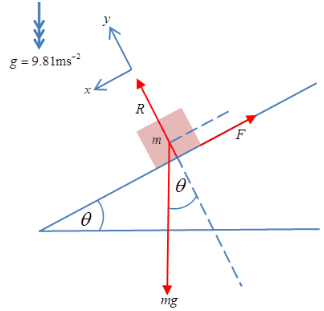
NOTE: No rotation, moments, torque. Only constant velocity circular motion. 1 & 2D constant acceleration motion.

Forces & Motion

Forces & Newton's Laws

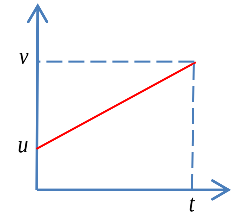
$$m\mathbf{a} = \sum_i \mathbf{f}_i$$

Resolving forces & Equilibrium

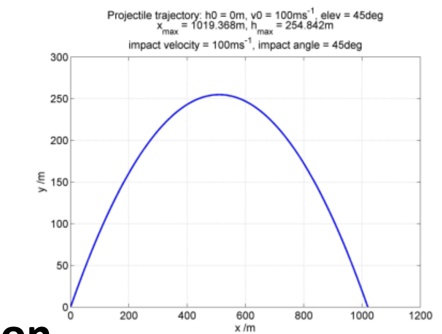


Constant acceleration motion (1D)

$$v^2 = u^2 + 2ax$$
$$v = u + at$$
$$x = ut + \tfrac{1}{2}at^2$$
$$x = \tfrac{1}{2}(u+v)t$$

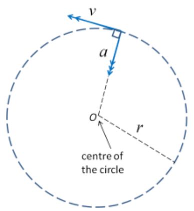


Projectile motion

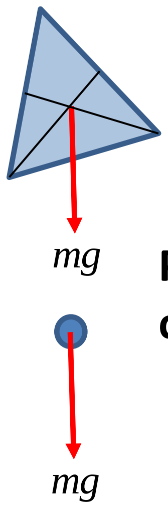


Circular motion

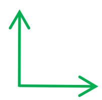
$v = 2\pi r/T \quad a = v^2/r$



Particles & centre of mass



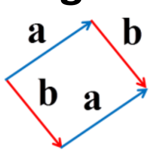
Frames of reference



Work & power $P = Fv$

$W = F\Delta x$

Using vectors



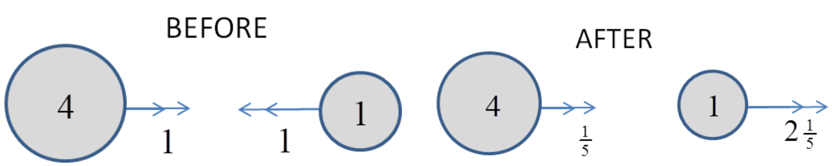
Conservation of momentum and collisions

Impulse

$\mathbf{I} = \Delta(m\mathbf{v})$

Restitution

$$C = \frac{\text{speed of separation}}{\text{speed of approach}}$$



Conservation of energy $E = \tfrac{1}{2}mv^2 + mgh + \tfrac{1}{2}kx^2$

Forces

Air resistance & lift

$F = Kv^2$

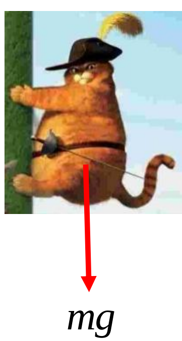
Friction & normal contact forces

$F \le \mu R$

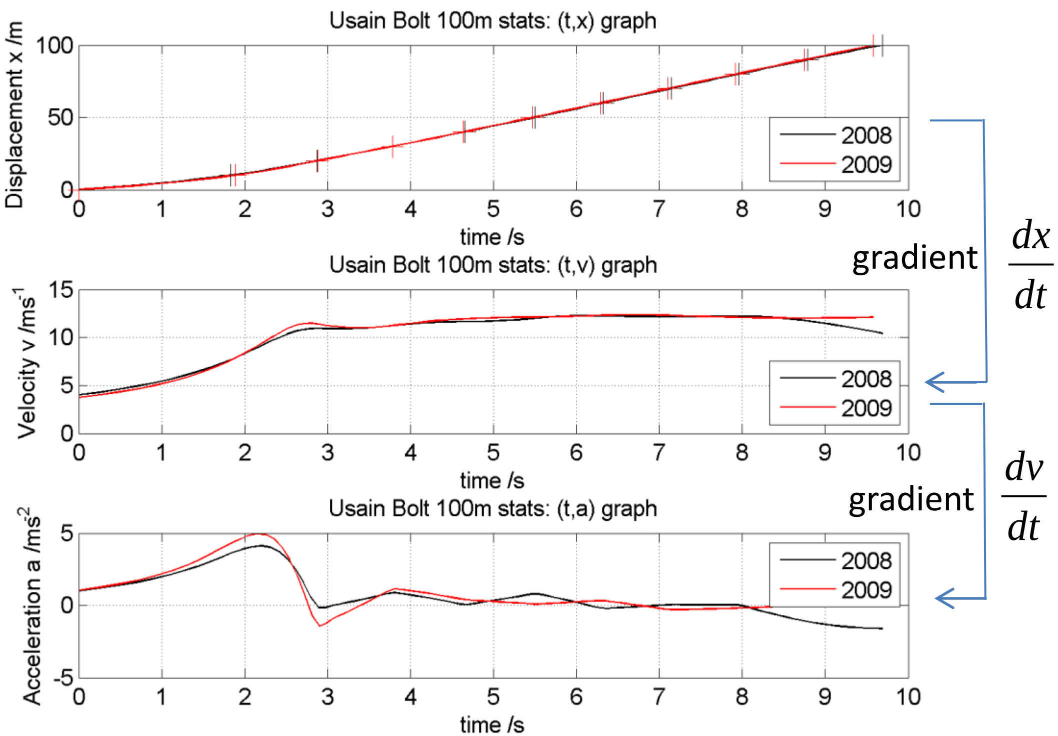
Elasticity $F = kx$

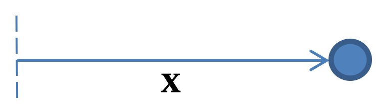
$E = \tfrac{1}{2}kx^2$

Gravity & weight

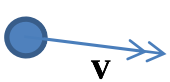


Relationship between displacement, velocity and acceleration

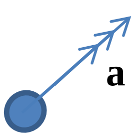




Displacement is the vector between a fixed origin and the point of interest. If an object is moving, the displacement will vary with time *t*



Velocity is the *rate of change of displacement*. If velocity is in the same direction as
displacement, it is the gradient of a *(t,x)* graph.



Acceleration is the *rate of change of velocity*. If acceleration is in the same direction as velocity, it is the gradient of a *(t,v)* graph.

Useful speed conversions:

1 ms^{-1} = 2.24 miles per hour

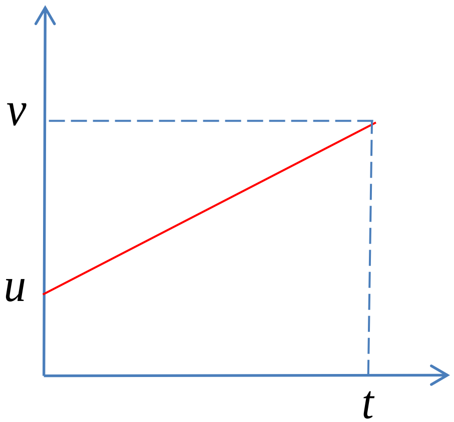
1 ms^{-1} = 3.6 km per hour

$$t\,/\,\mathrm{min} = 60 \times \frac{x\,/\,\mathrm{miles}}{v\,/\,\mathrm{mph}}$$

Speed in mph	Time in minutes per 10 miles
10	60
20	30
30	20
40	15
50	12
60	10
70	8.57

Constant acceleration motion

It is almost *always* a good idea to start with a *(t,v)* graph.
Let velocity increase at the same rate *a* from *u* to *v* in *t* seconds.



The acceleration is the gradient: $a = \dfrac{v-u}{t} \quad \therefore \quad \boxed{v = u + at}$

The area under the graph is the displacement.
Since this a trapezium shape: $\boxed{x = \tfrac{1}{2}(u+v)t}$

We can work out other useful relationships for constant acceleration motion

$$x = \tfrac{1}{2}(u + u + at)t$$

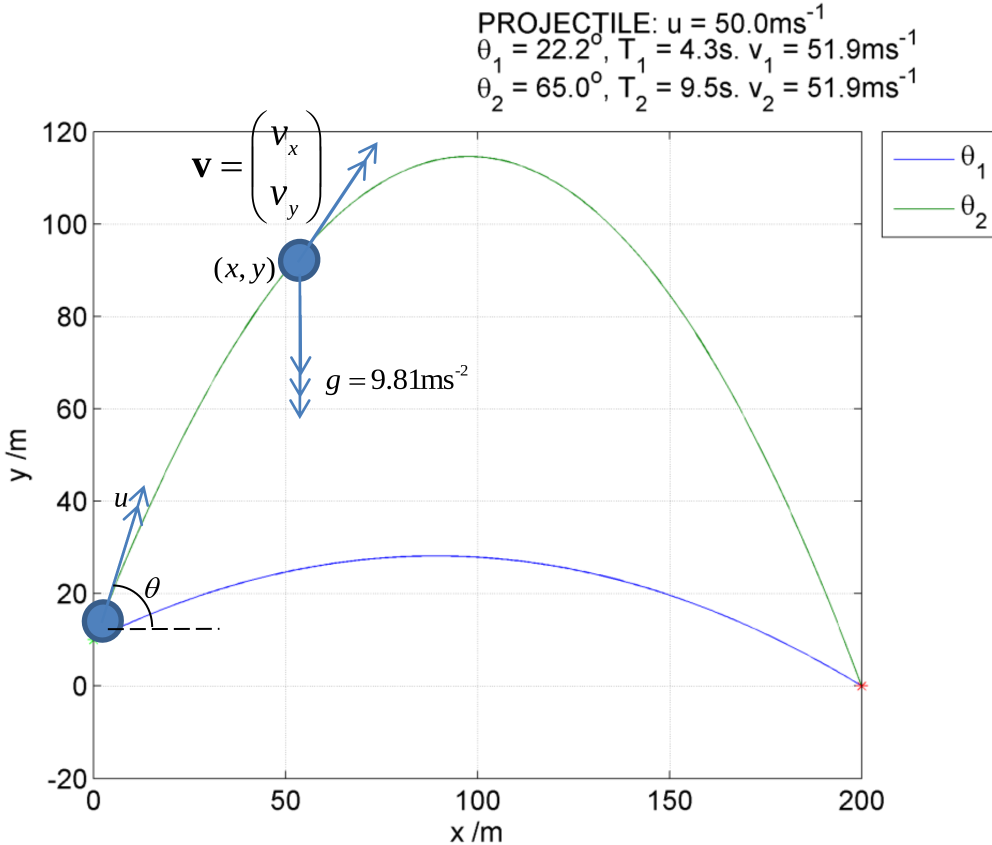
$$\boxed{x = ut + \tfrac{1}{2}at^2}$$

$$x = ut + \tfrac{1}{2}at^2$$

$$2ax = 2uat + a^2t^2$$

$$v^2 = (u+at)^2 = u^2 + 2uat + a^2t^2$$

$$\therefore \boxed{v^2 = u^2 + 2ax}$$



$$\boxed{\begin{aligned} v_x &= u\cos\theta \\ v_y &= u\sin\theta - gt \\ v_y^2 &= u^2\sin^2\theta - 2g(y - y_0) \\ x &= ut\cos\theta \\ y &= y_0 + ut\sin\theta - \tfrac{1}{2}gt^2 \end{aligned}}$$

$$x = ut\cos\theta$$

$$\therefore\ t = \frac{x}{u\cos\theta}$$

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$\therefore\ \boxed{y = y_0 + x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2}$$

Trajectory equation
An inverted parabola!

