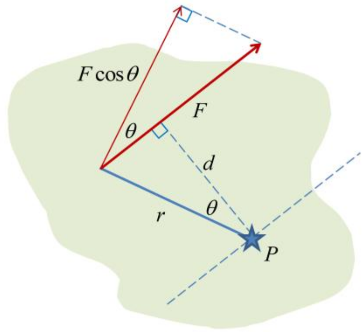
All mechanical systems referred to in this sheet will be in ***equilibrium***. This means:

(1) **The vector sum of all forces = zero**, which means by *Newton's Second Law*, that the *acceleration* of the ***centre of mass*** of the system is *zero*. This means the *centre of mass moves at constant velocity*, which *could* mean it is at rest. Geometrically, a ***zero vector sum of forces*** means that ***adding force vectors tip-to-tail forms a closed shape.***

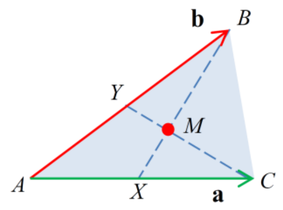
(2) **The sum of torques (i.e. turning moments) about *any* location in the system (i.e. not simply the centre of mass) is zero.** This means the *angular momentum* of the system is constant, which could mean the system is at rest in a rotational sense.



Clockwise turning moment about P:
$F\cos\theta \times r = F \times r\cos\theta = Fd$

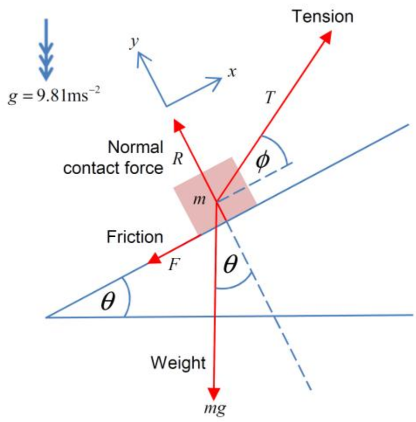
Torque (or **turning moment**) is the **force** $\times$ **perpendicular distance** to a line, that is parallel to the force vector, that passes through a point that moments are taken about, Alternatively it is the product of the magnitude of the displacement vector between the point of action of a force to the point moments are taken about, multiplied by the component of the force perpendicular to the displacement vector.

The ***centre of mass*** $\overline{\mathbf{r}}$ of a system of N masses at displacement $\mathbf{r}_i$ from some coordinate origin is: $\overline{\mathbf{r}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i$.



The centre of mass of a *triangular lamina* is $\overrightarrow{AM} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$ where the vectors which form the triangle sides from vertex A position $\overrightarrow{OA}$ are $\mathbf{a},\mathbf{b}$.

For a mass on an inclined plane (i.e. a slope), the mass will not slide if $F < \mu R$ where F is the frictional force parallel to the slope, R is the normal contact force (perpendicular to the slope) and μ is the coefficient of static friction between the mass and the surface of the slope. When $F = \mu R$ the mass 'is on the point of sliding.'



Note when the mass *does* begin to slide, the friction force becomes *constant* at $F = \mu_D R$. In many situations, the *dynamic* coefficient of friction $\mu_D = \mu$.

Unless told otherwise, assume the strength of gravity is: $g = 9.81\text{N/kg}$.

Question 1 **** ALWAYS DRAW A DIAGRAM FOR EVERY MECHANICS PROBLEM FIRST ****

(i) An artwork of weight 48N in a gallery in Samos, Greece is held in place on a vertical wall via two cables. One cable has a tension of 55N and points left. Calculate the magnitude of the cable tension in the second cable, and determine the angle from the vertical that the cable points.

(ii) Alice, Bethany and Clarissa are all mass m . Alice sits 0.5m from a see saw pivot, Bethany sits 0.4m behind her and Clarissa sits 0.3m behind Bethany. Dennis has a mass of $3m$ and sits on the other side of the see-saw, balancing the girls. How far from the pivot must Dennis sit?

(iii) Show that if a mass m is placed on a slope of inclination angle θ , the mass won't slide as long as $\tan\theta < \mu$.

