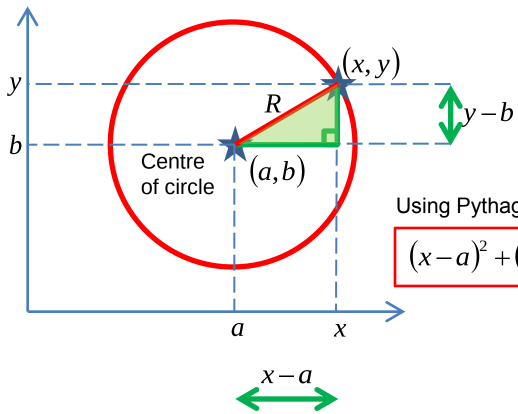
The **Cartesian equation of a circle** describes the locus of coordinates *(x,y)* that are on a circle of radius *R* and centre *(a,b).* It is a Cartesian equation because it involves the *orthonormal* coordinates *(x,y)*



Using Pythagoras's Theorem

$$\boxed{(x-a)^2+(y-b)^2=R^2}$$

Expanded form

$$(x-4)^2+(y-3)^2=4$$

$$\boxed{x^2-8x+y^2-6y+21=0}$$

Factorized form

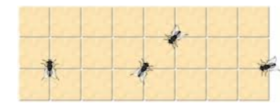
$$x^2-8x+y^2-6y+21=0$$

$$(x-4)^2-16+(y-3)^2-9+21=0$$

$$\boxed{(x-4)^2+(y-3)^2=4}$$

Complete the square

René Descartes
Cartesian x,y coordinate system linking algebra and geometry
(1596-1650)

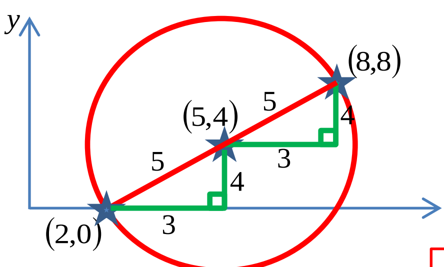


Cartesian (x,y) geometry was apparently inspired by ants moving on a tiled wall!

Equation of a circle from two points on the diameter

Work out the centre from the mean average of the points on the diameter

$$\tfrac{1}{2}\begin{pmatrix}2\\0\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}8\\8\end{pmatrix}=\begin{pmatrix}5\\4\end{pmatrix}$$



$$\begin{pmatrix}8\\8\end{pmatrix}-\begin{pmatrix}5\\4\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$$

$$R=3^2+4^2$$

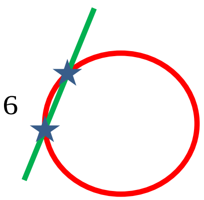
$$R=5$$

Then find the radius using one point and the centre

$$\boxed{(x-5)^2+(y-4)^2=25}$$

Intersection of a line with a circle

Find the coordinates of the points where the line $y=3x+6$ intersects the circle $x^2-8x+y^2+4y-30=0$



Solve algebraically by substitution

$$x^2-8x+(3x+6)^2+4(3x+6)-30=0$$

$$x^2-8x+9x^2+36x+36+12x+24-30=0$$

$$10x^2+40x+30=0$$

$$x^2+4x+3=0$$

$$(x+1)(x+3)=0$$

$$\boxed{x=-3,-1}$$

Therefore using $y=3x+6$

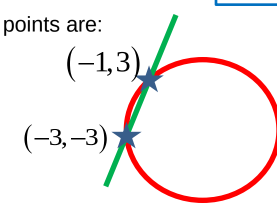
$$\boxed{y=-3,3}$$

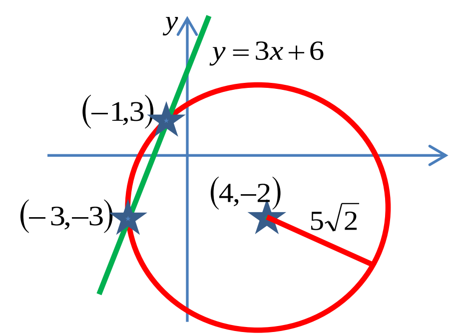
$$x^2-8x+y^2+4y-30=0$$

$$(x-4)^2-16+(y+2)^2-4-30=0$$

$$(x-4)^2+(y+2)^2=50=\left(5\sqrt{2}\right)^2$$

Intersection points are:





Normals and tangents to a circle

$$x^2+y^2+10x+2y+13=0$$

$$(x+5)^2-25+(y+1)^2-1+13=0$$

$$\boxed{(x+5)^2+(y+1)^2=13}$$

centre (-5,-1)
radius $\sqrt{13}$

Gradient of the normal is $\frac{3}{2}$

$$\therefore y_N=\tfrac{3}{2}x+c$$

Using (-3,2): $2=\tfrac{3}{2}(-3)+c$

$$2+4.5=c$$

$$c=6\tfrac{1}{2}$$

$$\boxed{y_N=\tfrac{3}{2}x+6\tfrac{1}{2}}$$

Hence: $y_T=-\tfrac{2}{3}x+d$

$$2=-\tfrac{2}{3}(-3)+d$$

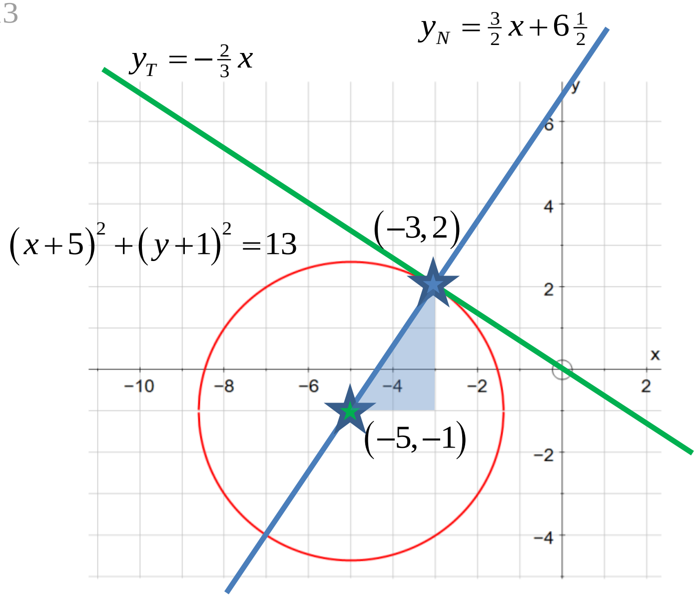
$$2-2=d$$

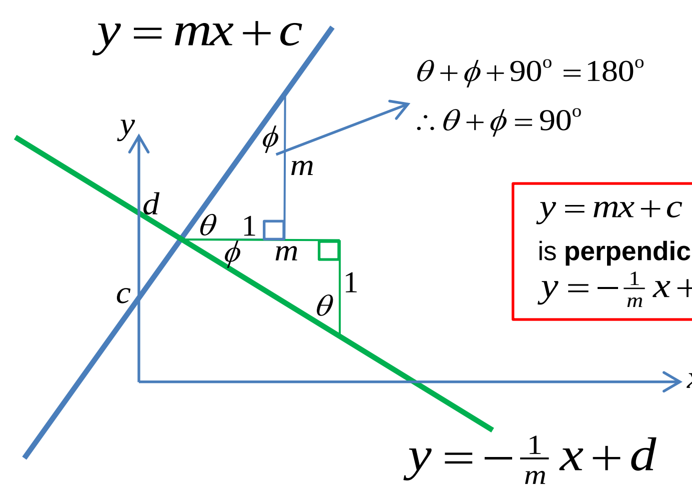
$$d=0$$

$$\boxed{y_T=-\tfrac{2}{3}x}$$

What is the equation of the tangent through (-3,2) ?

