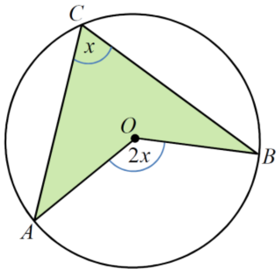
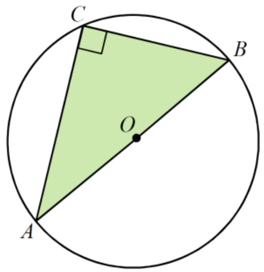
Circle theorems

There are five main circle theorems, which relate to triangles or quadrilaterals drawn inside the circumference of a circle.



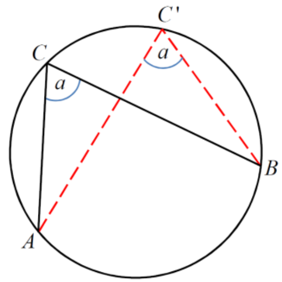
'Arrowhead' theorem

An angle at the **centre** of a circle is **twice** (the size of) the **angle on the circumference** if they are both **subtended** by the same **arc**.



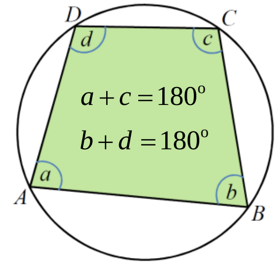
'Right-angle diameter' theorem

Any angle (**inscribed**) in a **semicircle** is a **right angle.**



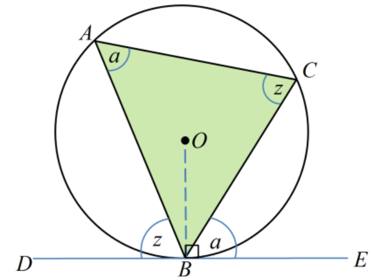
'Mountain' or 'bow-tie' theorem

The angles in the **same segment** (**subtended by the same arc or arcs of the same size**) are equal.



'Cyclic quadrilateral' theorem

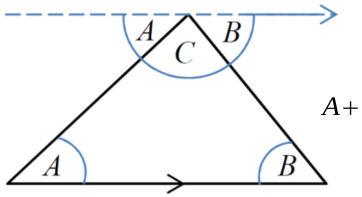
A quadrilateral ABCD is **cyclic** if and only if (it is convex and) **both pairs of opposite angles are supplementary**



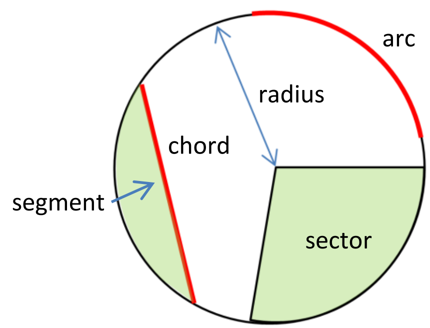
Chord-tangent or Alternate segment theorem

If a line drawn through the end point of a **chord** forms an angle equal to the angle subtended by the chord in the **alternate segment** then the line is a **tangent** (chord-tangent or alternate segment theorem)

Internal angles of any triangle sum to 180°



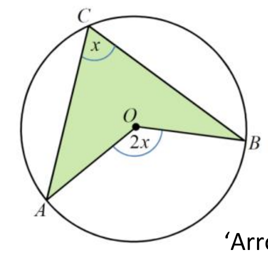
$$A + B + C = 180^\circ$$



Proof of 'Right-angle diameter' theorem

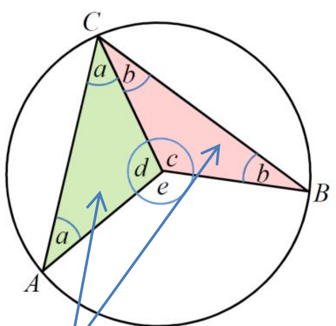
This is a special case of the 'Arrowhead' theorem:

When $2x$ = 180°
this means the arrowhead angle x is half this, i.e. x = 90°.