The *apogee* of the trajectory is when v_y=0

$$v_y = u\sin\theta - gt \quad \therefore \quad v_y = 0 \Rightarrow \boxed{t_a = \frac{u\sin\theta}{g}}$$

$$v_y^2 = u^2\sin^2\theta - 2g(y - y_0) \quad \therefore \quad v_y = 0 \Rightarrow \boxed{y_a = y_0 + \frac{u^2\sin^2\theta}{2g}}$$

$$x_a = ut_a\cos\theta \quad \therefore \quad \boxed{x_a = \frac{u^2\sin\theta\cos\theta}{g}}$$

So max range when

$$\theta = 45^{\circ}$$

Since parabola is *symmetric:*
When *y = y₀, x = R*

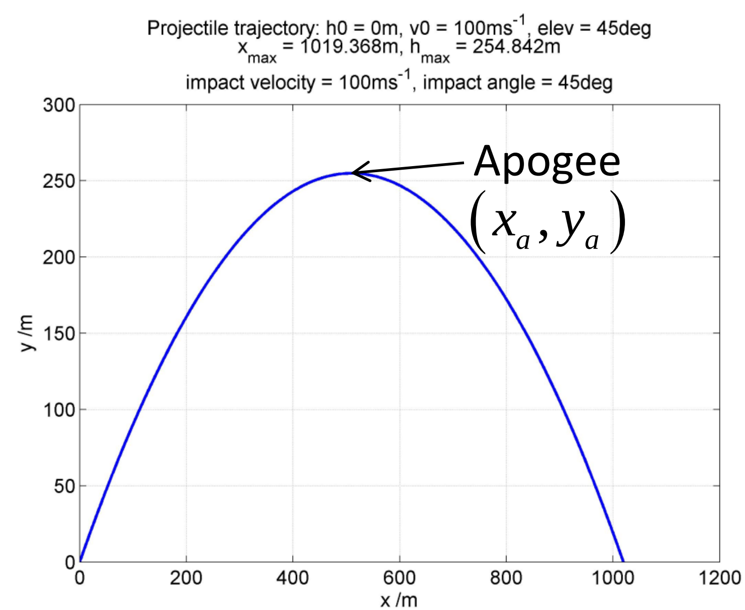
$$R = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g}$$

The speed *v* of the projectile is:

$$v = \sqrt{v_x^2 + v_y^2}$$

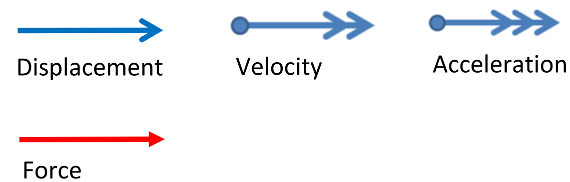
$$v = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta - 2g(y - y_0)}$$

$$\boxed{v = \sqrt{u^2 - 2g(y - y_0)}}$$



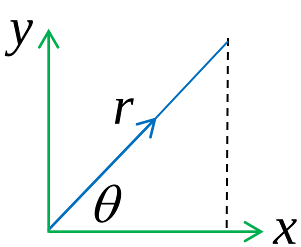
Vector and scalar quantities

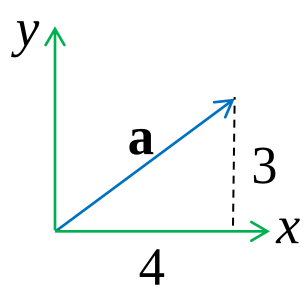
Vector quantities		Units	Scalar quantities	Units
Displacement	$\mathbf{x}$	$\mathrm{mm, m, km}$	Mass m	kg
Velocity	$\mathbf{v}$	$\mathrm{ms^{-1}, kmh^{-1}}$	Time t	$\mathrm{s, mins, h}$
Acceleration	$\mathbf{a}$	$\mathrm{ms^{-2}}$	Speed s	$\mathrm{ms^{-1}, kmh^{-1}}$
Momentum	$\mathbf{p} = m\mathbf{v}$	$\mathrm{kgms^{-1}}$	Length l	$\mathrm{mm, m, km}$
Force	$\mathbf{f}$	N (Newtons)		



A vector has both *magnitude* and *direction*.

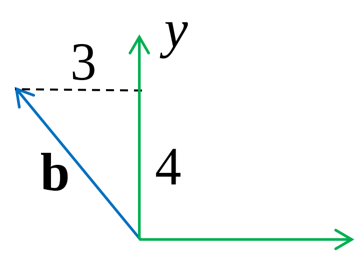
$\mathbf{r}$ $\quad |\mathbf{r}| = r$ $\quad \theta$





$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$|\mathbf{a}| = 5$$



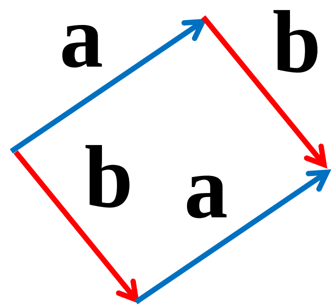
$$\mathbf{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$|\mathbf{b}| = 5$$

Addition and scalar multiplication using vectors

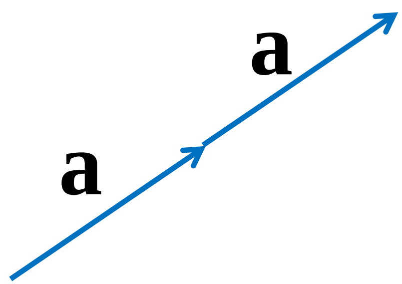
The algebra of vectors is very similar to scalars. Except *vector multiplication is very different*. This will not be discussed in this course!

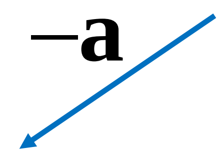
$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$



Vectors add ‘tip to tail’

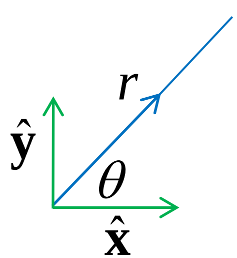
$$2\mathbf{a}=\mathbf{a}+\mathbf{a}$$





Components of a vector are with respect to a **coordinate system**

These are **unit vectors** in the x and y directions



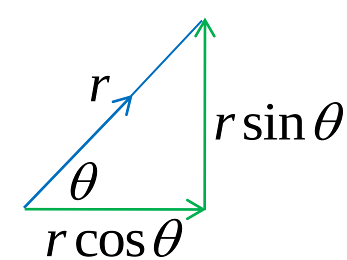
$$|\hat{\mathbf{y}}|=|\hat{\mathbf{x}}|=1$$

$$\mathbf{r}=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$r\cos\theta$ ← x component

$r\sin\theta$ ← y component

We often speak of ‘resolving’ a vector into components $\mathbf{r}=r\cos\theta\,\hat{\mathbf{x}}+r\sin\theta\,\hat{\mathbf{y}}$

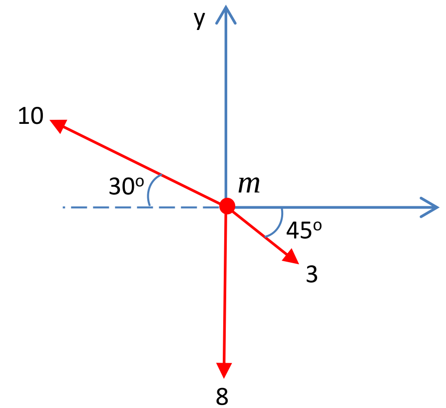


Newton's three laws of motion

Newton's First Law

If **no net force, acceleration is zero**, which means **velocity = constant**

$$\mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v} = \text{constant}$$



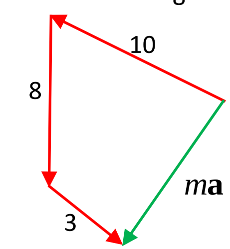
Newton's Second Law

$$m\mathbf{a} = \sum_i \mathbf{f}_\mathbf{i}$$

mass x acceleration = vector sum of forces

'inertia'

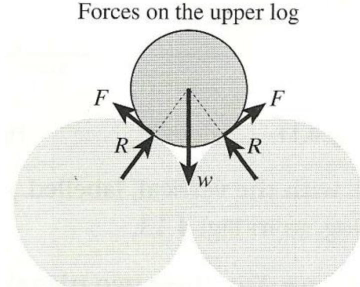
Resolving forces



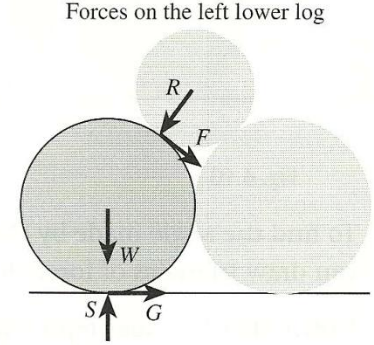
Newton's third law

If an object *A* exerts a force on an object *B*, then *B* exerts a force on *A* of the same magnitude along the same line but in the opposite direction.