Strategy: Find the *normal* through (-3,2) first.
Note : *All normals pass through the circle centre.*

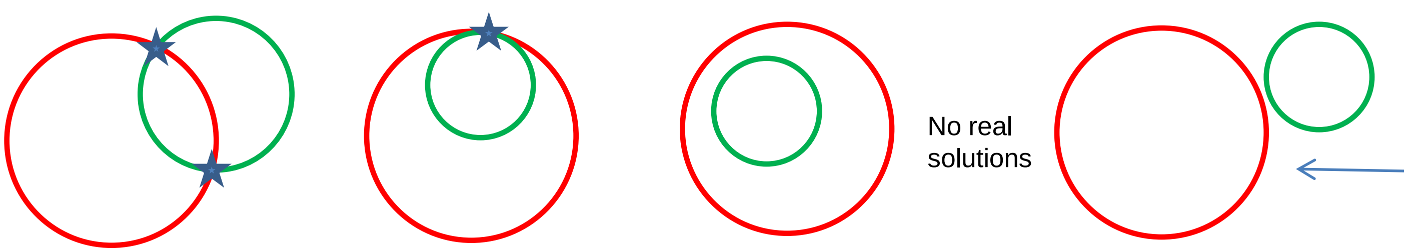




$y=mx+c$
is **perpendicular** to
$y=-\frac{1}{m}x+d$

Example: $y=-4x+7$ is perpendicular to $y=\frac{1}{4}x-4$

Intersection of two circles



Two solutions

One solution

No real solutions

Compare separation of centres to radii to determine between these cases

Example(s) 1. Write down the Cartesian equation of the circle in factorized form

(i) Circle centre (0,0) radius 12

$$\boxed{x^2 + y^2 = 144}$$

(ii) Circle centre (1,2) radius 13

$$\boxed{(x-1)^2 + (y-2)^2 = 169}$$

(iii) Circle centre (-1,3) radius 11

$$\boxed{(x+1)^2 + (y-3)^2 = 121}$$

(iv) Circle centre (0,-5) radius 7

$$\boxed{x^2 + (y+5)^2 = 49}$$

Example 2. Complete the square to work out the centre and radius of the following circles defined by their expanded Cartesian equations

(i) $x^2 - 2x + y^2 - 4y - 4 = 0$

$$(x-1)^2 - 1 + (y-2)^2 - 4 - 4 = 0$$

$$(x-1)^2 + (y-2)^2 = 9$$

Centre (1,2) radius 3

(ii) $x^2 + 4x + y^2 - 10y + 4 = 0$

$$(x+2)^2 - 4 + (y-5)^2 - 25 + 4 = 0$$

$$(x+2)^2 + (y-5)^2 = 25$$

Centre (-2,5) radius 5

Example 3. Intersection of two circles

$$x^2 + y^2 = 16$$

$$(x-2)^2 + y^2 = 9$$

To solve we have to solve these simultaneously
i.e. eliminate one variable first then solve the resulting equation

$$x^2 + y^2 = 16 \quad (1)$$

$$(x-2)^2 + y^2 = 9 \quad (2)$$

Note in general we will have linear *and* quadratic terms in x and y.

(1) – (2)

$$x^2 - (x-2)^2 = 7$$

$$x^2 - \{x^2 - 4x + 4\} - 7 = 0$$

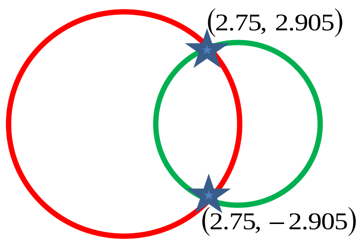
$$4x - 4 - 7 = 0$$

$$x = \tfrac{11}{4} = 2.75$$

$$x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

$$\therefore y = \pm\sqrt{16 - \left(\tfrac{11}{4}\right)^2} \Rightarrow y = \pm 2.905$$



Two solutions