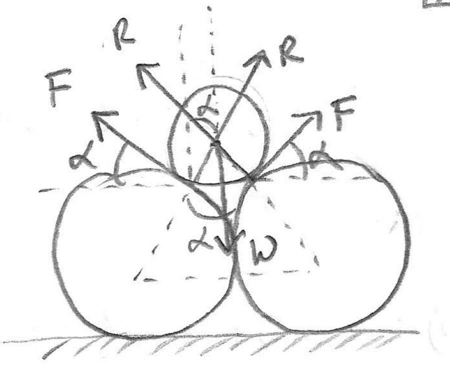
$$\therefore \quad \boxed{h < r\sqrt{\frac{3\rho_1}{\rho_2}}}$$ as required.

So if $\rho_1 = \rho_2$, $\boxed{h < r\sqrt{3}}$

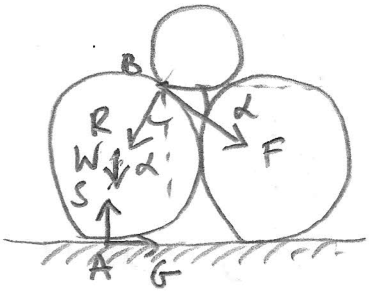
5/



overall geometry (Fig 1)



Forces on upper log (Fig 2)

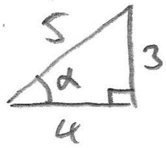


Forces on lower log (Fig 3)

Geometry: $5a\sin\alpha = 3a$

$\boxed{\sin\alpha = 3/5}$

Assume equilibrium



$\therefore$ $\boxed{\cos\alpha = 4/5}$

NII for upper cylinder:

$//y: \quad 0 = 2R\cos\alpha + 2F\sin\alpha - W \quad \Rightarrow \quad \frac{8}{5}R + \frac{6F}{5} = W$

$\Rightarrow \quad \boxed{5W = 8R + 6F}$ ①

NII for lower cylinder:

$//x: \quad 0 = G - R\sin\alpha + F\cos\alpha \quad \Rightarrow \boxed{5G = 3R - 4F}$ ②

NII for lower cylinder:

$$// y: \quad 0 = S - R\cos\alpha - F\sin\alpha - W$$

$$\therefore \quad \boxed{5S = 4R + 3F + 5W} \qquad ③$$

$\circlearrowright$+ moments about centre of lower cylinder

$$0 = F \times 3a - G \times 3a \quad \Rightarrow \quad \boxed{F = G} \qquad ④$$

So in summary:

$$5w = 8R + 6F \qquad ①$$
$$5G = 3R - 4F \qquad ②$$
$$5S = 4R + 3F + 5W \qquad ③$$
$$F = G \qquad ④$$

using ④ in ②: $5F = 3R - 4F$ $\quad \therefore \quad 3R = 9F$

$$\boxed{R = 3F}$$

$\therefore$ in ①: $5w = 8 \times 3F + 6F$

$$5w = 30F$$

$$\boxed{\frac{1}{6}w = F} \qquad \therefore \quad \boxed{G = \frac{1}{6}w}$$

$$\therefore \quad \boxed{R = \frac{1}{2}w}$$

$$\therefore \quad 5S = 4\frac{1}{2}w + 3\frac{1}{6}w + 5W$$

$$5S = \frac{5}{2}w + 5W \quad \Rightarrow \quad \boxed{S = \frac{1}{2}w + W}$$

Now for no slip at A (by, ground interface)

$$G < \mu_A S \quad \Rightarrow \quad \frac{1}{6}w < \mu_A\left(\frac{1}{2}w + W\right)$$

$$\Rightarrow \quad \boxed{\mu_A > \frac{w}{3w + 6W}}$$

For no slip at by-bg interface B:

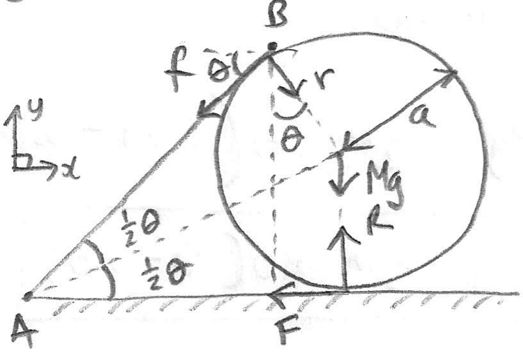
$$F < \mu_B R$$

$$\therefore \quad \tfrac{1}{6}w < \mu_B \tfrac{1}{2}w \quad \Rightarrow \quad \boxed{\mu_B > \tfrac{1}{3}}$$

