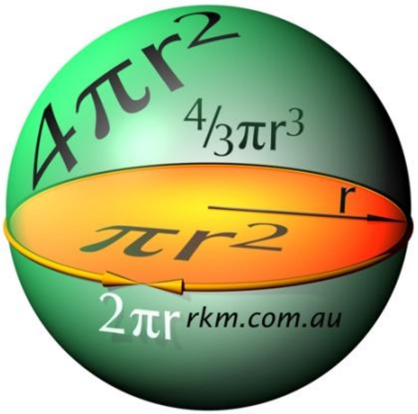
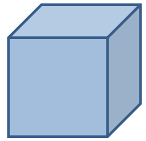
Surface areas of basic solids

Sphere of radius r

$\boxed{A = 4\pi r^2}$

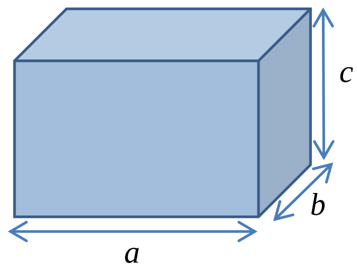


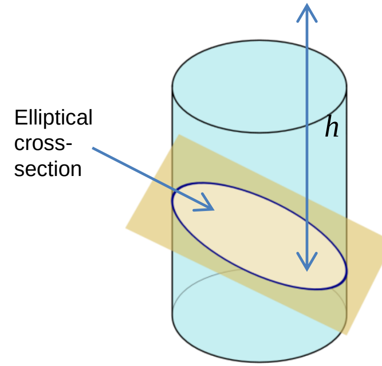
Cube of side a



$\boxed{A = 6a^2}$

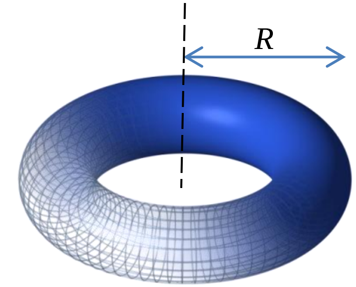
Cuboid $\boxed{A = 2(ab + bc + ca)}$





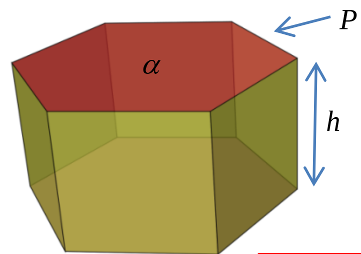
Cylinder, with radius r and height h

$\boxed{A = 2\pi r^2 + 2\pi rh}$

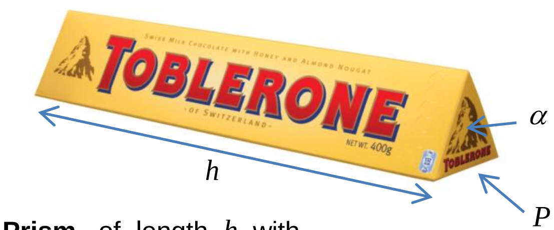


Torus, with a circular cross section of radius r

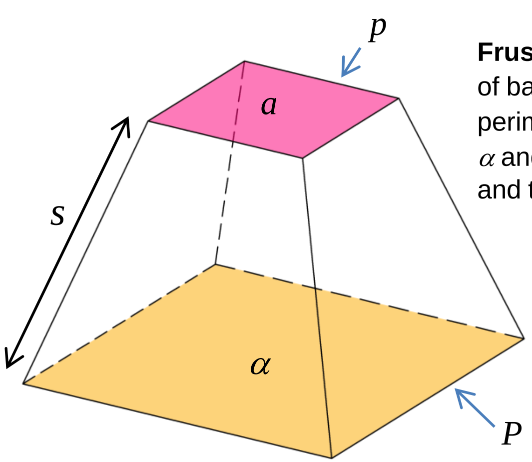
$\boxed{A = 2\pi r \times 2\pi R = 4\pi^2 rR}$



$\boxed{A = 2\alpha + Ph}$



Prism, of length h with uniform cross sectional area α with perimeter P



$\boxed{A = a + \alpha + \frac{1}{2}s(p + P)}$

Frustum, (a truncated pyramid) of base perimeter P, top area perimeter p and slant height s. α and a are the respective base and top areas

Assume all slants are the same length

Otherwise we have an *oblique* frustum or pyramid and these formulae will not hold