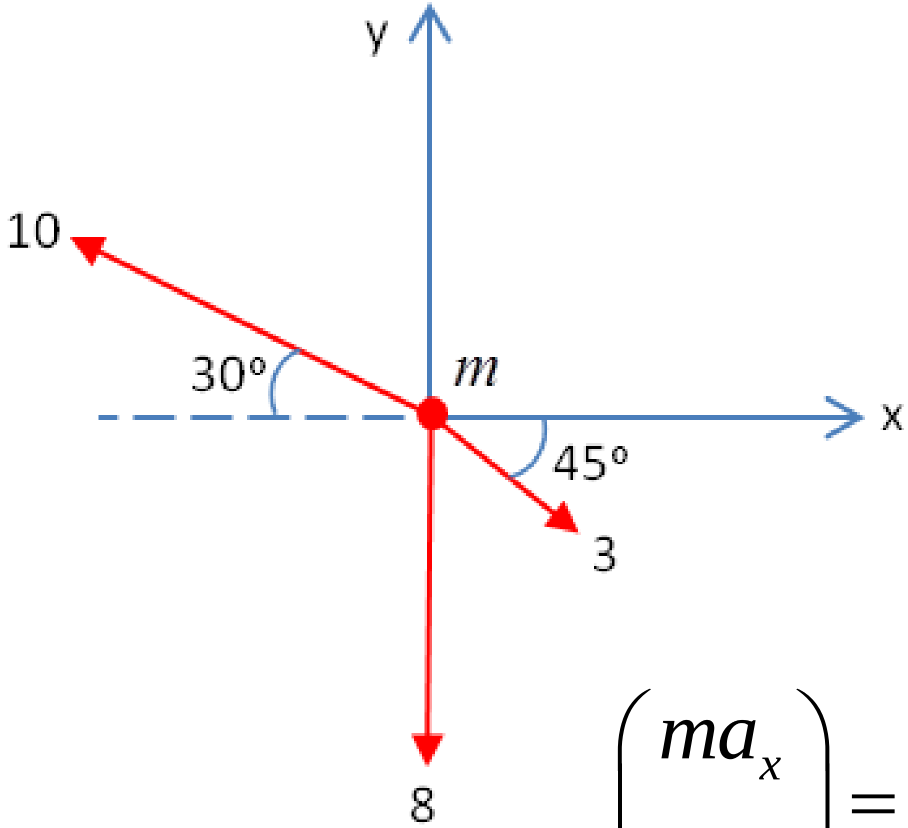
Forces on the upper log



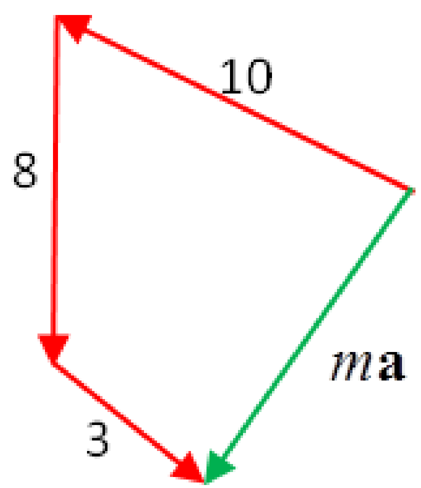
Forces on the left lower log



Resolving forces



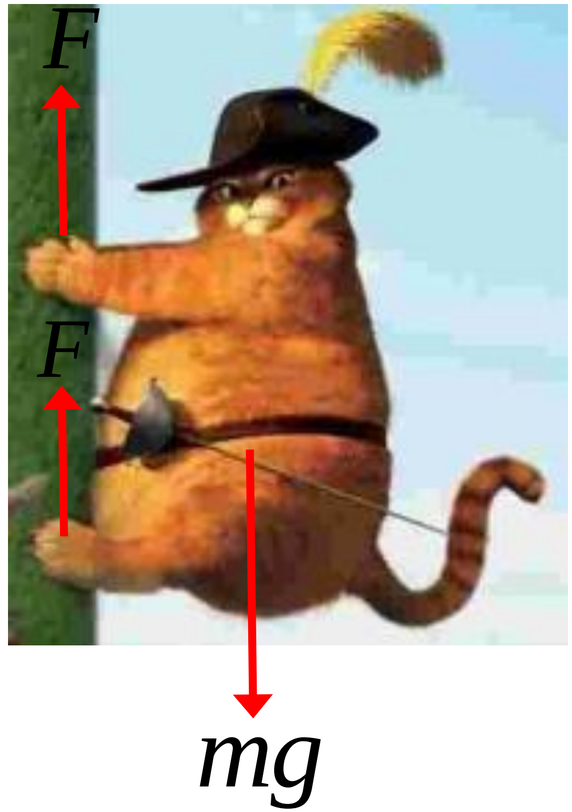
$$\begin{pmatrix} ma_x \\ ma_y \end{pmatrix} = \begin{pmatrix} -10\cos 30^{\circ} + 3\cos 45^{\circ} \\ 10\sin 30^{\circ} - 3\sin 45^{\circ} - 8 \end{pmatrix}$$



Newton II:
mass x acceleration = vector sum of forces

Gravity & weight

A weighty puss indeed....



$$g = 9.81\text{ms}^{-2}$$

gravitational field strength on the surface of the Earth

The gravitational force *mg* on a mass of *m* kg is called its ***weight***.

It is measured in Newtons.

Therefore a 70kg man weighs 686.7N *on Earth*.

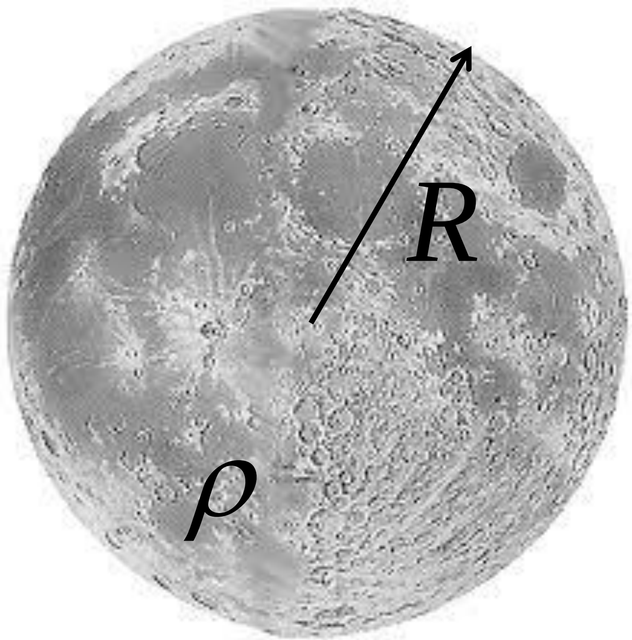
g depends on the **mass** and **radius** of a planet

$$g_{mars} = 3.71\text{ms}^{-2}$$

$$g_{moon} = 1.63\text{ms}^{-2}$$

The force due to gravity upon a mass of *m* kg is ***mg*** where *g* is the *gravitational field strength*.

Amazingly, 'gravitational mass' appears to be the **same** as the **inertia** in Newton II i.e. **inertia x acceleration = vector sum of forces.** Therefore gravitational field strength is the **acceleration of a particle freely falling** (i.e. where other forces such as drag are not acting).



Newton's law of universal gravitation states that the gravitational field strength at a distance *R* from a spherical object is proportional to the mass contained within a sphere of radius *R* centred on the object and inversely proportional to R^2

$$\boxed{g = \frac{GM}{R^2}}$$

$G = 6.67 \times 10^{-11}\ \mathrm{m^3\ kg^{-1} s^{-2}}$

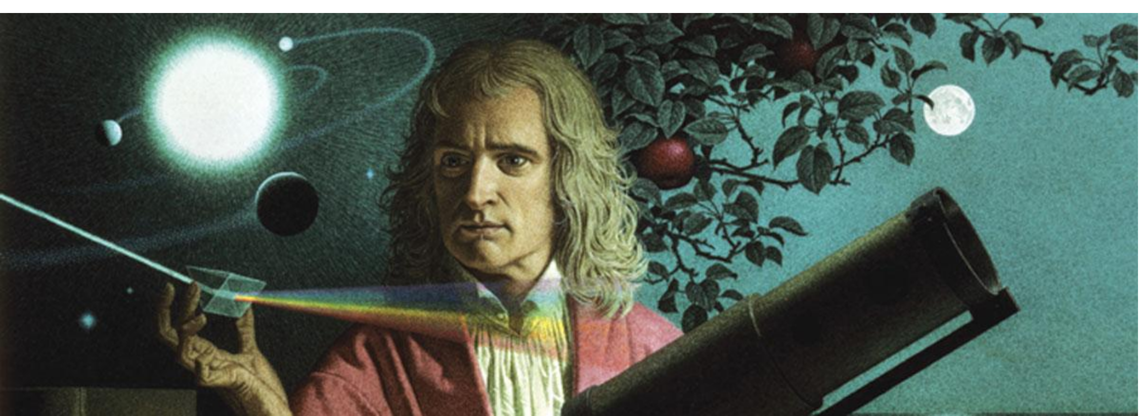
If a planet has *uniform density* ρ $\quad \boxed{M = \tfrac{4}{3}\pi R^3 \rho}$

$$\therefore g = \frac{G}{R^2}\tfrac{4}{3}\pi R^3 \rho$$

$$\Rightarrow \boxed{g = \tfrac{4}{3}\pi G \rho R}$$

$$R_{moon} = 1.737 \times 10^6\ \mathrm{m}$$
$$g_{moon} = 1.63\mathrm{ms^{-2}}$$
$$\rho = 3359\ \mathrm{kgm^{-3}}$$

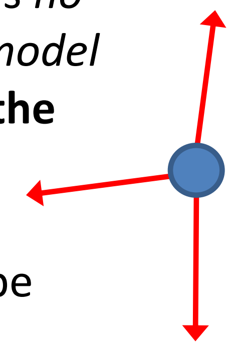
$$R_{earth} = 6.371 \times 10^6\ \mathrm{m}$$
$$g_{earth} = 9.81\mathrm{ms^{-2}}$$
$$\therefore \rho = 5511\mathrm{kgm^{-3}}$$
$$R_{moon} = 1.737 \times 10^6\ \mathrm{m}$$
$$g_{moon} = 1.63\mathrm{ms^{-2}}$$
$$\therefore \rho = 3359\ \mathrm{kgm^{-3}}$$



Isaac Newton
1643-1727

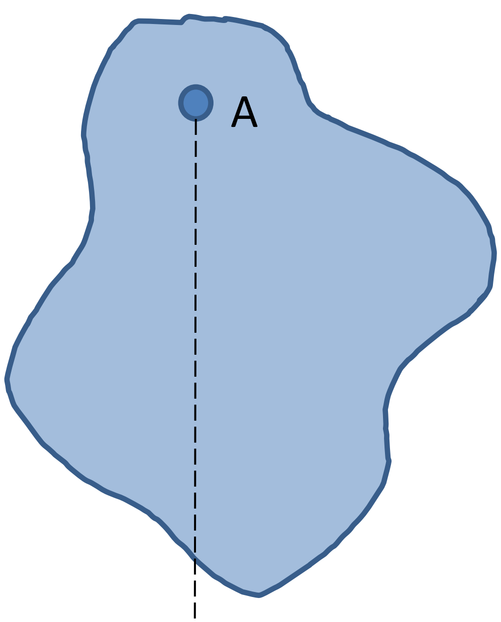
Particles & centre of mass

A *particle* is an object which has mass (and forces can act upon it) but it has *no extension.* i.e. it is located at a point in space. If objects are **rigid**, we can ‘*model them as particles’* since one can decompose motion into **displacement of the centre of mass + rotation of an object about the centre of mass.**



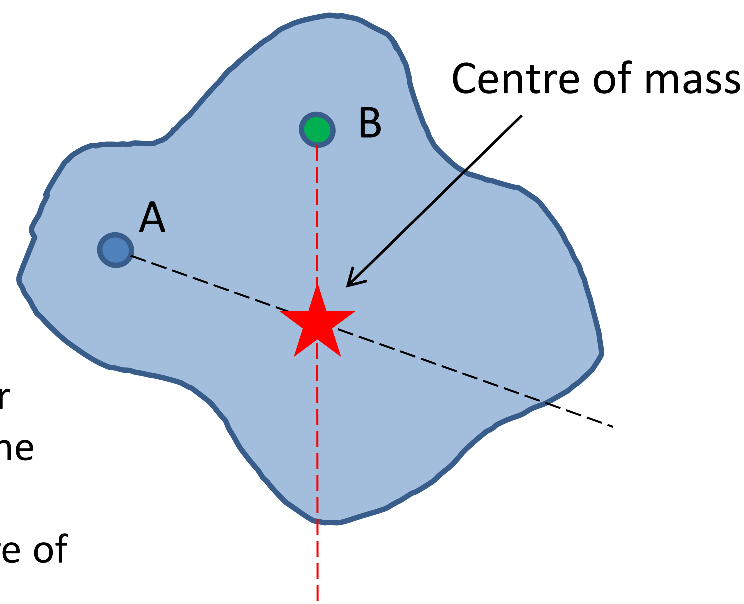
The centre of mass is the point where the entire weight of the object can be balanced without causing a turning moment about this point.