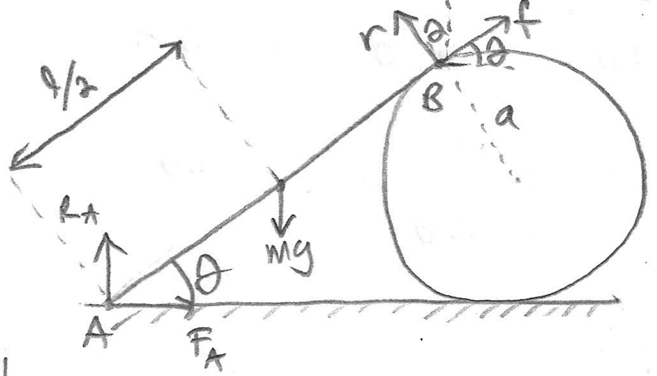
6/

Fig

Forces on cylinder of mass M and radius a



Forces on stick of mass m and length l



Newton II:

$//x: \; 0 = r\sin\theta - f\cos\theta - F$ ①

$//y: \; 0 = R - Mg - r\cos\theta - f\sin\theta$ ②

Moments ↺+ about centre of cylinder

$$0 = F\times a - f\times a$$

$\Rightarrow \boxed{F = f}$ ③

Newton II:

$//x: \; 0 = F_A + f\cos\theta - r\sin\theta$ ④

$//y: \; 0 = R_A - mg + r\cos\theta + f\sin\theta$ ⑤

Moments ↺+ about A

$$0 = mg\times \frac{l}{2}\cos\theta - rl$$

$\Rightarrow \boxed{r = \tfrac{1}{2}mg\cos\theta}$ ⑥

So from ①, and ③, and ⑥

$$0 = \underbrace{\tfrac{1}{2}mg\cos\theta\sin\theta}_{r} - f\cos\theta - \underbrace{f}_{F}$$

$$\Rightarrow \quad f(1+\cos\theta) = \tfrac{1}{2}mg\cos\theta\sin\theta$$

$$\Rightarrow \quad \boxed{f = \frac{\tfrac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}}$$

$$\therefore \quad \boxed{F = \frac{\tfrac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}}$$

∴ using ② : $R = Mg + r\cos\theta + f\sin\theta$

$$R = Mg + \frac{1}{2}mg\cos\theta + \frac{\frac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}\sin\theta$$

$$R = Mg + \frac{1}{2}mg\cos\theta\left(1 + \frac{\sin^2\theta}{1+\cos\theta}\right)$$

Now $\sin^2\theta = 1-\cos^2\theta = (1+\cos\theta)(1-\cos\theta)$

$\therefore \frac{\sin^2\theta}{1+\cos\theta} = 1-\cos\theta$ $\quad\therefore R = Mg + \frac{1}{2}mg\cos\theta(1+1-\cos\theta)$

$$\boxed{R = Mg + \frac{1}{2}mg\cos\theta(2-\cos\theta)}$$

in ④: $F_A = r\sin\theta - f\cos\theta$

$$F_A = \frac{1}{2}mg\cos\theta\sin\theta - \frac{\frac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}\cos\theta$$

$$F_A = \frac{1}{2}mg\cos\theta\sin\theta\left(1 - \frac{\cos\theta}{1+\cos\theta}\right)$$

$$F_A = \frac{1}{2}mg\cos\theta\sin\theta\left(\frac{1+\cos\theta-\cos\theta}{1+\cos\theta}\right)$$

$$\boxed{F_A = \frac{\frac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}}$$

In ⑤ $R_A = mg - r\cos\theta - f\sin\theta$

$$\therefore R_A = mg - \frac{1}{2}mg\cos\theta\cos\theta - \frac{\frac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}\sin\theta$$

$$\therefore R_A = mg - \frac{1}{2}mg\cos\theta\left(\cos\theta + \frac{\sin^2\theta}{1+\cos\theta}\right) = mg - \frac{1}{2}mg\cos\theta(\cos\theta + 1 - \cos\theta)$$

$$\therefore \boxed{R_A = mg - \frac{1}{2}mg\cos\theta}$$

In summary :

$$\boxed{\begin{array}{ll} r = \frac{1}{2} mg\cos\theta & \\ f = \dfrac{\frac{1}{2} mg\cos\theta\sin\theta}{1+\cos\theta} & F = \dfrac{\frac{1}{2} mg\cos\theta\sin\theta}{1+\cos\theta} \\ R = Mg + \frac{1}{2} mg\cos\theta(2-\cos\theta) & \end{array}}$$