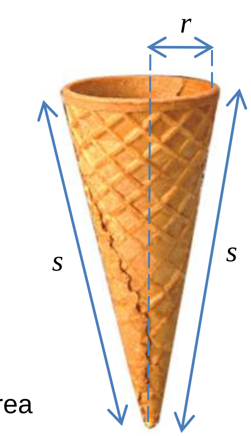
Pyramid, with base perimeter P and slant height s

$\boxed{A = \frac{1}{2}Ps}$

Cone, with base radius r and slant height s

$\boxed{A = \pi rs}$

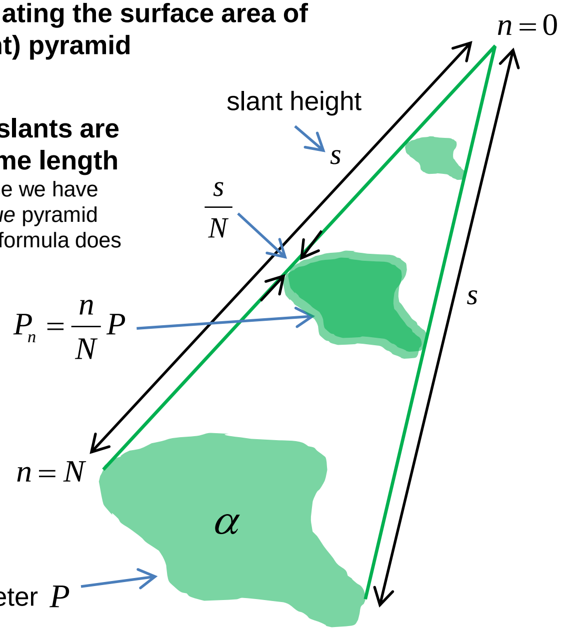
This is the *curved* surface area i.e. an 'open' cone



Calculating the surface area of a (right) pyramid

i.e. **all slants are the same length**

Otherwise we have an *oblique* pyramid and this formula does not hold



A pyramid of perpendicular slant height s and base perimeter P can be thought of being composed of an (infinitely) large number of **similar** *laminae*.

The *enlargement factor* of each lamina *is in direct proportion to the slant distance from the apex of the pyramid.* This is true since the sides of the pyramid are straight lines with a constant gradient.

Let there be N laminae of thickness s/N. Since all laminae are *similar*, the perimeters scale as n/N where n is the lamina number from the apex. The exterior area of each lamina is:

$$A_n = P_n \times \frac{s}{N}$$

$$A_n = \frac{n}{N}P \times \frac{s}{N}$$

$$A_n = \frac{Ps}{N^2}n$$

Therefore the total exterior of the pyramid is the sum of the exterior areas of the N laminae

$$A_N = \frac{Ps}{N^2}1 + \frac{Ps}{N^2}2 + ... + \frac{Ps}{N^2}N$$

$$A_N = \frac{Ps}{N^2}(1+2+....+N)$$

Now the sum of the first N integers is

$$1+2+....+N = \tfrac{1}{2}N(N+1)$$

Hence:

$$A_N = \frac{Ps}{N^2}\tfrac{1}{2}N(N+1)$$

$$A_N = \tfrac{1}{2}Ps\frac{N^2+N}{N^2}$$

$$A_N = \tfrac{1}{2}Ps\left(1+\tfrac{1}{N}\right)$$

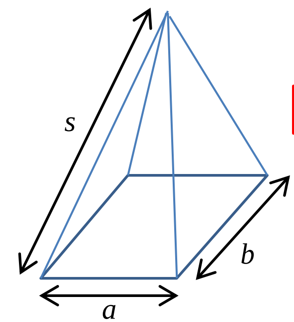
Therefore as N becomes infinite, the area tend towards

$$\boxed{A = \tfrac{1}{2}Ps}$$

Add the base area α to make the total surface area

i.e. *"the total area of a pyramid is the base plus half the product of the base perimeter and the slant height"*

