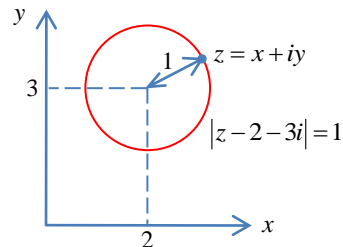
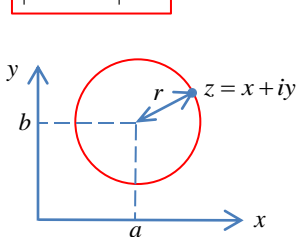


Argand Diagram loci

Regions of the complex plane (the Argand Diagram) can be defined via algebraic expressions involving **complex numbers**.

Circle, centre (a,b) , radius r

$$|z - a - ib| = r$$



$$|z - a - ib|^2 = r^2$$

$$|x + iy - a - ib|^2 = r^2$$

$$|(x - a) + i(y - b)|^2 = r^2$$

$$(x - a)^2 + (y - b)^2 = r^2$$

It is easy to show that this form results in the Cartesian equation of the circle

Another similar form of modulus-based equation also yields a **circular locus**

$$|z - a - ib| = k|z - c - id|$$

$$|z - a - ib|^2 = k^2|z - c - id|^2$$

$$(x - a)^2 + (y - b)^2 = k^2(x - c)^2 + k^2(y - d)^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = k^2(x^2 - 2cx + c^2 + y^2 - 2dy + d^2)$$

$$x^2(1 - k^2) + y^2(1 - k^2) - 2x(a + k^2c) - 2y(b + k^2d) + a^2 + b^2 - k^2(c^2 + d^2) = 0$$

$$x^2 + y^2 - 2x \frac{a + k^2c}{1 - k^2} - 2y \frac{b + k^2d}{1 - k^2} + \frac{a^2 + b^2 - k^2(c^2 + d^2)}{1 - k^2} = 0$$

$$\left(x - \frac{a + k^2c}{1 - k^2}\right)^2 + \left(y - \frac{b + k^2d}{1 - k^2}\right)^2 = \left(\frac{b + k^2d}{1 - k^2}\right)^2 + \left(\frac{a + k^2c}{1 - k^2}\right)^2 - \frac{a^2 + b^2 - k^2(c^2 + d^2)}{1 - k^2}$$

$$\text{Circle: centre } \left(\frac{a + k^2c}{1 - k^2}, \frac{b + k^2d}{1 - k^2}\right) \text{ radius } \sqrt{\left(\frac{b + k^2d}{1 - k^2}\right)^2 + \left(\frac{a + k^2c}{1 - k^2}\right)^2 - \frac{a^2 + b^2 - k^2(c^2 + d^2)}{1 - k^2}}$$

$$z = x + iy$$

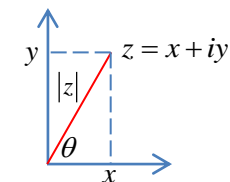
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg z = \tan^{-1} \frac{y}{x}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$z = |z|e^{i\theta}$$

$$\arg z = \theta$$



The exception is when $k = 1$, which defines a **straight line locus**

$$|z - a - ib| = |z - c - id|$$

$$|z - a - ib|^2 = |z - c - id|^2$$

$$(x - a)^2 + (y - b)^2 = (x - c)^2 + (y - d)^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = x^2 - 2cx + c^2 + y^2 - 2dy + d^2$$

$$x(2c - 2a) + a^2 + b^2 - c^2 - d^2 = y(2b - 2d)$$

$$y = \frac{c - a}{b - d}x + \frac{a^2 + b^2 - c^2 - d^2}{2b - 2d}$$

$$\text{Straight line with gradient } \frac{c - a}{b - d} \text{ and y intercept } \frac{a^2 + b^2 - c^2 - d^2}{2b - 2d}$$