$[(2-\cos\theta)(1+\cos\theta)$

$= 2 + \cos\theta -$

$$\boxed{\begin{array}{l} F_A = \dfrac{\frac{1}{2} mg\cos\theta\sin\theta}{1+\cos\theta} \\ \\ R_A = mg - \frac{1}{2} mg\cos\theta \end{array}}$$

{ ie $F_A = F$ which makes sense if you consider the whole system}

For no slip at A: $F_A < \mu_A R_A$

$$\therefore \quad \frac{\frac{1}{2} mg\cos\theta\sin\theta}{1+\cos\theta} < \mu_A mg\left(1 - \tfrac{1}{2}\cos\theta\right)$$

$$\frac{\cos\theta\sin\theta}{1+\cos\theta} < \mu_A(2-\cos\theta)$$

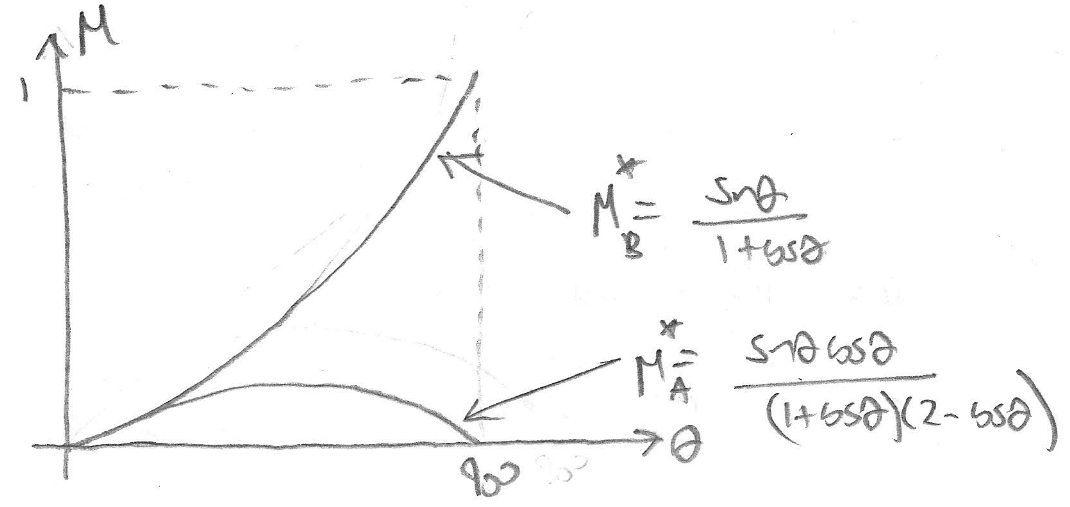
$$\therefore \quad \boxed{\mu_A > \frac{\cos\theta\sin\theta}{(1+\cos\theta)(2-\cos\theta)}}$$ as required

For no slip at B: $f < \mu_B r$

$$\therefore \quad \frac{\frac{1}{2} mg\cos\theta\sin\theta}{1+\cos\theta} < \mu_B \tfrac{1}{2} mg\cos\theta$$

$$\boxed{\frac{\sin\theta}{1+\cos\theta} < \mu_B}$$

b)



$\mu_B^* = \frac{\sin\theta}{1+\cos\theta}$

$\mu_A^* = \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}$

No slip if $\mu_B > \mu_B^*$ and $\mu_A > \mu_A^*$

Now $$\frac{\mu_B^* \cos\theta}{2-\cos\theta} = \mu_A^*$$

when $\theta \to 0$ $$\mu_A^* \to \frac{\mu_B^* \cos(0)}{2-\cos(0)} = \mu_B^*$$

ie $\mu_B^* = \mu_A^*$ (ie same line)