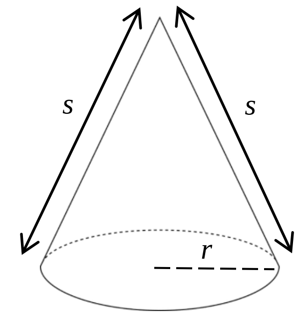
Area of a *right* rectangular based pyramid



$$A = ab + \tfrac{1}{2}s(2a+2b)$$

$$\boxed{A = ab + s(a+b)}$$

Area of a *right* cone

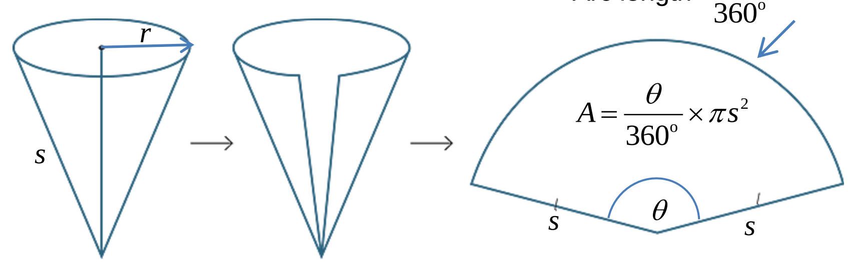


$$A = \pi r^2 + \tfrac{1}{2}2\pi rs$$

$$\boxed{A = \pi r^2 + \pi rs}$$

Note we can find the **curved area of a *right* cone** by noting how it can be constructed from a circle with a sector missing

Arc length $\frac{\theta}{360^\circ} \times 2\pi s = 2\pi r$

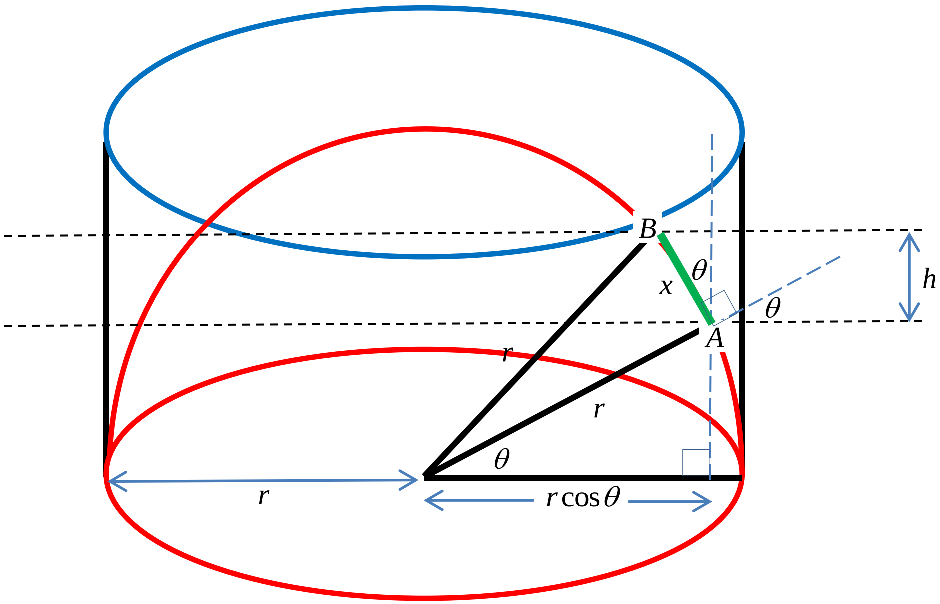


$$\therefore \frac{\theta}{360^\circ} = \frac{r}{s}$$

$$\therefore A = \frac{r}{s} \times \pi s^2$$

$$\therefore \boxed{A = \pi rs}$$

Calculating the area of a sphere *without* using calculus



Consider the **tangent** to a circular cross section of a hemisphere at point A, which is at elevation angle θ from the base of a hemisphere of radius r.

Point B is a distance away x *along the tangent* such that $x\cos\theta = h$, where h is the projected width of the hemisphere on a cylinder of radius r. (The cylinder width between points at the heights of positions A and B).

Now if we shrink h, then position B will move **closer to the circle**, and hence in this limit we would expect the **arc length** between A and B to become:

$$x = \frac{h}{\cos\theta}$$

The area of a **circular strip** with width x between A and B on the hemisphere is approximately:

$$\Delta A = 2\pi r\cos\theta \times x$$

$$\therefore \Delta A = 2\pi r\cos\theta \frac{h}{\cos\theta} = 2\pi rh$$

So we can conclude that the strip area, in the limit of small h, **does not depend on angle θ.**

i.e. each strip on the hemisphere 'maps' (i.e. can be projected onto) a cylindrical strip of (small) width h.