'Arrowhead' theorem

Proof of the 'Arrowhead' theorem



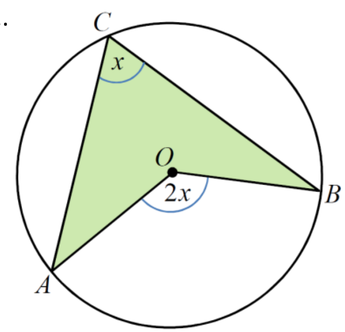
$$\left.\begin{aligned} 2a + d &= 180^\circ \\ 2b + c &= 180^\circ \end{aligned}\right\} \text{Add these together ...}$$

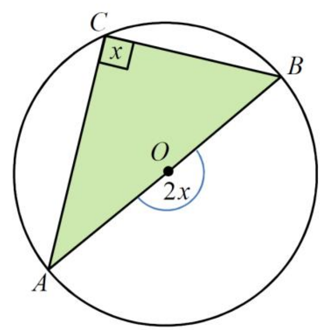
$$\therefore 2(a+b) + d + c = 360^\circ$$

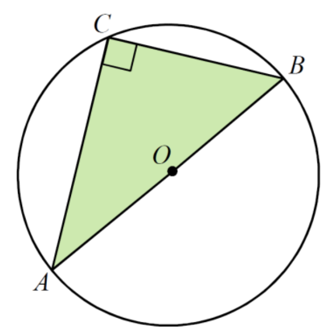
$$d + c + e = 360^\circ$$

$$\therefore d + c + e = 2(a+b) + d + c$$

$$\therefore \boxed{e = 2(a+b)}$$







These are *isosceles triangles*
since they both meet at the *origin* of the circle, and therefore two edges of each triangle are circle radii.

Proof of the 'Mountain' theorem

Consider two arrowheads drawn from the same points *A* and *B* on the circle perimeter.

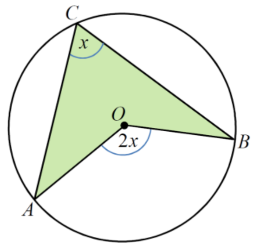
The obtuse angle AOB = 2*a* is the *same* for both arrowheads.

By the 'Arrowhead' theorem, the arrowhead angle must be half this, i.e. *a*.

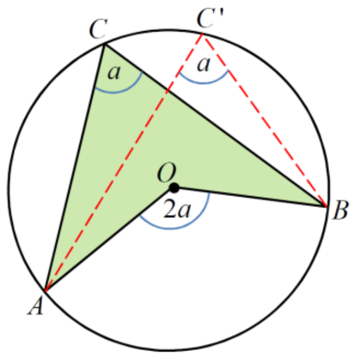
Hence the arrowhead angles at C and C' **must both be *a*.**

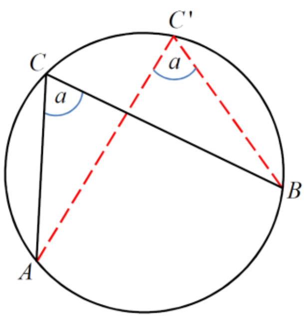
The 'Mountain' theorem is so named because the angles at C and C' look a little like the snowy peaks of mountains!

The 'Searchlight', or 'bow-tie' theorem is another popular name, for similar visual reasons.