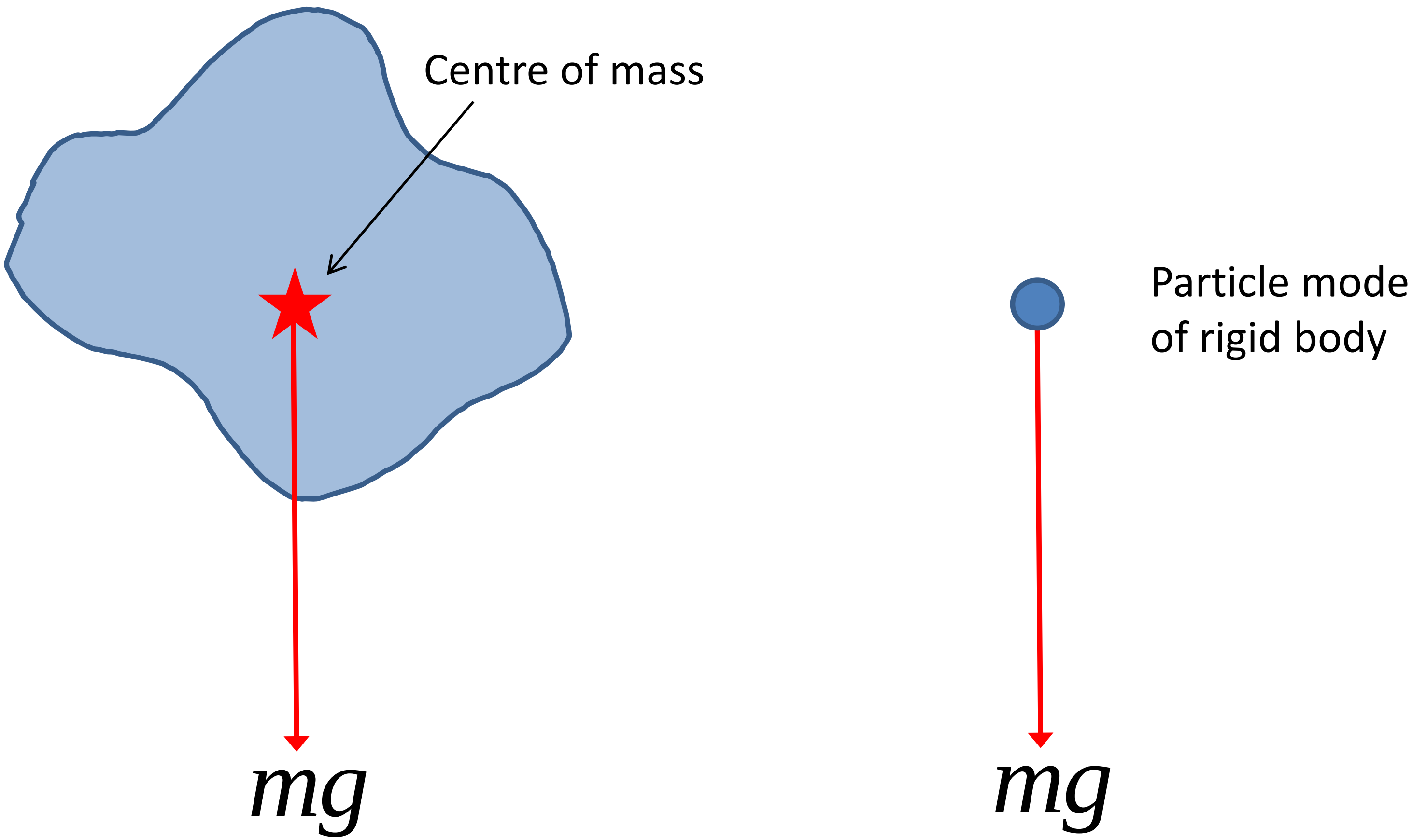
It can be found practically by hanging a 2D object from various positions and working out where the plumb lines intersect.



Hang object from position A and draw on plumb line

Hang object from position B and draw on another plumb line. Where the two plumb lines intersect is the centre of mass.

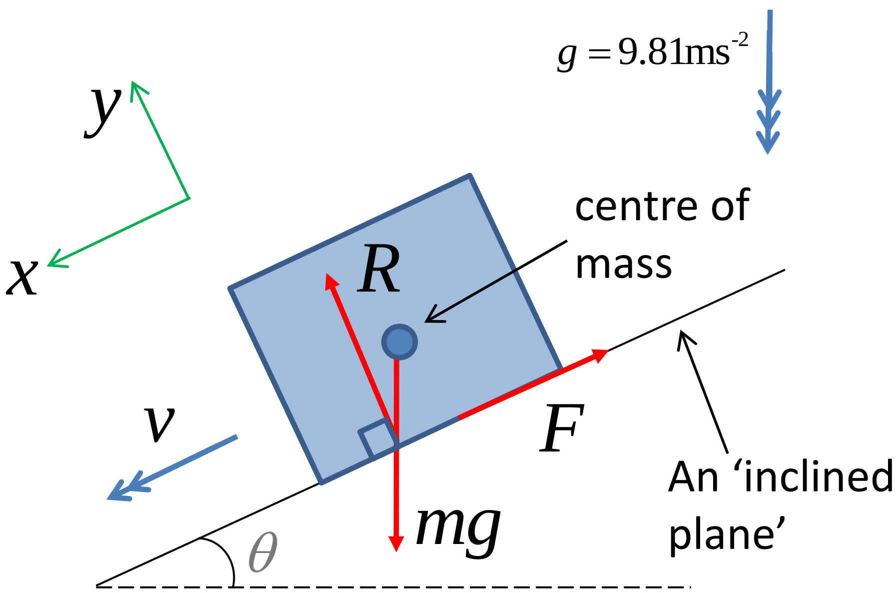




The entire weight of a rigid object effectively acts upon its **centre of mass**.

If rotation is ignored, we can model a rigid object as a **particle** i.e. just consider the motion of the centre of mass

Friction & Normal contact forces



By Newton's Third Law, if you push against a surface with force R, the surface will push back at you with a force of the same magnitude, but in the opposite direction

Contact forces can be usefully decomposed into **normal contact** (perpendicular to a surface) and **friction** (parallel to the surface), which always *opposes* motion.

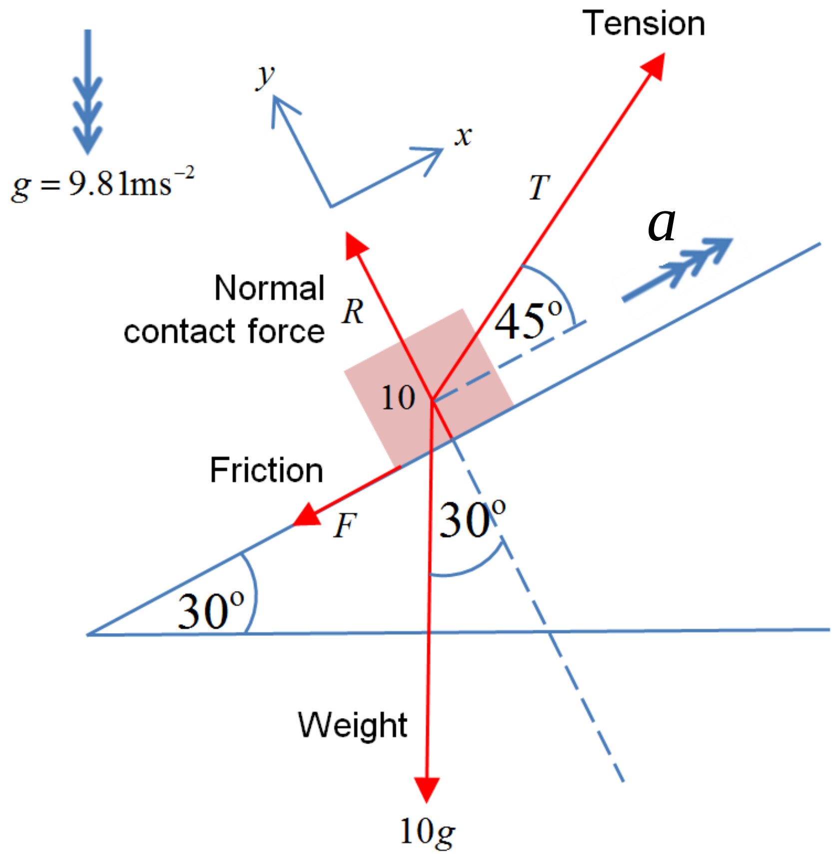
The normal contact force 'acts' at the point of intersection of a vertical '*plumb line*' from the *centre of mass* of the object.

