Transformations using matrices: Invariant Lines

Generalized reflections, stretches and shears based on an *invariant line* can be derived by combining transformations parallel to the x axis with pre and post rotations.

An *invariant line* is one which is unaffected by the transformation. In other words, coordinates along the invariant lines are transformed to their original locations.

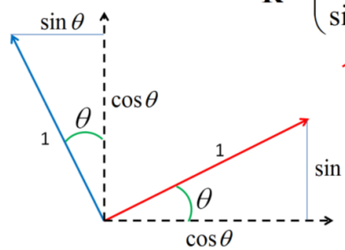
Generalized reflection in line $y = x\tan\theta$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Anticlockwise rotation by θ — Reflection in the x axis — Clockwise rotation by θ

Rotation anticlockwise about (0,0) by angle θ

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

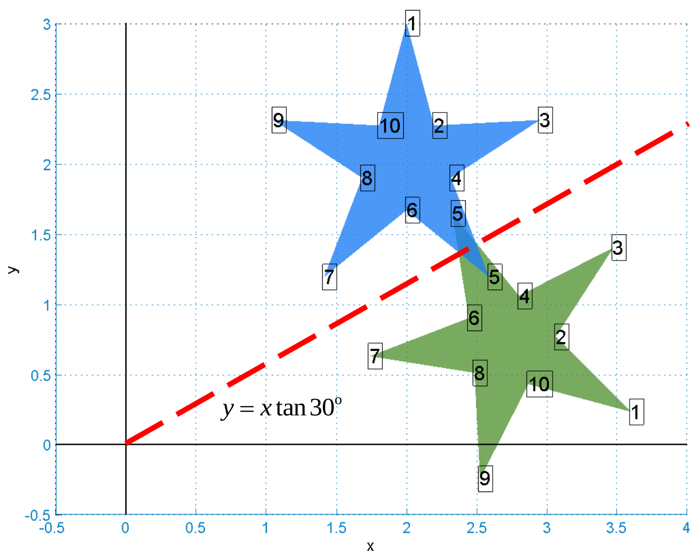


$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}$$

$$\boxed{\mathbf{R} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}}$$



Generalized shear (factor k) parallel to line in line $y = x\tan\theta$ i.e. invariant line is $y = x\tan\theta$

$$\mathbf{S} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

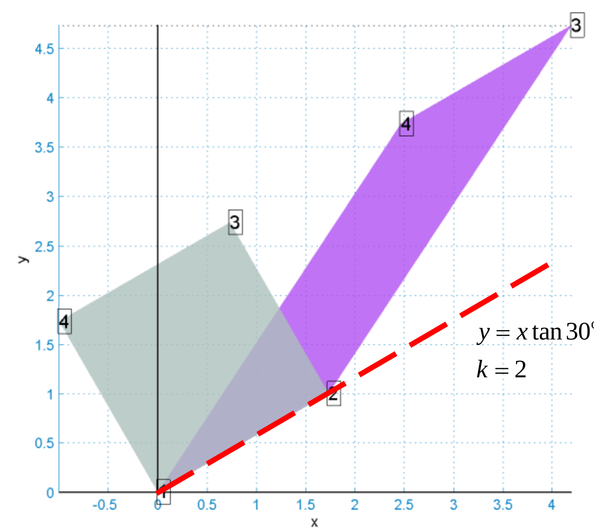
Anticlockwise rotation by θ — Shear (factor k) x axis invariant — Clockwise rotation by θ

$$\mathbf{S} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta - k\sin\theta & \sin\theta + k\cos\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \cos^2\theta - k\cos\theta\sin\theta + \sin^2\theta & \cos\theta\sin\theta + k\cos^2\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - k\sin^2\theta - \cos\theta\sin\theta & \sin^2\theta + k\sin\theta\cos\theta + \cos^2\theta \end{pmatrix}$$

$$\boxed{\mathbf{S} = \begin{pmatrix} 1 - \tfrac{1}{2}k\sin 2\theta & k\cos^2\theta \\ -k\sin^2\theta & 1 + \tfrac{1}{2}k\sin 2\theta \end{pmatrix}}$$



Generalized stretch parallel to line in line $y = x\tan\theta$
i.e. Invariant line is $y = -x\cot\theta$

$$\mathbf{E} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Anticlockwise rotation by θ