Max value of $y = \frac{\cos\theta}{2-\cos\theta}$ is when $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = \frac{(2-\cos\theta)(-\sin\theta) - \cos\theta(\sin\theta)}{(2-\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{-\sin\theta(2-\cos\theta+\cos\theta)}{(2-\cos\theta)^2} = \frac{-2\sin\theta}{(2-\cos\theta)^2}$$

So $\frac{dy}{d\theta} = 0$ when $\theta = 0$ or 90°

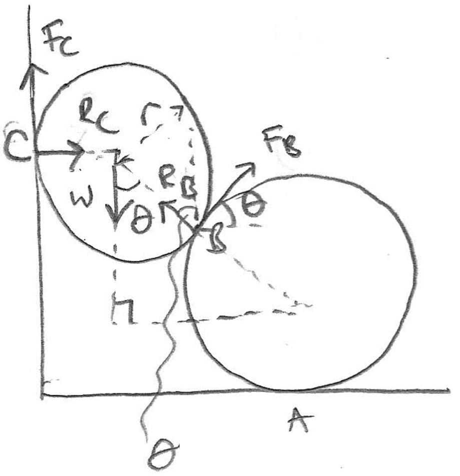
$$\left.\begin{array}{l} \text{when } \theta = 0, \ \frac{\cos\theta}{2-\cos\theta} = 1 \\ \theta = 90^\circ, \ \frac{\cos\theta}{2-\cos\theta} = 0 \end{array}\right\} \Rightarrow \frac{\cos\theta}{2-\cos\theta} \leq 1$$

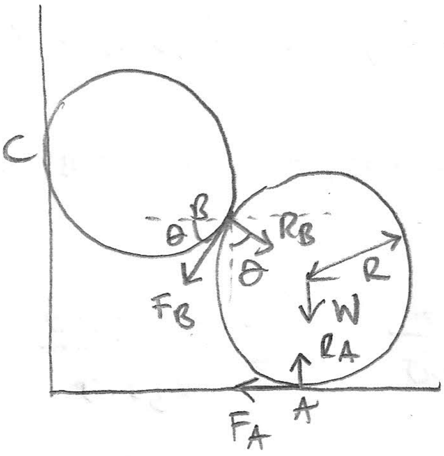
$$\Rightarrow \boxed{\mu_B^* \geq \mu_A^*}$$

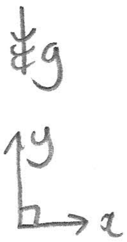
So if $\mu_A = \mu_B$ and $\mu_B > \frac{\sin\theta}{1+\cos\theta}$

$\Rightarrow \quad \mu_A > \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}$ ie If stick doesn't slip against the cylinder it won't slip against the ground if $\mu_A = \mu_B$.

7/







[Assume equilibrium]

Forces on $\boxed{\text{entire system}}$

$//y: \quad 0 = F_C + R_A - w - W \quad \therefore \quad \boxed{F_C + R_A = w + W}$ ①

$\circlearrowright +$ moments about centre of cylinder of radius r

$0 = F_C \times r - F_B \times r \Rightarrow \boxed{F_C = F_B}$ ②

$\circlearrowright +$ moments about centre of cylinder of radius R

$\boxed{F_A = F_B}$ ③ ie $\boxed{F_A = F_B = F_C = F}$

Forces on upper cylinder

NII: $//y: \quad 0 = F_C - w + R_B\cos\theta + F_B\sin\theta$ ④

$//x: \quad 0 = R_C - R_B\sin\theta + F_B\cos\theta$ ⑤

" " lower cylinder

NII: $//y: \quad 0 = R_A - W - F_B\sin\theta - R_B\cos\theta$ ⑥

$//x: \quad 0 = R_B \sin\theta - F_B \cos\theta - F_A$ ⑦

To summarize: $F(1+\cos\theta) = R_B \sin\theta$ ⑦

$W + F\sin\theta = R_A - R_B \cos\theta$ ⑥

$F\cos\theta = R_B \sin\theta - R_C$ ⑤

$F(1+\sin\theta) = w - R_B \cos\theta$ ④

④/⑦ $$\frac{1+\sin\theta}{1+\cos\theta} = \frac{w - R_B\cos\theta}{R_B \sin\theta}$$

$$R_B\left(\frac{1+\sin\theta}{1+\cos\theta}\right)\sin\theta + R_B\cos\theta = w$$

$$R_B\left(\frac{\sin\theta + \sin^2\theta + \cos\theta + \cos^2\theta}{1+\cos\theta}\right) = w$$