Models of friction & sliding

$F < \mu_{static} R$	No sliding, and object is in **static equilibrium**
$F = \mu_{static} R$	Object is on the point of sliding – friction is 'limiting'
$F = \mu_{slide} R$	$v > 0$ i.e. object is sliding

μ **Coefficients of friction**. Typically <<1. We often assume $\mu_{static} \approx \mu_{slide}$

Resolving forces and applying Newton's Second Law



A mass of 10kg is being pulled up a rough slope by a tow rope which provides tension *T*

It accelerates up the slope with acceleration *a*

To calculate this we would need a model for the friction force *F*

e.g. $F = 0.1R$

coefficient of friction

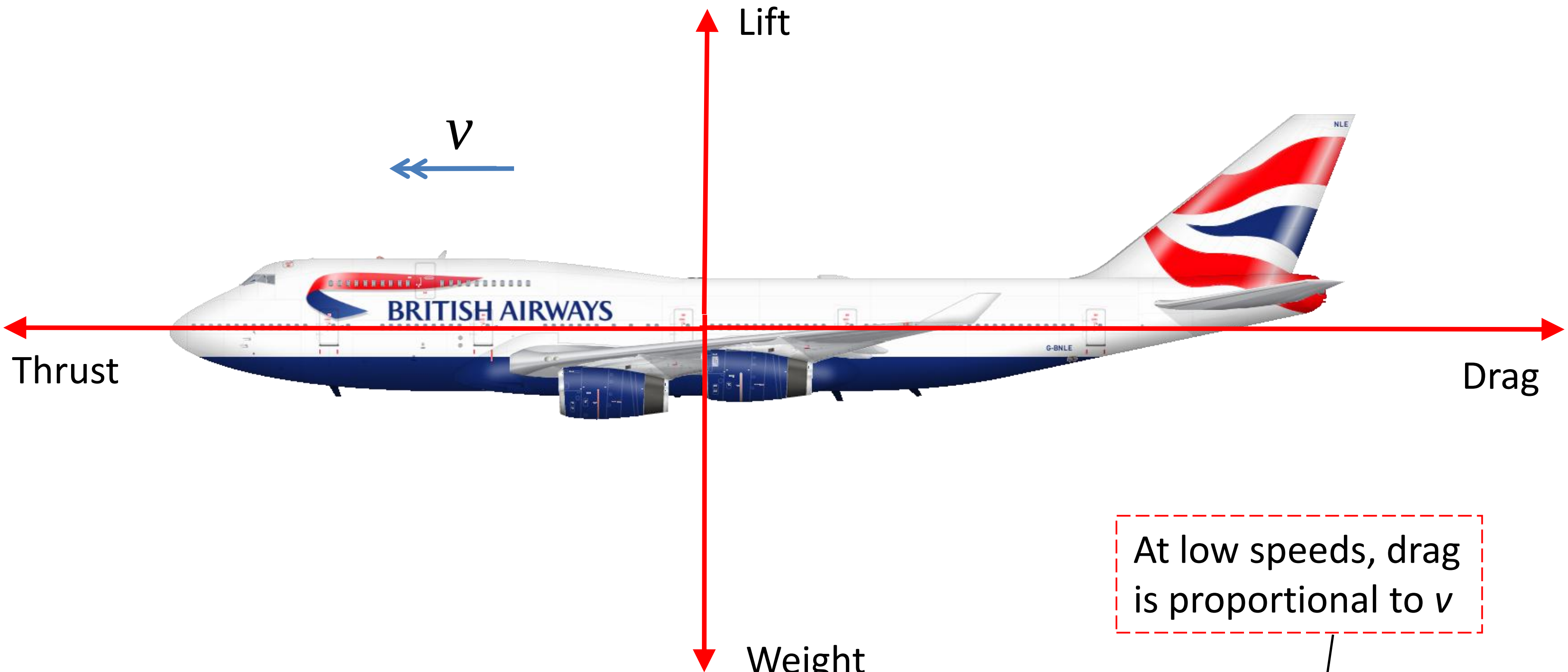
Resolve parallel to x and y directions →

$$// x: 10a = T\cos 45^\circ - F - 10g\sin 30^\circ$$

$$// y: 0 = R + T\sin 45^\circ - 10g\cos 30^\circ$$

Air resistance & lift

If an aircraft has a constant airspeed then it is not accelerating. Therefore the vector sum of all forces must be zero



At low speeds, drag is proportional to *v*

At 'modest speeds' (i.e. several ms^{-1}), both lift an drag forces are typically $\propto v^2$

$$F_{drag} = \tfrac{1}{2} c_D \rho A v^2$$

Drag coefficient
Typically << 1

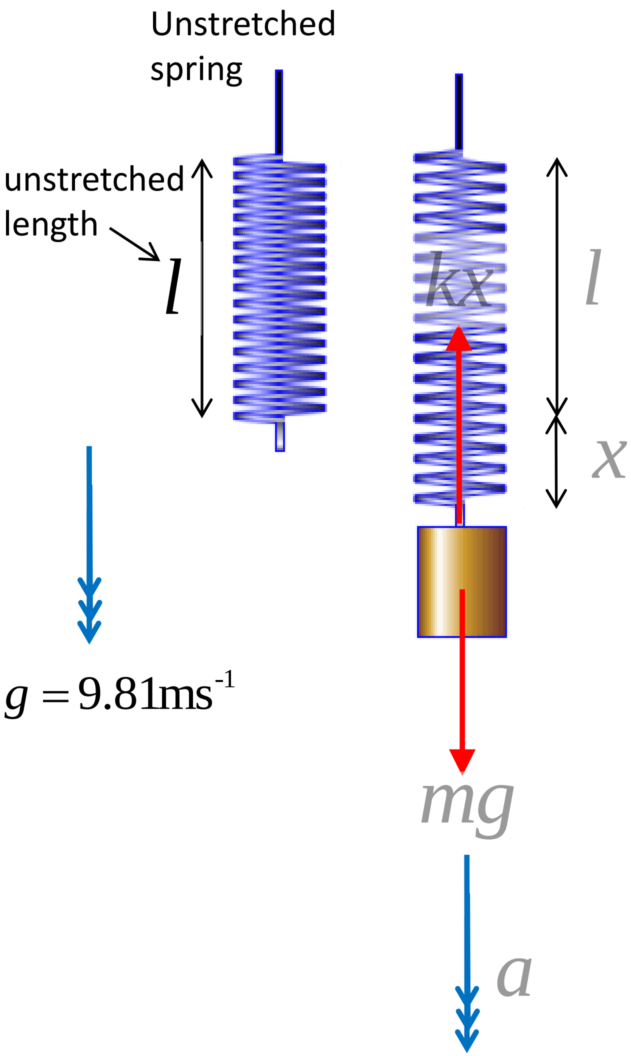
Density of air

Cross sectional area of aircraft perpendicular to velocity

Aerodynamics of a sportscar (and driver!) being analysed using a wind tunnel

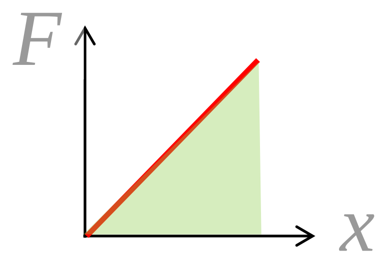
Elasticity

Elastic materials can be modelled by springs. Hooke’s law means the *restoring force* due to a spring stretched by extension *x* is *proportional to the extension*



By Newton II applied to the mass attached to the spring:

$$ma = mg - kx$$



The work done by the restoring force, if ‘left to its own devices’ is called the **elastic potential energy**. This is the area under the (displacement, force) graph. Since triangular in shape for a ‘Hookean spring’ :

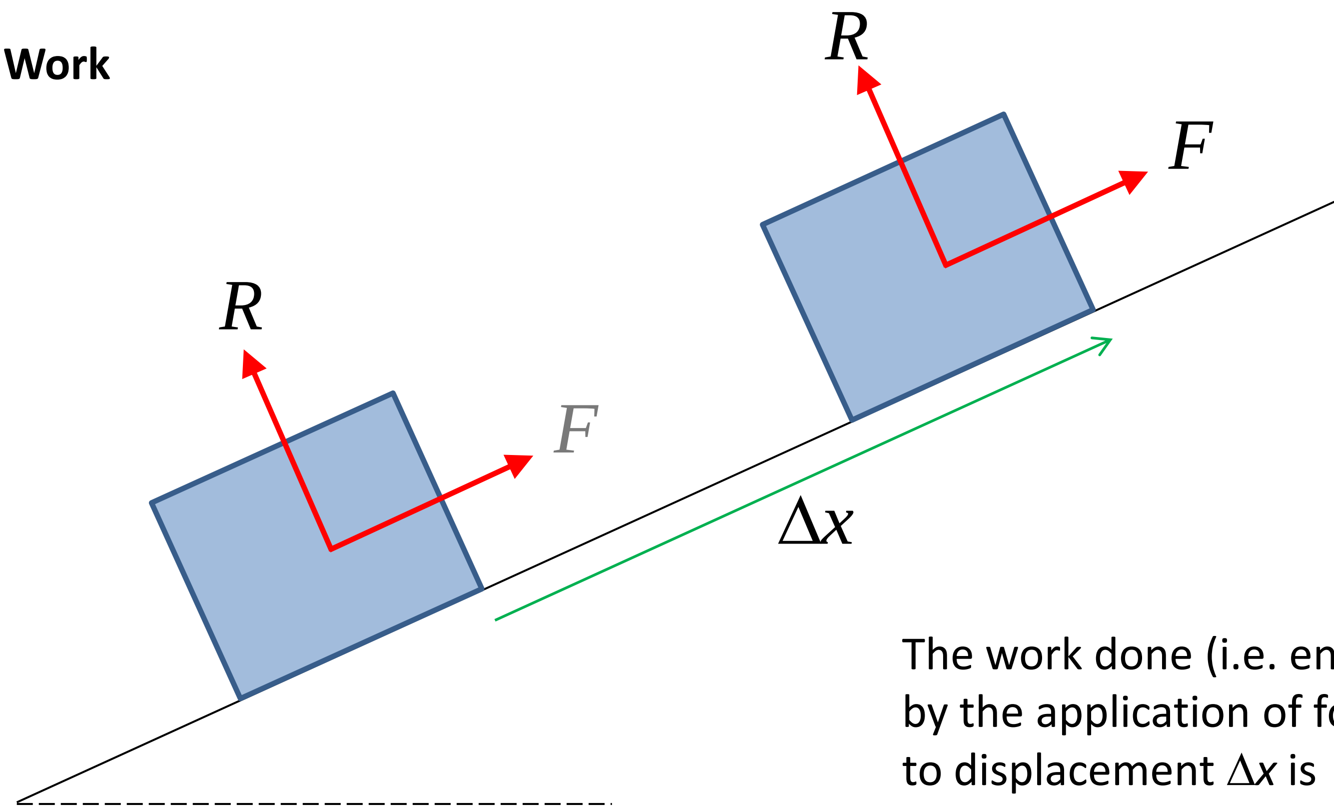
$$\boxed{E = \tfrac{1}{2}kx^2}$$

Hookes’ Law

k is the spring constant, alternatively expressed in terms of an *elastic modulus* λ

$$\boxed{F = kx = \frac{\lambda}{l}x}$$

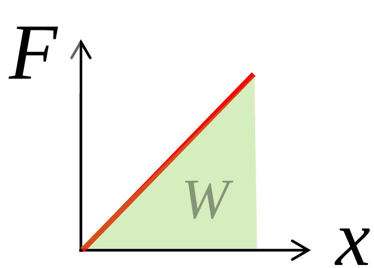
Work



The work done (i.e. energy transferred) by the application of force *F* parallel to displacement Δx is