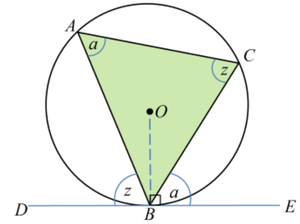


'Arrowhead' theorem





Proof of the Alternate segment theorem



From the diagram

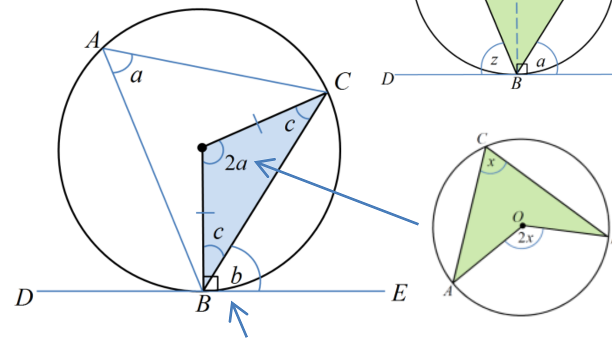
$2a + 2c = 180^\circ$

$\therefore a + c = 90^\circ$

$c + b = 90^\circ$

$\therefore c + b = a + c$

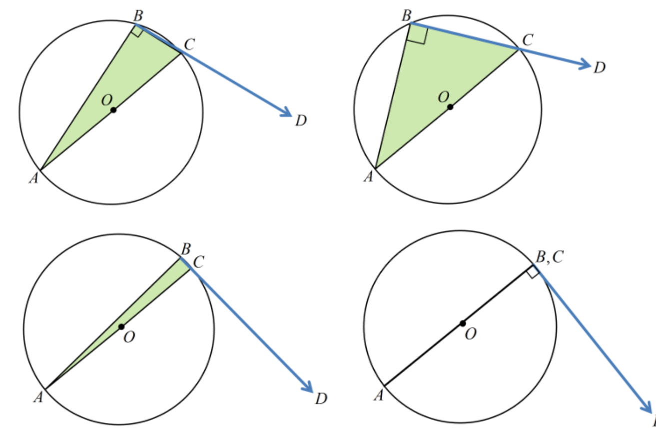
$\therefore \boxed{b = a}$



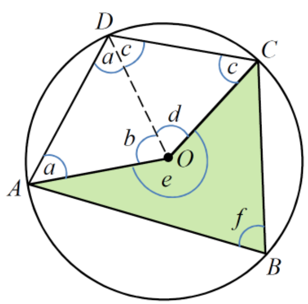
$c + b = 90^\circ$

Note *DE* is a *tangent* to the circle at point *A* hence $c + b = 90^\circ$

This can be proven by application of the 'right angle diameter' theorem . In the picture sequence, BD is a constant, but the chord BC tends to zero.



Proof of the 'Cyclic quadrilateral' theorem



$b + d + e = 360^\circ$

$2a + b = 180^\circ$

$2c + d = 180^\circ$

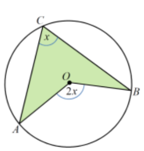
$\therefore 2(a+c) + b + d = b + d + e$

$\therefore\ 2(a+c) = e$

Which essentially shows the 'Arrowhead' theorem *generalizes* for any 'external' angle at AOC. i.e. reflex angles as well as obtuse or indeed acute varieties.

From the 'Arrowhead' theorem

$2f = b + d$



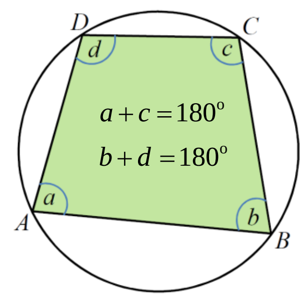
Putting these results together

$b + d + e = 360^\circ$

$\therefore 2f + 2(a+c) = 360^\circ$

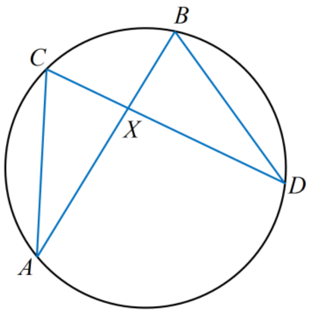
$\therefore \boxed{f + a + c = 180^\circ}$

i.e. **the opposite angles of a cyclic quadrilateral sum to 180°**



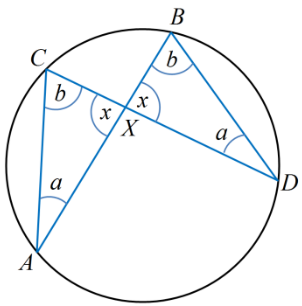
There are two other circle theorems in addition to the main five