(iv) A climber of mass 75kg takes on an overhang. During one move, the climber's legs dangle and all his weight is supported by his arms. The left arm extends to an angle of 45° from the vertical whereas the right arm grips a hold requiring the arm to be outstretched to an angle of 30° to the vertical. Calculate the tension in the left and right arms of the climber.

(v) A suitcase of mass 20kg is placed on a ramp in an airport. The suitcase begins to slide when the ramp is inclined at an angle of 30° from the horizontal. What is the coefficient of friction? Calculate the magnitudes of the normal contact force R and the frictional force F on the suitcase when the ramp is at angle of 20° .

(vi) John (70kg), Paul (80kg), George (75kg) and Ringo (80kg) are sitting on a see-saw. John is 1.5m behind the pivot on the left, and Paul is 0.5m behind him. George sits 30% further back than Ringo on the right side of the pivot. If the see-saw is balanced, calculate the separation between Paul and George (in m)

(vii) A climber of mass 60kg is taking a rest on a long slab climb. The angle of the slope is 60° from the horizontal, and the climber is supported from her harness via a rope at angle 10° to the slope from her belaying partner above. If the coefficient of friction between the climbers shoes and the rock face is $\mu = 1.23$, calculate and inequality for the tension in the climbing rope (in N).

(viii) A mathematical cult calling themselves the *Neo-Pythagoreans* worship a symbol of π. One acolyte makes this symbol out of iron rods with mass per unit length of 31.4kg per metre. The two uprights of his symbol are vertical and 20cm long, and 15cm apart. The horizontal cap, perpendicular to the uprights is 19cm long.

(a) Calculate the location of the centre of mass from the bottom left upright.

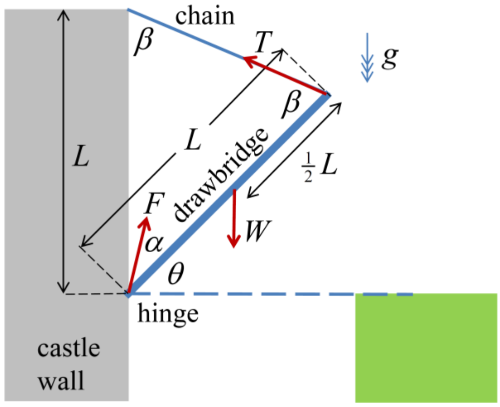
(b) Once the symbol is welded together it is temporarily hung over a bar protruding from a wall. Calculate the angle of tilt from the vertical of the uprights.

(ix) Show that the centre of mass of a uniform solid *paraboloid* of x, y cross section $y = \frac{h}{r^2}x^2$ is $\frac{2}{3}h$ from the apex, along the symmetry axis of the paraboloid.

(x) A uniform metal ladder of length L and mass m is placed against a frictionless vertical ice wall. The coefficient of friction between the base of the ladder and the rough gravel below the wall is μ and the angle of the ladder from the horizontal is θ. A climber of mass M climbs the ladder. Show that the ladder does not slip as long as:

$$\mu > \frac{M + \frac{1}{2}m}{M + m}\frac{1}{\tan\theta}$$

Question 2 A drawbridge of length L and weight W is lowered by a *pair* of chains, providing tension T at angle β to the plane of the drawbridge. Assume the drawbridge is lowered so slowly that it can be considered to be in equilibrium.



(a) Explain from the diagram why $\beta = 45^0 + \frac{1}{2}\theta$.

(b) By taking moments from the centre of the drawbridge, show that $F\sin\alpha = T\sin\beta$. Draw a vector triangle of the forces acting on the drawbridge, and hence show that $\alpha = 45^0 - \frac{1}{2}\theta$.

(c) Take moments about the hinge to show that $T = \frac{1}{2}W\cos\theta/\sin\beta$.