$$\boxed{W = F\Delta x}$$

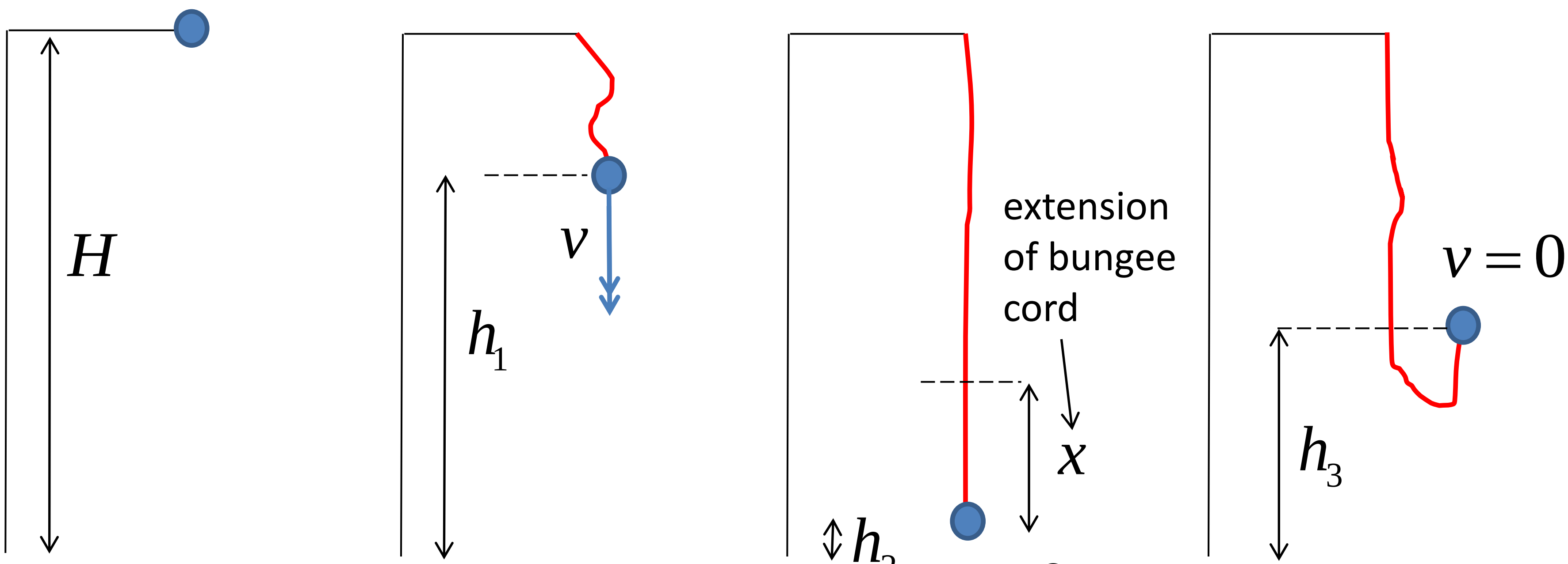
For *varying forces*, the work done is more generally the **area under the (displacement, force) graph**



Note there is **no work done** by any component of a force **perpendicular** to the **displacement.** i.e. force *R* does no work.

Conservation of energy $E = \text{constant}$

$$E = \underbrace{\tfrac{1}{2}mv^2}_{\text{Kinetic*}} + \underbrace{\tfrac{1}{2}kx^2}_{\text{Elastic potential energy}} + \underbrace{mgh}_{\text{Gravitational potential energy}} + \overbrace{\text{losses}}^{\text{Drag, friction etc}}$$



$E = mgH$

$E = mgh_1 + \tfrac{1}{2}mv^2$

$E = mgh_2 + \tfrac{1}{2}kx^2$

$E = mgh_3 + \text{losses}$

*Not just movement of the centre of mass, in general we must include vibration, rotation etc

The *rate* of work done is **power**

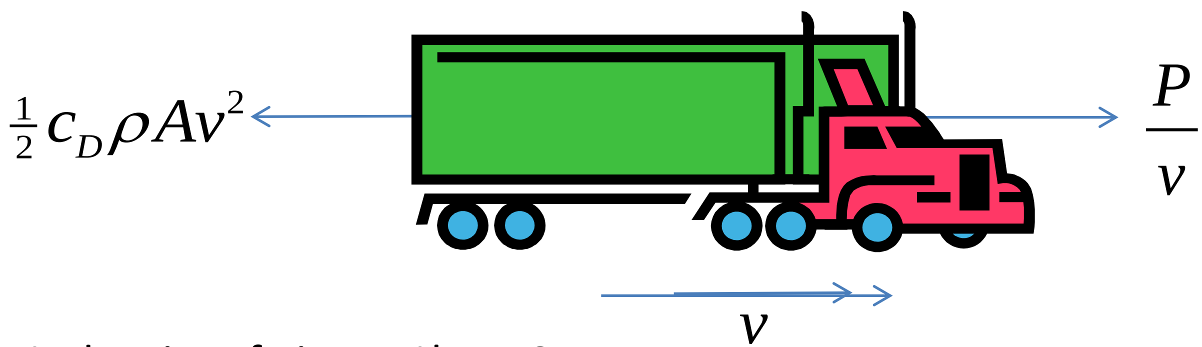
$$P = F\frac{\Delta x}{\Delta t} \qquad \therefore \boxed{P = Fv}$$

A lorry is travelling a constant speed of 60 mph. If friction between the tyres and the road can be ignored at this speed, and internal losses such has heating etc can be ignored, the *driving force* of the engine is balanced by *air resistance*. If the cab has a cross section of 8 m^2, estimate the engine power *P*.

Since lorry is in equilibrium, **driving force = air resistance**

$$\tfrac{1}{2}c_D\rho A v^2 = \frac{P}{v}$$

$$\therefore P = \tfrac{1}{2}c_D\rho A v^3$$

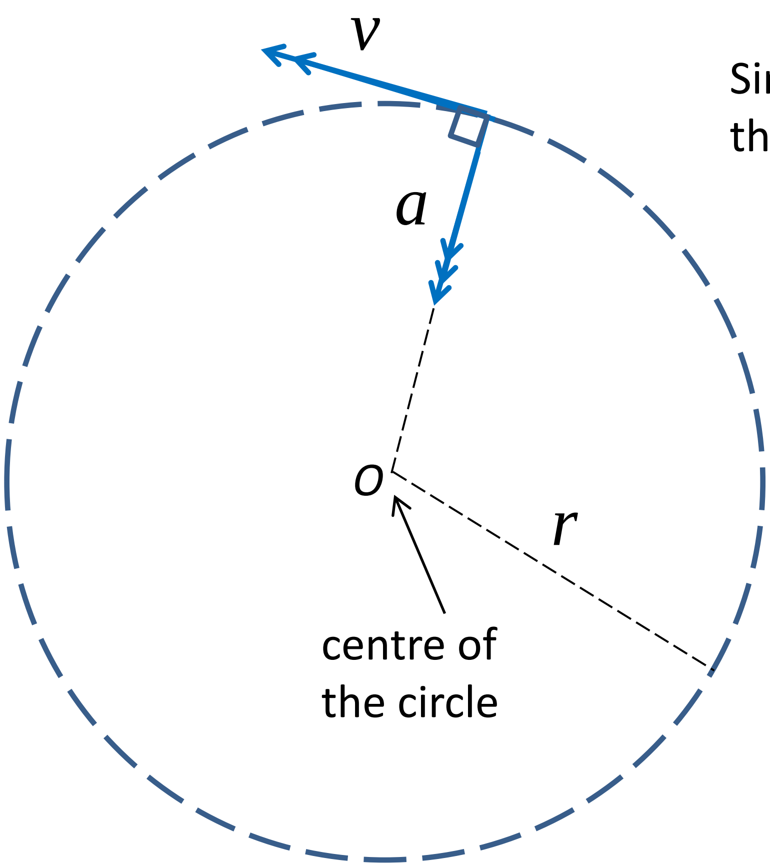


Assume drag coefficient c_D = 1, density of air ρ = 1kgm-3
v = 60/2.34 = 25.64ms^{-1}

$$P = \tfrac{1}{2}\times 1\times 1\times 8\times 25.64^3$$

$$\boxed{P \approx 67.4\text{kW}}$$

Motion in a horizontal circle



A particle moves around a circle of radius *r* at a constant speed *v*.

Since the *direction* of the velocity changes constantly, the particle must be **accelerating**

$$\boxed{v = \frac{2\pi r}{T}}$$

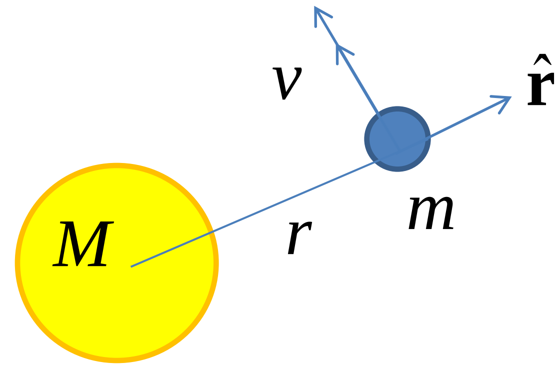
Time taken for one complete revolution

$$\boxed{a = \frac{v^2}{r}}$$

Centripetal acceleration – always towards the centre of the circle

What is the orbital speed of the Earth about the Sun, assuming a circular orbit? How does orbital radius and period vary?

Assume a *circular* orbit (ellipses are more accurate, but circular orbits are a good approximation for many planets in the solar system)



Newton II in the radial direction:

$$\text{mass x acceleration} \rightarrow \frac{mv^2}{r} = \frac{GMm}{r^2} \leftarrow \text{Newton's model of gravitational force}$$

$$\therefore v = \sqrt{\frac{GM}{r}}$$

$$v = \frac{2\pi r}{T}$$

$$\therefore \frac{4\pi^2}{T^2} r = \frac{GM}{r^2}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM} r^3} \leftarrow \text{Kepler's Third Law}$$