$$R_B\left(\frac{1+\sin\theta+\cos\theta}{1+\cos\theta}\right) = w$$

$$\therefore \boxed{R_B = \frac{w(1+\cos\theta)}{1+\sin\theta+\cos\theta}}$$

$$\therefore F_A = F_B = F_C = F = \frac{W(1+\cos\theta)}{1+\sin\theta+\cos\theta}\,\frac{\sin\theta}{1+\cos\theta}$$ using ⑦

$$\therefore \boxed{F = \frac{W\sin\theta}{1+\sin\theta+\cos\theta}}$$

In ⑥: $R_A = W + F\sin\theta + R_B\cos\theta$

$$\therefore R_A = W + \frac{w\sin^2\theta}{1+\sin\theta+\cos\theta} + \frac{w(1+\cos\theta)\cos\theta}{1+\sin\theta+\cos\theta}$$

$$R_A = W + \frac{w\left(\sin^2\theta + \cos\theta + \cos^2\theta\right)}{1+\sin\theta+\cos\theta}$$

$$\boxed{R_A = W + \frac{w(1+\cos\theta)}{1+\sin\theta+\cos\theta}}$$

In ⑤: $R_C = R_B\sin\theta - F\cos\theta$

$$R_C = \frac{w(1+\cos\theta)}{1+\sin\theta+\cos\theta}\sin\theta - \frac{w\sin\theta\cos\theta}{1+\sin\theta+\cos\theta}$$

$$R_C = \frac{w\sin\theta}{1+\sin\theta+\cos\theta}\left(1+\cos\theta-\cos\theta\right)$$

$$\boxed{R_C = \frac{w\sin\theta}{1+\sin\theta+\cos\theta}}$$

For no slip at A: $F_A < \mu_A R_A$

$$\Rightarrow \frac{w\sin\theta}{1+\sin\theta+\cos\theta} < \mu_A \frac{\left(W+W\sin\theta+W\cos\theta+w+w\cos\theta\right)}{1+\sin\theta+\cos\theta}$$

$$\Rightarrow \boxed{\mu_A > \frac{w\sin\theta}{W(1+\cos\theta+\sin\theta)+w(1+\cos\theta)}}$$

For no slip at B: $F_B < \mu_B R_B$

$$\frac{w\sin\theta}{1+\sin\theta+\cos\theta} < \mu_B \frac{w(1+\cos\theta)}{1+\sin\theta+\cos\theta}$$

$$\Rightarrow \quad \boxed{\mu_B > \frac{\sin\theta}{1+\cos\theta}}$$

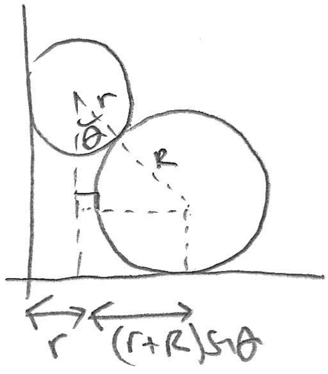
For no slip at C: $F_c < \mu_c R_c$

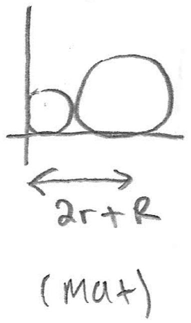
$$\frac{w\sin\theta}{1+\sin\theta+\cos\theta} < \mu_c \frac{w\sin\theta}{1+\sin\theta+\cos\theta}$$

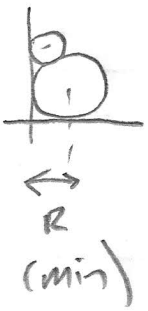
$$\Rightarrow \quad \boxed{\mu_c > 1}$$

What are the bounds for θ?

limits of $r+(r+R)\sin\theta$ are

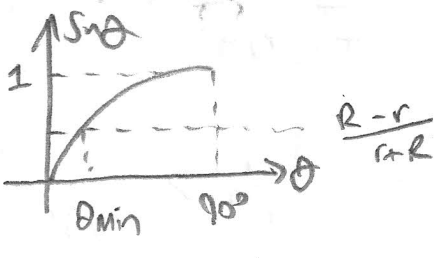






so $r+(r+R)\sin\theta \geq R$ and $r+(r+R)\sin\theta \leq 2r+R$

$$\therefore \quad \boxed{\sin\theta \geq \frac{R-r}{r+R}} \qquad \boxed{\sin\theta \leq 1}$$