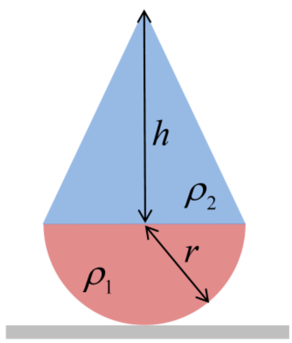
(d) Hence the addition formulae for sine and cosine to show that:

$T = \frac{1}{\sqrt{2}}W\left(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\right)$ and $F = \frac{1}{\sqrt{2}}W\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)$.

(e) Plot T, F, α vs θ assuming $L = 10.0\text{m}$, $W = 22{,}000\text{N}$.

Question 3 (a) Prove that the centre of mass of a *triangular lamina* is $\overrightarrow{AM} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$ where the vectors which form the triangle sides from vertex A position $\overrightarrow{OA}$ are $\mathbf{a}, \mathbf{b}$. (b) A uniform triangular lamina with base length $b = 30\text{cm}$ and adjecent side of length $a = 20\text{cm}$ at an elevation angle of 30°, is hung from the vertex where sides a, b meet. Determine the resulting angle of the base from the vertical.

Question 4 A *Weeble* consists of a solid hemisphere of density ρ_1 and radius r topped with a cone of density ρ_2, radius r and height h. When placed on a horizontal surface, the Weeble will always self-right to a vertical position.

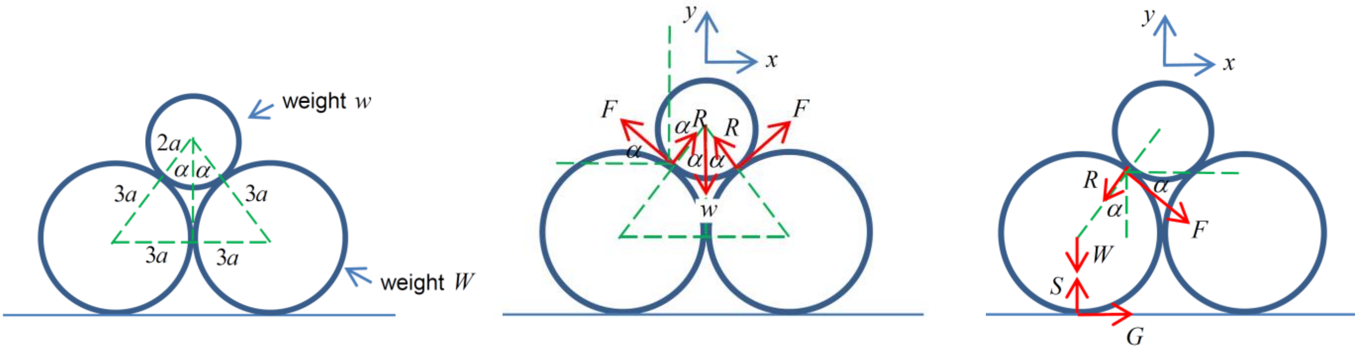


(a) Show that the centre of mass of the hemisphere is $\frac{3}{8}r$ from circular face of area πr^2.

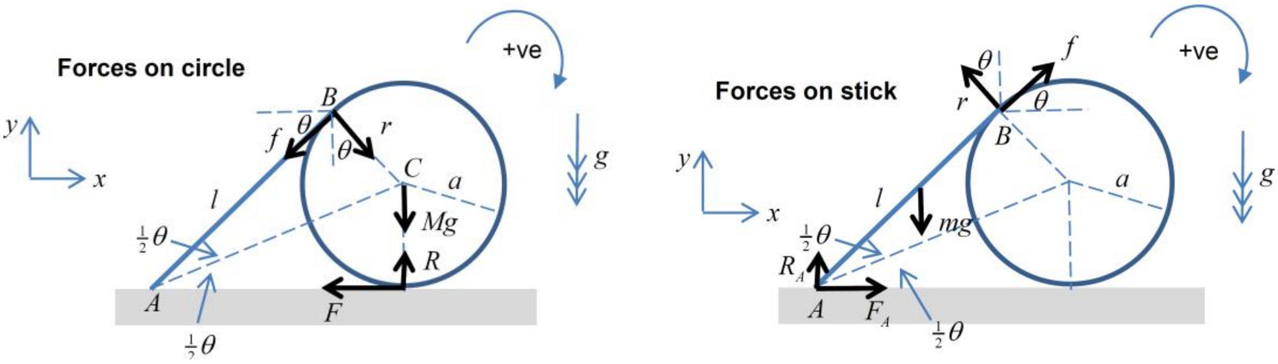
(b) Show that the centre of mass of the cone is $\frac{1}{4}h$ from the base.