Stretch scale factor k parallel to the x axis

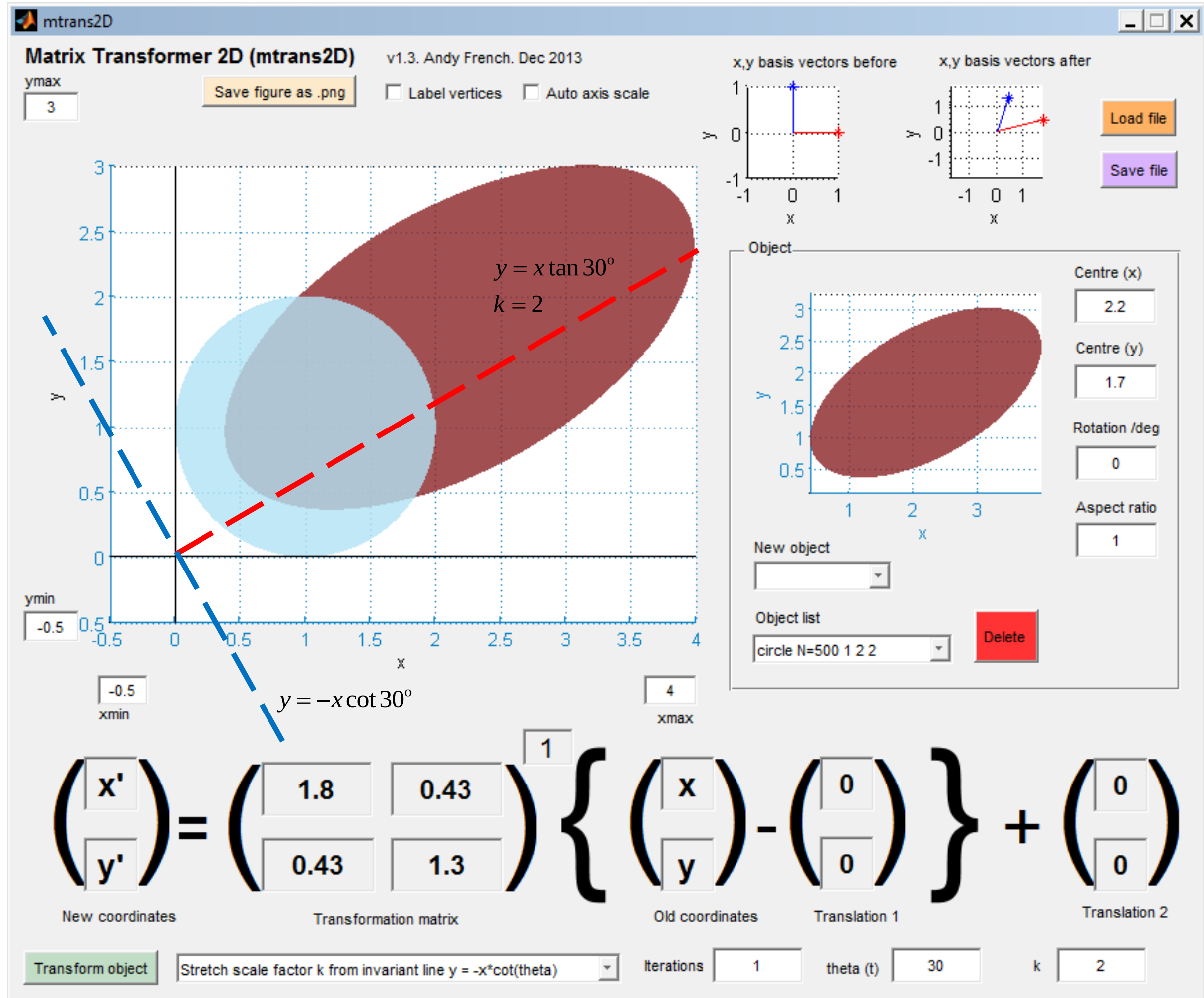
Clockwise rotation by θ

$$\mathbf{E} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k\cos\theta & k\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} k\cos^2\theta + \sin^2\theta & k\cos\theta\sin\theta - \sin\theta\cos\theta \\ k\cos\theta\sin\theta - \cos\theta\sin\theta & k\sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$\boxed{\mathbf{E} = \begin{pmatrix} k\cos^2\theta + \sin^2\theta & \tfrac{1}{2}(k-1)\sin 2\theta \\ \tfrac{1}{2}(k-1)\sin 2\theta & k\sin^2\theta + \cos^2\theta \end{pmatrix}}$$



Rotation anticlockwise about (0,0) by angle θ

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$