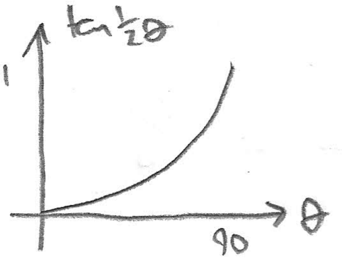
Summarizing (again!) For no slip:

$$\mu_A > \frac{w\sin\theta}{W(1+\cos\theta+\sin\theta)+w(1+\cos\theta)}$$

$$\mu_B > \frac{\sin\theta}{1+\cos\theta}$$

$$\mu_C > 1$$

Now $\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta}$ so $\mu_B > \tan\frac{1}{2}\theta$



∴ minimum μ_B for no slip is when $\theta = \theta_{min} = \boxed{\sin^{-1}\left(\frac{R-r}{r+R}\right)}$

ie $\boxed{\mu_B > \tan\frac{1}{2}\theta_{min}}$

Use Quadling M3&4 Ex 4A p100 Q17

$W = 400N$ $R = 35cm$

$w = 300N$ $r = 30cm$

$$\therefore \theta_{min} = \sin^{-1}\left(\frac{35-30}{35+30}\right)$$

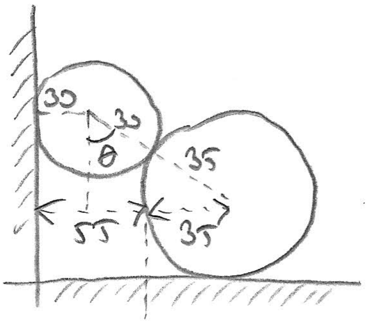
$$= \boxed{4.41°}$$

$$\therefore \boxed{\mu_B > 0.039}$$

$$\mu_A > \frac{300\sin\theta_{min}}{400(1+\cos\theta_{min}+\sin\theta_{min})+300(1+\cos\theta_{min})}$$

$$\mu_A > \boxed{0.016}$$

But in the question it specifies this geometry:

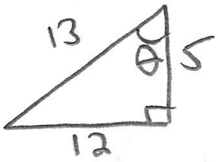


$$\text{so}\quad 65\sin\theta + 30 = 55 + 35$$

$$\sin\theta = \frac{60}{65}$$

$$\boxed{\sin\theta = \frac{12}{13}}$$

Now



ie $13^2 = 12^2 + 5^2$

$$\text{so}\quad \boxed{\cos\theta = \frac{5}{13}}$$

$$\therefore \frac{\sin\theta}{1+\cos\theta} = \frac{12/13}{1+5/13} = \frac{12/13}{18/13} = \frac{2}{3}$$

$$\Rightarrow \boxed{\mu_B > \frac{2}{3}}$$

$$1 + \sin\theta + \cos\theta = \frac{13+12+5}{13} = \frac{30}{13}$$

$$\therefore \mu_A > \frac{300 \times 12/13}{400\left(\frac{30}{13}\right) + 300\left(\frac{18}{13}\right)}$$

$$\Rightarrow \mu_A > \frac{3600}{12000 + 5400}$$

$$\boxed{\mu_A > \frac{6}{29}}$$

Answers give $\frac{1}{13}, \frac{1}{5}, 1$ however....

$\uparrow$ min μ_A $\quad\uparrow$ min μ_B $\quad\nwarrow$ min μ_C

ie from axis of lower drum to wall

$\downarrow$

But if 55 is instead

$$55 - 35 = \boxed{20}$$

$$65\sin\theta + 30 = 55$$

$$\Rightarrow \boxed{\theta = \sin^{-1}\left(\frac{5}{13}\right)}$$

$$\therefore \boxed{\cos\theta = \frac{12}{13}}$$

$$\therefore \frac{\sin\theta}{1+\cos\theta} = \frac{5/13}{\frac{13+12}{13}}$$

$$= \boxed{\frac{1}{5}} \checkmark \quad \text{so} \quad \boxed{\mu_B > \frac{1}{5}}$$

$$\text{and } \mu_A > \frac{300 \times 5}{400 \times 30 + 300 \times 25}$$

$$\mu_A > \boxed{1/13} \checkmark$$