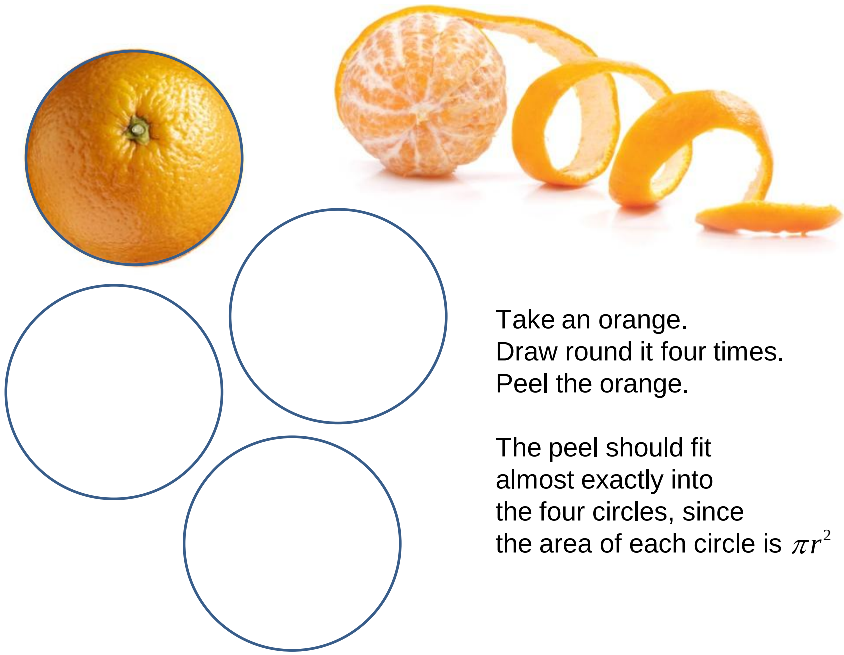
This means the **curved surface of the hemisphere must equal that of the enclosing cylinder.**

Hence the curved surface area of the hemisphere must be:

$$2\pi r \times r = 2\pi r^2$$

Therefore the **surface area of a sphere** is:

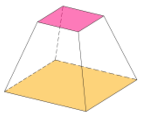
$$\boxed{A = 4\pi r^2}$$



Take an orange.
Draw round it four times.
Peel the orange.

The peel should fit almost exactly into the four circles, since the area of each circle is πr^2

A *frustum* is a *truncated pyramid*



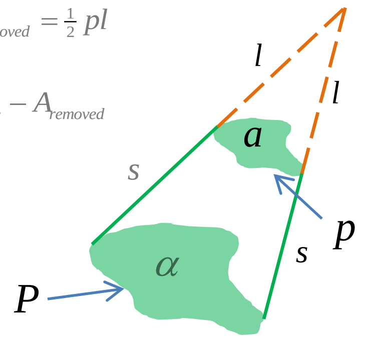
Curved area of entire pyramid (assume NOT oblique)

$$A_{entire} = \tfrac{1}{2}P(s+l)$$

Since cap is *similar* to entire pyramid, curved of removed 'cap' pyramid is: $A_{removed} = \tfrac{1}{2}pl$

Therefore frustum curved area is $A = A_{entire} - A_{removed}$

$$A = \tfrac{1}{2}P(s+l) - \tfrac{1}{2}pl$$



Now since the top and base of the frustum are *similar*

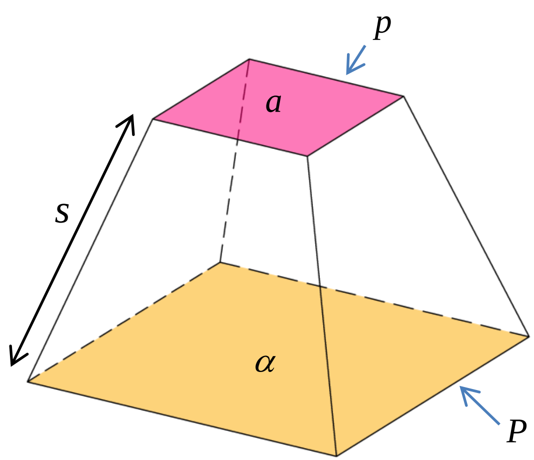
$$\frac{P}{p} = \frac{s+l}{l}$$
$$\frac{P}{p} = \frac{s}{l} + 1$$
$$\frac{P}{p} - 1 = \frac{s}{l}$$
$$\frac{P-p}{p} = \frac{s}{l}$$
$$l = \frac{ps}{P-p}$$