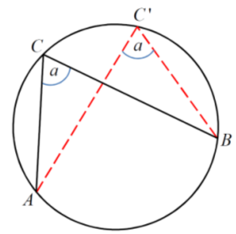
Intersecting chords theorem



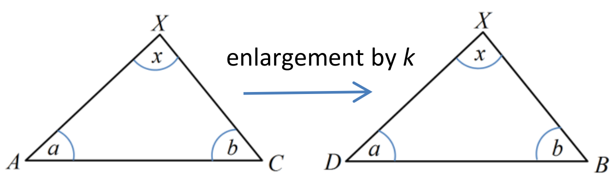
$\boxed{AX \times BX = CX \times DX}$

One can easily prove this result using the 'Mountain Theorem' to label the internal angles





Triangles ACX and DBX are therefore *similar*

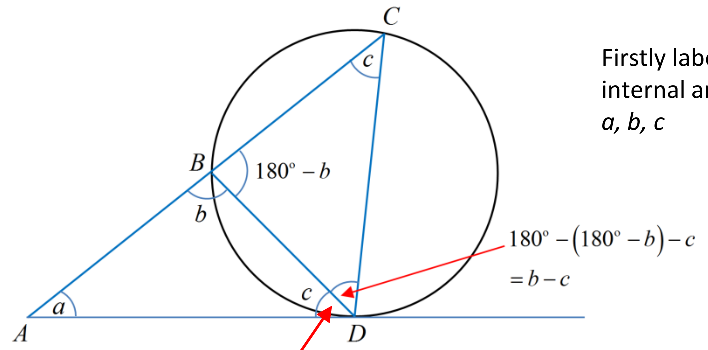


Hence the enlargement factor *k*
between corresponding sides must be the same

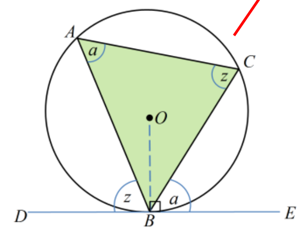
$$k = \frac{BX}{CX} = \frac{DX}{AX}$$

$$\therefore \boxed{AX \times BX = CX \times DX}$$

Secant / Tangent theorem $\boxed{AC \times BA = AD^2}$



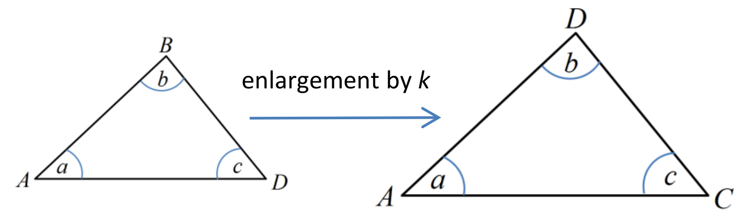
Firstly label internal angles *a, b, c*



Use the Alternate segment theorem to show that angle ADB is also *c*

Hence angle ADC is b

Triangles ABD and ADC are therefore *similar*

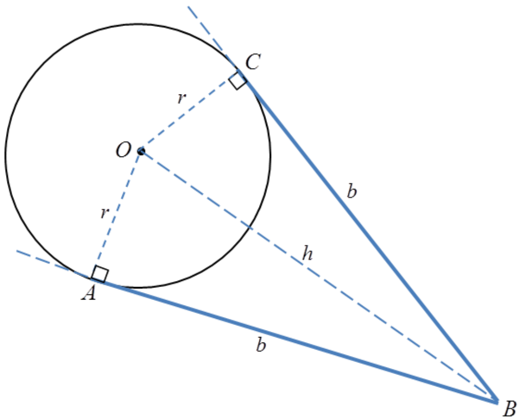


Hence the enlargement factor *k*
between corresponding sides must be the same

$$k = \frac{AD}{BA} = \frac{AC}{AD}$$

$$\therefore \boxed{AC \times BA = AD^2}$$

Further circle theorem notes



Tangents from an external point are equal in length.

This is perhaps obvious on symmetry grounds, but can be proven formally since triangles OCB and OAB have the following properties:

(i) A right angle at, respectively, A and C since lines AB and CB are tangents to the circle
(ii) The sides OC and OA are circle radii so must be the same length
(iii) The side OB is common to both triangles

Hence using *Pythagoras' Theorem*, $h^2 = r^2 + b^2$ the tangent lengths CB and AB must be the same.