Example: Find the locus defined by $|z - 2i| \leq |z + 2|$

$$|z - 2i|^2 \leq |z + 2|^2$$

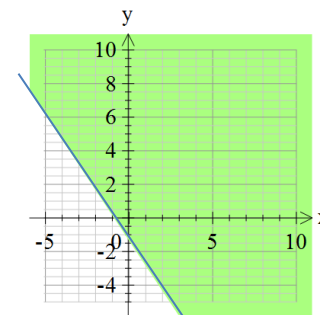
$$x^2 + (y - 2)^2 \leq (x + 2)^2 + y^2$$

$$x^2 + y^2 - 4y + 4 \leq x^2 + 6x + 9 + y^2$$

$$0 \leq 4y + 6x + 5$$

$$y \leq \frac{-6x - 5}{4}$$

$$y \leq -\frac{3}{2}x - \frac{5}{4}$$

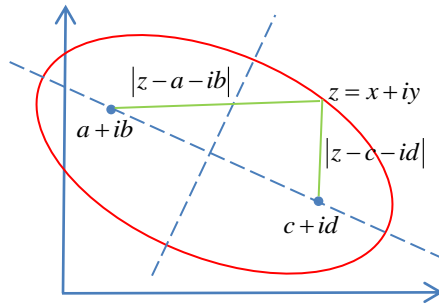


$$|z - 2i| \leq |z + 2|$$

$$y = -\frac{3}{2}x - \frac{5}{4}$$

Elliptical loci

$$|z - a - ib| + |z - c - id| = k \quad k > 0$$



i.e. foci of the ellipse are the coordinates (a, b) and (c, d)

A defining property of an ellipse is the direct distance from one focus, to a point on the ellipse, to the focus is always a *constant*.

Example: Consider the locus defined by $|z + \sqrt{a^2 - b^2}| + |z - \sqrt{a^2 - b^2}| = 2a$ ← Note any ellipse with long axis of symmetry (the “semi-major axis”) can be described in this way if we shift the origin of the Argand diagram to the geometric centre of the ellipse.

If the solution is an *ellipse* centred on $(0, 0)$ then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\begin{aligned} |z + \sqrt{a^2 - b^2}| + |z - \sqrt{a^2 - b^2}| &= 2a \\ \sqrt{(x + \sqrt{a^2 - b^2})^2 + y^2} + \sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2} &= 2a \\ (x + \sqrt{a^2 - b^2})^2 + y^2 &= \left(2a - \sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2}\right)^2 \\ (x + \sqrt{a^2 - b^2})^2 + y^2 &= 4a^2 - 4a\sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2} + (x - \sqrt{a^2 - b^2})^2 + y^2 \\ (x + \sqrt{a^2 - b^2})^2 - (x - \sqrt{a^2 - b^2})^2 - 4a^2 &= -4a\sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2} \\ (2x)(2\sqrt{a^2 - b^2}) - 4a^2 &= -4a\sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2} \\ (x\sqrt{a^2 - b^2} - a^2)^2 &= a^2\left((x - \sqrt{a^2 - b^2})^2 + y^2\right) \end{aligned}$$

Elliptical geometry

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1 \quad \text{Cartesian equation i.e. a two-way stretch of a unit circle}$$

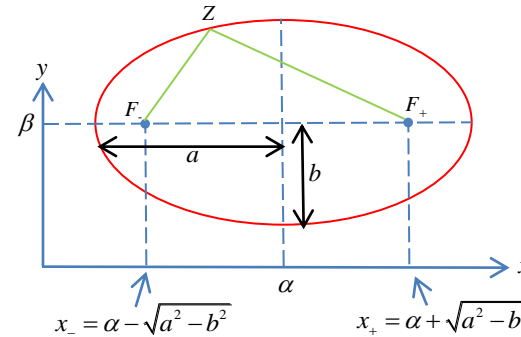
$$a \geq b$$

$$\begin{aligned} x_{\pm} &= \alpha \pm \sqrt{a^2 - b^2} \\ y_{\pm} &= \beta \end{aligned}$$

Coordinates of foci

$$\text{Eccentricity } \varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

A circle has zero eccentricity since $a = b$



$$F_1Z + ZF_2 = 2a$$

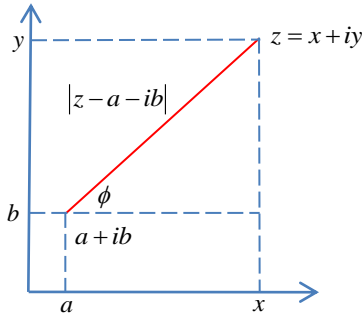
This is a *defining* property of an ellipse
See below!