Let the Earth be mass *m* and the Sun mass *M*

M = 2 x 10^{30} kg

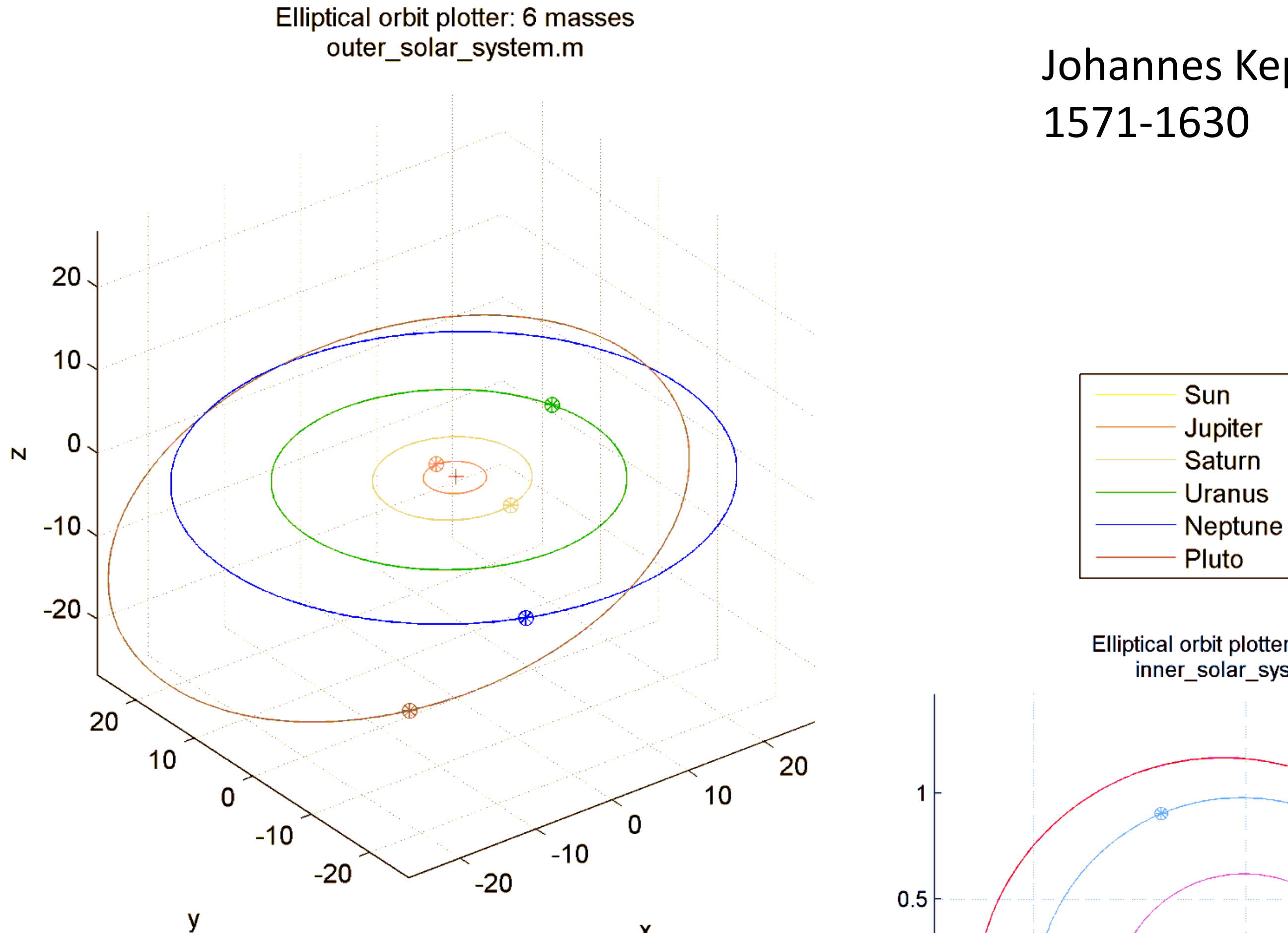
G = 6.67 x 10^{-11} m^3 $kg^{-1}s^{-2}$

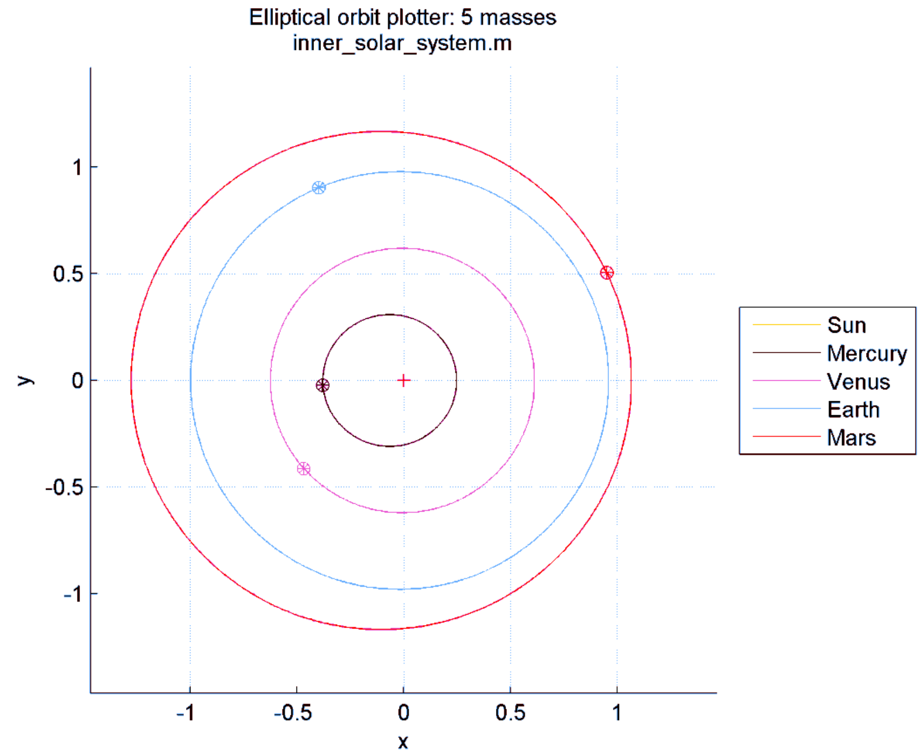
T = 365 days = 3.154 x10^7s

r = **150 million km**

v = **29.8 kms^{-1}**

Johannes Kepler
1571-1630



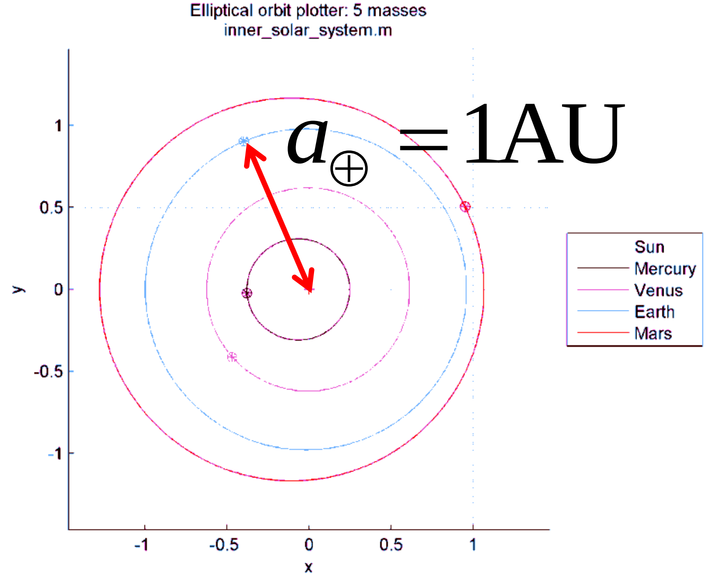


The Solar System

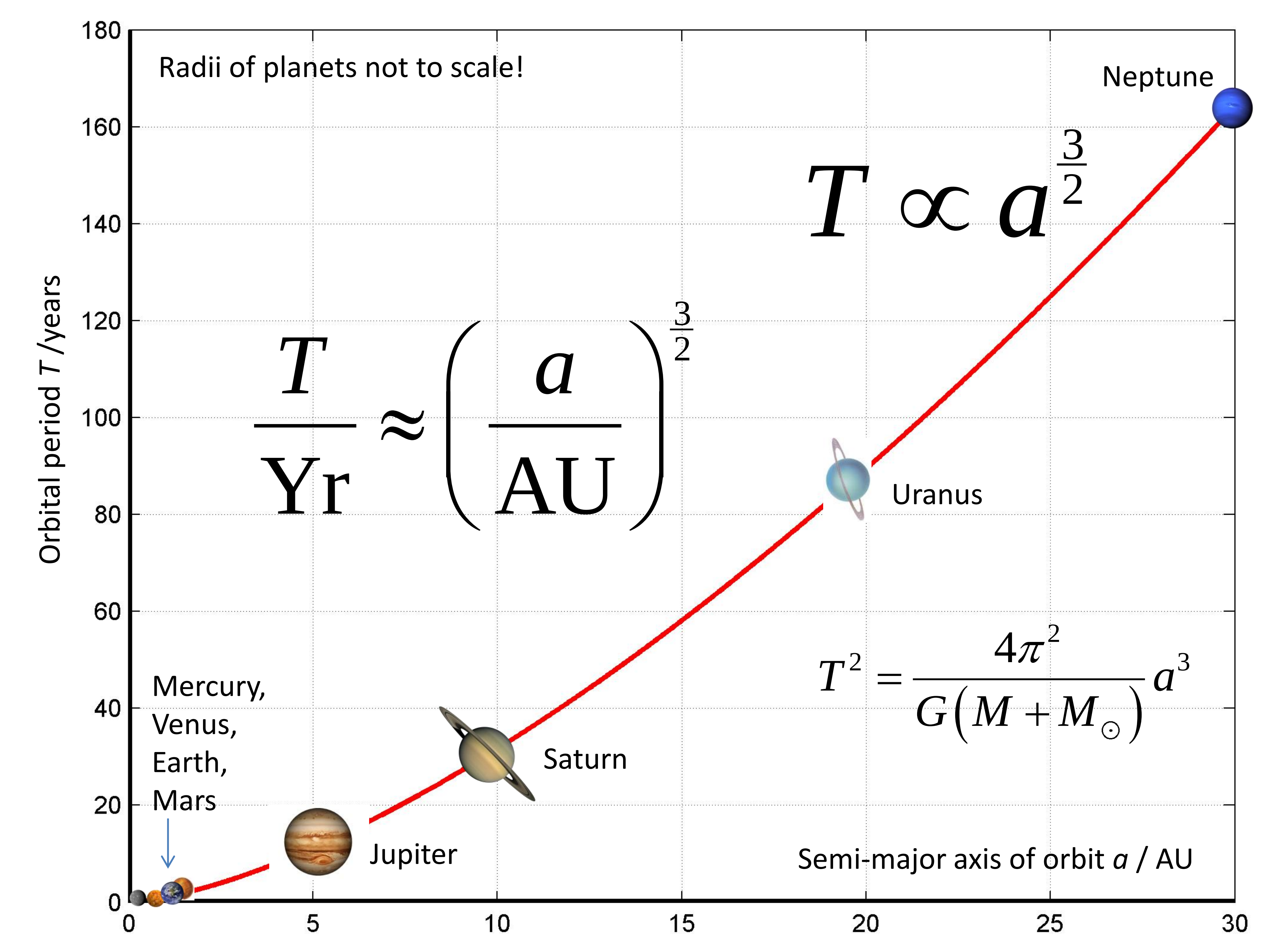
Orbits of the planets are *ellipses*
i.e. ‘squashed circles’

Object	$M/M_{\oplus}$	a /AU	$R/R_{\oplus}$	T_{rot} / days	T /Yr
Sun	332,837	-	109.123	-	-
Mercury	0.055	0.387	0.383	58.646	0.241
Venus†	0.815	0.723	0.949	243.018	0.615
Earth	1.000	1.000	1.000	0.997	1.000
Mars	0.107	1.523	0.533	1.026	1.881
Jupiter	317.85	5.202	11.209	0.413	11.861
Saturn	95.159	9.576	9.449	0.444	29.628
Uranus†	14.500	19.293	4.007	0.718	84.747
Neptune	17.204	30.246	3.883	0.671	166.344
Pluto†	0.003	39.509	0.187	6.387	248.348

Kepler's Third Law of planetary motion relates the 'radius'* of the orbit to the time taken to complete the orbit (the period)



*since the orbits are ellipses, the orbital radius is *not* constant. a is actually the 'semi-major axis' of the ellipse.



