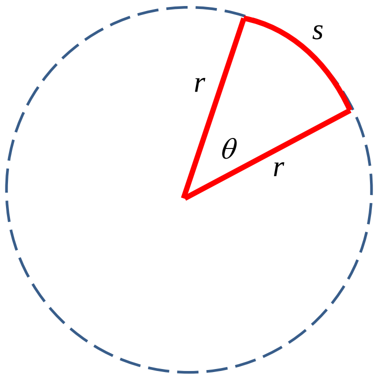
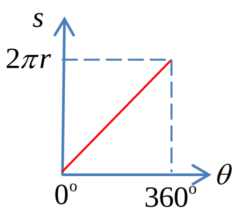
Arcs, sectors, radians, trigonometry

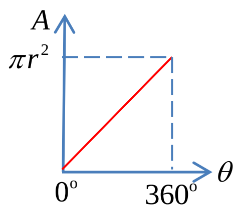


Consider a *sector* of a circle of angle θ.

The *arc length* of the sector is s and the *sector area* is A

Now since the circle radius is constant, one would expect *both* arc length and sector area to *increase linearly* with arc angle θ. If this was not the case, then one part of the circle would be different from another.





Hence: $\boxed{\dfrac{\theta}{360^\circ} = \dfrac{s}{2\pi r} = \dfrac{A}{\pi r^2}}$

So if θ is measured in degrees:

$$\boxed{\begin{aligned} s &= \frac{2\pi r\theta}{360^\circ} \\ A &= \frac{\pi r^2\theta}{360^\circ} \end{aligned}}$$

Let us define an alternative angular measure to simplify the above.

This will prove *very useful* when considering the gradients of trigonometric functions like sine and cosine!

Define the **radian** via the following formula:

$$\boxed{\pi \text{ radians} = 180^\circ}$$

Hence the following conversions:

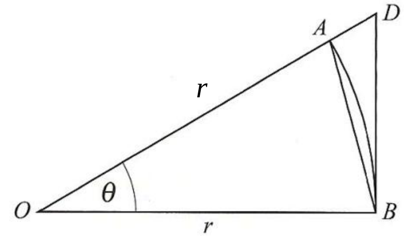
$$\boxed{\begin{aligned} \theta_{\text{rad}} &= \frac{\pi}{180}\theta_{\text{deg}} \\ \theta_{\text{deg}} &= \frac{180}{\pi}\theta_{\text{rad}} \end{aligned}}$$

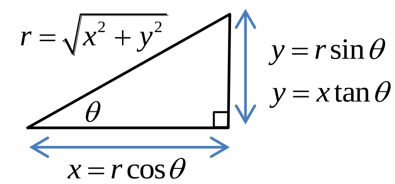
Therefore if we define the sector angle θ in radians:

$$s = \frac{2\pi r\theta}{2\pi} \Rightarrow \boxed{s = r\theta}$$

$$A = \frac{\pi r^2\theta}{2\pi} \Rightarrow \boxed{A = \tfrac{1}{2}r^2\theta}$$

Now consider a sector OAB of radius *r*. A right angled triangle ODB is drawn around it, and isosceles triangle OAB is drawn inside it.





Reminder of basic trigonometric relationships

If we use *radians* for our angular measure (π radians = 180°) the areas of these three shapes are:

$A_{ODB} = \frac{1}{2}r^2\tan\theta$	Right angled triangle
$A_{OAB} = \frac{1}{2}r^2\theta$	Sector
$A_{\Delta OAB} = \frac{1}{2}r^2\sin\theta$	Isosceles triangle

Now from the diagram:

$$A_{ODB} > A_{OAB} > A_{\Delta OAB}$$

$$\tfrac{1}{2}r^2\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta$$

$$\therefore \quad \tan\theta > \theta > \sin\theta$$

As θ becomes *small*, the diagram justifies the result:

$\boxed{\lim_{\theta\to 0}(\tan\theta) = \lim_{\theta\to 0}(\sin\theta) = \theta}$ i.e. $\boxed{\tan\theta \approx \theta \approx \sin\theta}$

where θ is in radians and $\ll 1$

This is a very useful approximation which leads to, amongst many other things, the analysis of *small oscillations* and also the calculus results for the *gradient* of sine and cosine functions

$\boxed{\dfrac{d}{d\theta}\sin\theta = \cos\theta \qquad \dfrac{d}{d\theta}\cos\theta = -\sin\theta}$ but *only* when θ is in *radians*