(c) Hence show that if 'Weebles wobble but won't fall down' : $h < r\sqrt{3\rho_1/\rho_2}$

Question 5 Two solid cylindrical logs of radius $3a$ are placed next to each other, and another cylindrical log of the same density, but radius $2a$ is placed on top as shown below. The logs are in static equilibrium. The weight of the larger logs is W, and w for the smaller logs. Show that forces: $F=\frac{1}{6}w,\ G=\frac{1}{6}w,\ R=\frac{1}{2}w,\ S=W+\frac{1}{2}w$ and show for no slip at the ground-log interface, coefficient of friction $\mu_A >= \dfrac{w}{6W+3w}$. Also show for no slip at the log-log interface, coefficient of friction $\mu_B > \frac{1}{3}$. (Adapted from Quadling, *Mechanics M3&4* pp90-93)



Question 6 A rigid stick of length l and mass m rests on a cylinder of mass M and radius a.

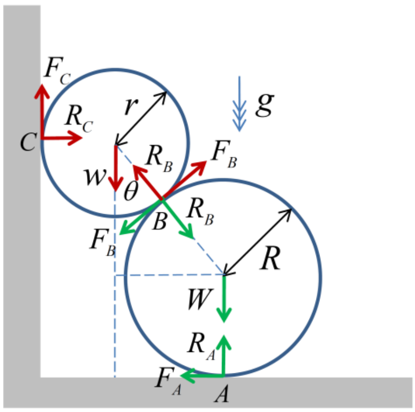


(a) Show that: $r=\frac{1}{2}mg\cos\theta,\ f=F=F_A=\dfrac{\frac{1}{2}mg\cos\theta\sin\theta}{1+\cos\theta}$, $R=Mg+\frac{1}{2}mg\cos\theta(2-\cos\theta), R_A=\frac{1}{2}mg(2-\cos\theta)$.

(b) Hence show the coefficient of static friction between stick and ground: $\mu_A > \dfrac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}$ and the coefficient of static friction between the stick and the cylinder: $\mu_B > \dfrac{\sin\theta}{1+\cos\theta}$.

(b) Sketch $\dfrac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}$ and $\dfrac{\sin\theta}{1+\cos\theta}$ vs θ. Hence show that if the stick does not slip against the cylinder, it will never slip against the ground if $\mu_A=\mu_B$. (Adapted from Morin, *Introduction to Classical mechanics*, pp38)

Question 7 A cylindrical cask of weight W and radius R rests on a rough horizontal floor. The coefficient of friction is μ_A. A smaller cask of weight w and radius r rests between the larger drum and a vertical wall. The coefficient of friction between the pair of casks is μ_B and the coefficient of friction between the smaller cask and the wall is μ_C.



(a) Find expressions for all the forces on both cylinders in terms of θ, w, W and hence show: $\mu_A > \dfrac{w\sin\theta}{W(1+\cos\theta+\sin\theta)+w(1+\cos\theta)}$, $\mu_B > \dfrac{\sin\theta}{1+\cos\theta}$, $\mu_C > 1$, and that $\dfrac{R-r}{R+r} \le \sin\theta \le 1$. *Hint:* as in questions 5 and 6, consider forces and turning moments on the two casks separately. Note the direction of the *equal and opposite* normal contact forces at *B*.

(b) Show that $\mu_A > \frac{1}{13}, \mu_B > \frac{1}{5}$ if $W=400\text{N},\ w=300\text{N},\ r=30\text{cm},\ R=35\text{cm}$ and the edge of the lower cask is 20cm from the wall. (Adapted from Quadling, *Mechanics M3&4* pp100)