Radii of planets not to scale!
Neptune
$T \propto a^{\frac{3}{2}}$
$\frac{T}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$
Uranus
$T^2 = \frac{4\pi^2}{G\left(M + M_\odot\right)} a^3$
Mercury,
Venus,
Earth,
Mars
Saturn
Jupiter
Semi-major axis of orbit a / AU
Orbital period T /years
180
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30

Conservation of momentum and collisions

Momentum is a vector quantity

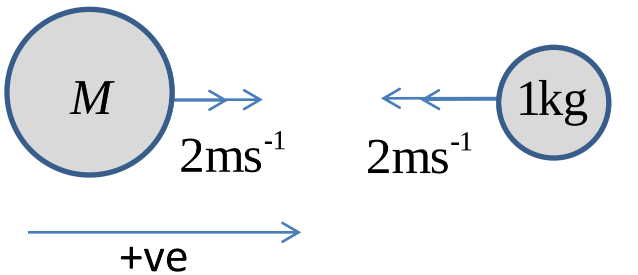
$$\mathbf{p} = m\mathbf{v}$$

Total momentum is *conserved* in collisions

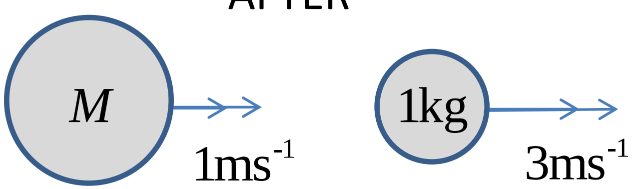
i.e. each mass receives an equal magnitude but opposite signed **impulse** which is a **change in momentum**

Example 1: Find the mass *M*, and then calculate the amount of kinetic energy lost in the collision.

BEFORE



AFTER



Note the ***coefficient of restitution*** is $C = 0.5$ in this case.

$$C = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$C = 1$ **ELASTIC** $C = 0$ **INELASTIC**

By conservation of momentum

$$2M - 2 = M + 3$$

$$\boxed{M = 5\text{kg}}$$

The amount of *kinetic energy lost* is

$$\Delta E = \tfrac{1}{2}(5)(2^2) + \tfrac{1}{2}(1)(2^2) - \tfrac{1}{2}(5)(1) - \tfrac{1}{2}(1)(3^2)$$

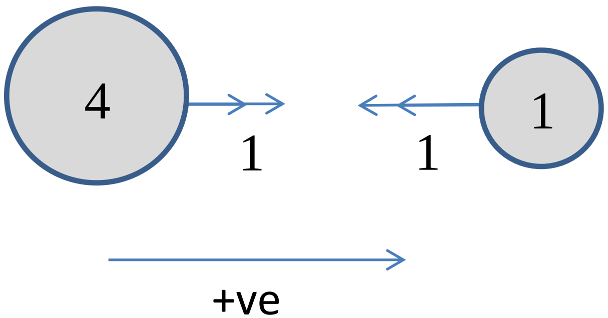
$$\Delta E = \tfrac{1}{2}\left(20 + 4 - 5 - 9\right)$$

$$\Delta E = \tfrac{1}{2}\left(20 + 4 - 5 - 9\right)$$

$$\boxed{\Delta E = 5\text{J}}$$

Example 2: Find the velocities post-collision
Assume the collision is elastic. Masses are in kg and velocities in ms^{-1}.

BEFORE



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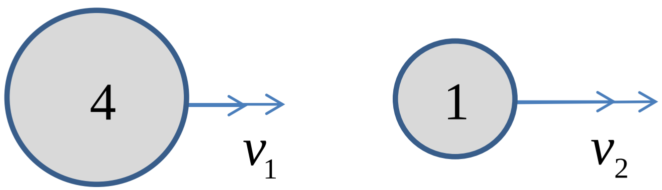

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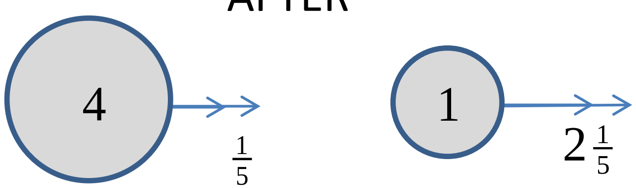


By conservation of momentum

$$4v_1 + v_2 = 4(1) - (1)(1)$$

$$\boxed{4v_1 + v_2 = 3}$$

Since collision is elastic i.e. $C = 1$

$$\frac{v_2 - v_1}{2} = 1$$

$$\Rightarrow \boxed{v_2 - v_1 = 2}$$

Subtracting these equations eliminates v_2

$$5v_1 = 1$$

$$\boxed{v_1 = \tfrac{1}{5}}$$

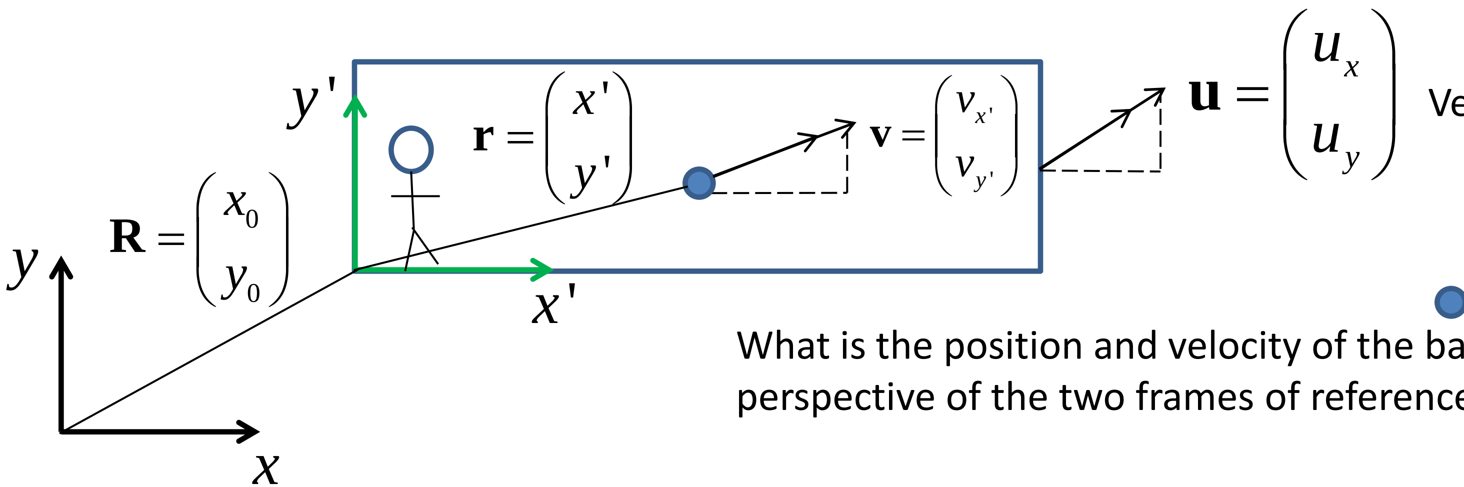
Hence:

$$v_2 = 2 + v_1$$

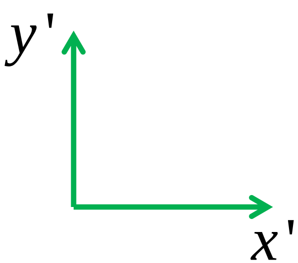
$$\boxed{v_2 = 2\tfrac{1}{5}}$$

Frames of reference are essentially **coordinate systems** used to describe the motion of an object. It is useful to be able to transform between different frames of reference to get a change in perspective. For example, how does the motion of a ball thrown on a moving train differ from (i) the person throwing the ball; (ii) a stationary observer watching the rain pass by?

When objects move *close to the speed of light*, the rules of converting between frames of reference become more complicated. This is called **Special Relativity**, developed by Albert Einstein. We will consider the *modest speed version*, which is often called **'Galilean Relativity'** after the great Renaissance Physicist Galileo. One major difference is that **time passes at the same rate** in the latter, regardless how fast a reference frame is moving relative to another.

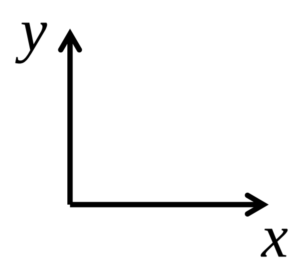


What is the position and velocity of the ball from the perspective of the two frames of reference?



Position $\begin{pmatrix} x' \\ y' \end{pmatrix}$

Velocity $\begin{pmatrix} v_{x'} \\ v_{y'} \end{pmatrix}$

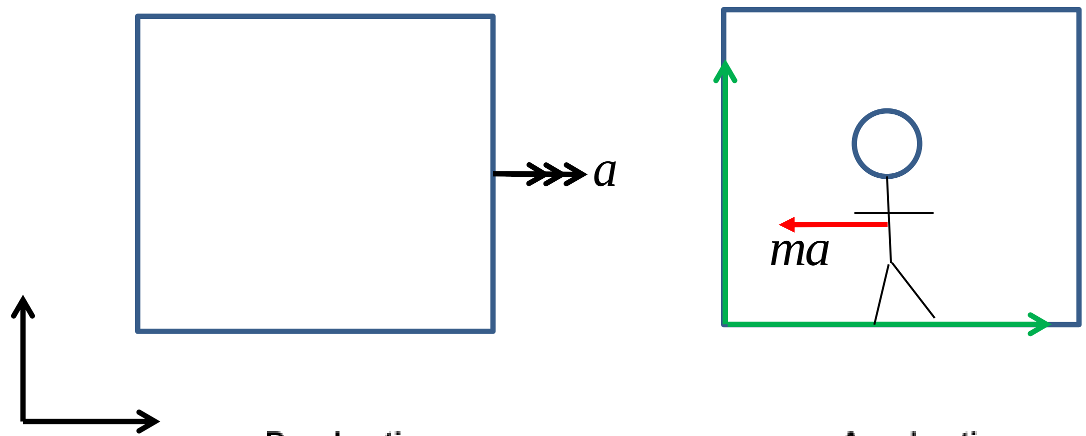


Position $\begin{pmatrix} x_0 + x' \\ y_0 + y' \end{pmatrix}$

Velocity $\begin{pmatrix} u_x + v_{x'} \\ u_y + v_{y'} \end{pmatrix}$

The effect of an **accelerating frame of reference** (these are called '*non inertial frames*')

If you are in an ***accelerating*** reference frame, you will experience a **force** with magnitude equal to the **acceleration of the frame x your mass**. This is because the frame is accelerating away from you, so, relative to the frame, you will experience a mass x acceleration in the *opposite* direction.



This explains why you get pushed into your seat when a car accelerates forward, and why you get thrown forward when a car breaks. (Which is why we use seat belts!)