Hence:

$$A = \tfrac{1}{2}P(s+l) - \tfrac{1}{2}pl$$
$$A = \tfrac{1}{2}Ps + \tfrac{1}{2}l(P-p)$$
$$A = \tfrac{1}{2}Ps + \tfrac{1}{2}\frac{ps}{P-p}(P-p)$$
$$\boxed{A = \tfrac{1}{2}s(P+p)}$$

Therefore total surface area of the frustum is:

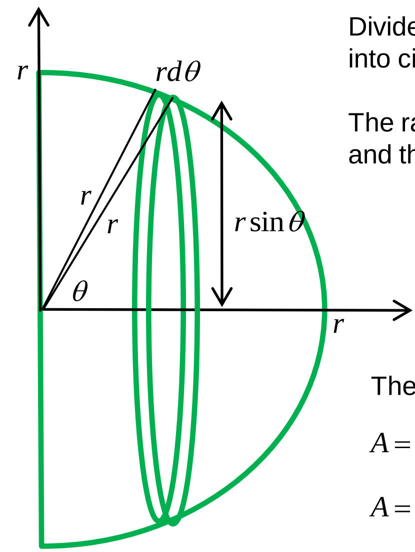
$$\boxed{A = a + \alpha + \tfrac{1}{2}s(p+P)}$$



This is only true for non oblique frustums!

Calculating the area of a sphere using calculus

Note all angles are in *radians* i.e. π radians = 180°.



Divide up the surface of a hemisphere into circular ribbons

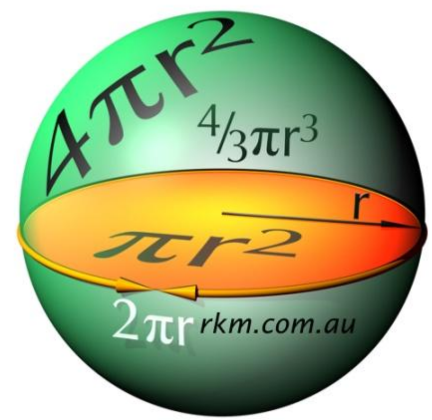
The radius of each ribbon is $r\sin\theta$ and the width is the arc length $rd\theta$

The area of each ribbon is

$$dA = 2\pi \times r\sin\theta \times rd\theta = 2\pi r^2 \sin\theta d\theta$$

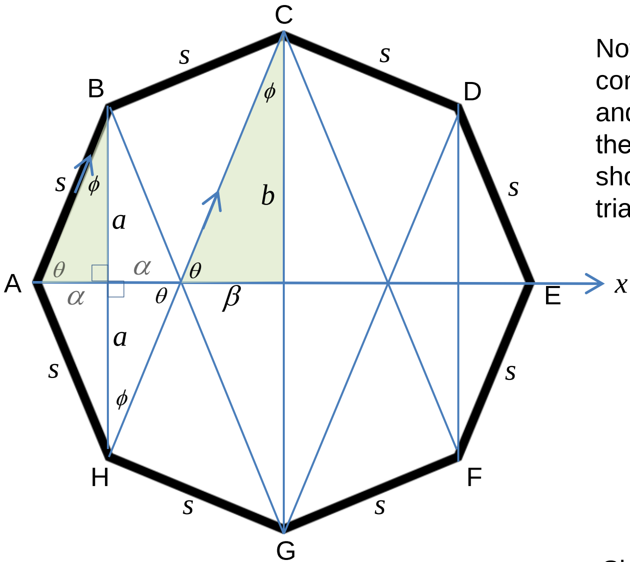
The total area of the *sphere* is therefore

$$A = 2 \times \int dA$$
$$A = 2 \times 2\pi r^2 \int_0^{\frac{1}{2}\pi} \sin\theta d\theta$$
$$A = 4\pi r^2 \left[-\cos\theta\right]_0^{\frac{1}{2}\pi}$$
$$A = 4\pi r^2 \left\{\left(-\cos\tfrac{1}{2}\pi\right) - \left(-\cos(0)\right)\right\}$$
$$A = 4\pi r^2 \left\{0 - (-1)\right\}$$
$$\boxed{A = 4\pi r^2}$$