$$\begin{aligned} (x\sqrt{a^2 - b^2} - a^2)^2 &= a^2\left((x - \sqrt{a^2 - b^2})^2 + y^2\right) \\ x^2(a^2 - b^2) + a^4 - 2xa^2\sqrt{a^2 - b^2} &= a^2\{x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + y^2\} \\ a^2x^2 - b^2x^2 + a^4 - 2xa^2\sqrt{a^2 - b^2} &= a^2x^2 - 2xa^2\sqrt{a^2 - b^2} + a^4 - a^2b^2 + a^2y^2 \\ a^2b^2 &= b^2x^2 + a^2y^2 \\ 1 &= \frac{x^2}{a^2} + \frac{y^2}{b^2} \end{aligned}$$

So the locus is indeed an ellipse centred on $(0, 0)$

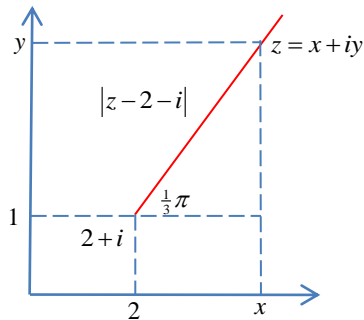
Straight line segments

$$\arg(z - a - ib) = \phi$$



Example: Sketch the locus of the equation

$$\arg(z - 2 - i) = \frac{1}{3}\pi$$



Intersection of line segment and circle example

Find z such that

$$|z - 1 - i| = 1$$

$$\arg(z + i) = \frac{\pi}{4}$$

Cartesian equations of intersecting loci are:

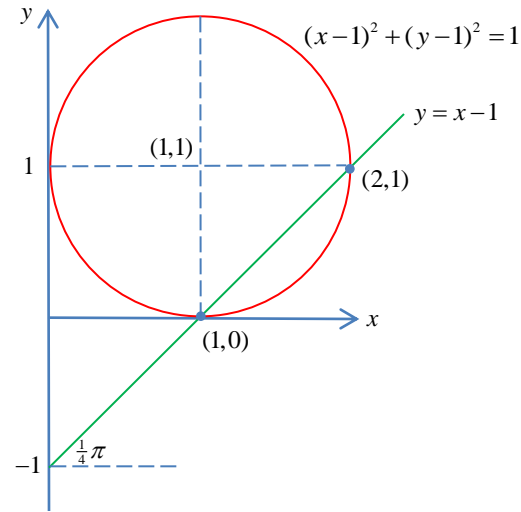
$$|z - 1 - i| = 1$$

$$(x - 1)^2 + (y - 1)^2 = 1$$

$$\arg(z + i) = \frac{\pi}{4}$$

$$y = x \tan \frac{\pi}{4} - 1$$

$$y = x - 1$$



Substituting for y

$$(x - 1)^2 + (y - 1)^2 = 1$$

$$(x - 1)^2 + (x - 2)^2 = 1$$

$$x^2 - 2x + 1 + x^2 - 4x + 4 = 1$$

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x - 1)(x - 2) = 0$$

$$\therefore x = 1, 2$$

$$\therefore y = 0, 1$$

Hence solutions to $|z - 1 - i| = 1$ are $z = 1$
 $\arg(z + i) = \frac{\pi}{4}$ $z = 2 + i$

Circle tangents example

What are the range of values of $\arg(z - 4 + 2i)$ such that $|z - 2 - 3i| = 2$?

From diagram $\phi_{\min} \leq \arg(z - 4 + 2i) \leq \phi_{\max}$

i.e. range of possible intersections of line segments from $(4, -2)$ with the circle are bounded by the tangents to the circle.

Pythagoras: $OA = \sqrt{2^2 + 5^2} = \sqrt{29}$

$$\sqrt{29} \sin \theta = 2$$

$$\therefore \theta = \sin^{-1}\left(\frac{2}{\sqrt{29}}\right)$$

$$180^\circ - 2\theta - \phi_{\min} = \tan^{-1} \frac{5}{2}$$

$$\therefore \phi_{\min} = 180^\circ - 2 \sin^{-1}\left(\frac{2}{\sqrt{29}}\right) - \tan^{-1} \frac{5}{2} \approx 68.2^\circ$$

$$\phi_{\max} = \phi_{\min} + 2\theta$$

$$\therefore \phi_{\max} = 180^\circ - \tan^{-1} \frac{5}{2} \approx 111.8^\circ$$