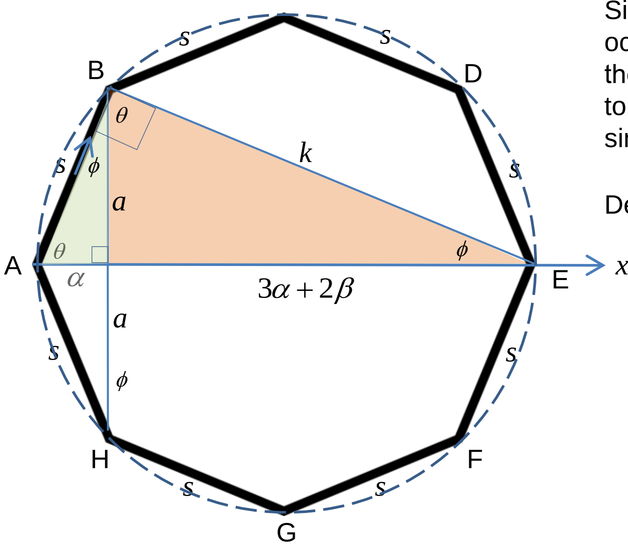
Archimedes' method for calculating the area of a sphere

Firstly this involves finding a formula for the surface area of a *volume of revolution* of a regular octagon about the x axis. Generalizing for regular even-number sided polygons of increasing large numbers of sides, we anticipate the resulting surface area to be that of a sphere.



Noting the parallel lines connecting vertices AB and HC, and by application of the 'Z-angle' theorem, one can show that the green right angled triangles highlighted are *similar*

Hence: $\boxed{\dfrac{b}{\beta} = \dfrac{a}{\alpha}}$



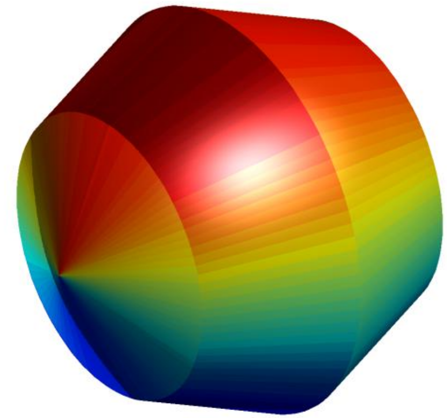
Since we can *circumscribe* the octagon with a circle, we can use the 'right angle diameter theorem' to show the orange circle is also similar to the green triangles.

Define $k = \text{BE}$

$$\therefore \frac{k}{s} = \frac{b}{\beta} \Rightarrow \boxed{b = \beta \frac{k}{s}}$$

$$\therefore \frac{k}{s} = \frac{a}{\alpha} \Rightarrow \boxed{a = \alpha \frac{k}{s}}$$

A volume of revolution of the octagon is the volume of **two cones** formed by the rotation of triangle ABH and **two frustums** formed by the rotation of trapezium BHGC



The total surface area is therefore:

$$A = 2\pi as + 2 \times \tfrac{1}{2} s\left(2\pi a + 2\pi b\right)$$

$$A = 2\pi as + 2\pi s\left(a + b\right)$$

$$A = 2\pi s\left(2a + b\right)$$

Using the results above

$$A = 2\pi k\left(2\alpha + \beta\right)$$

$$A = \pi k \times \left(4\alpha + 2\beta\right)$$

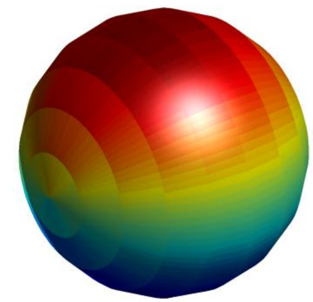
$$\boxed{A = \pi k \times \text{AE}}$$

Now we would expect the same result *for any even polygon*, but as the number of sides increase, **we expect k to approach AE** as s shrinks

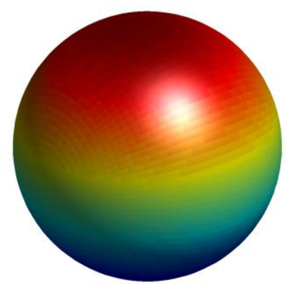
For a circle $\text{AE} = 2r$. Therefore if in this scenario $k = \text{AE}$

$$A = \pi 2r \times 2r$$

$$\boxed{A = 4\pi r^2}$$



20 sided polygon of revolution



100 sided polygon of revolution – this looks very much like a sphere!