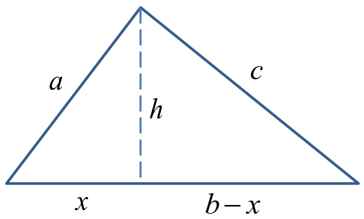
Proof of Hero's Formula* for the area of a triangle



$a^2 = x^2 + h^2$ Pythagoras' Theorem

$c^2 = (b-x)^2 + h^2$

$\therefore c^2 - a^2 = b^2 - 2bx + x^2 + h^2 - x^2 - h^2$

$\therefore c^2 - a^2 = b^2 - 2bx$

$\therefore x = \dfrac{a^2+b^2-c^2}{2b}$

$a^2 = x^2 + h^2$

$\therefore h^2 = a^2 - \dfrac{\left(a^2+b^2-c^2\right)^2}{4b^2}$

$h^2 = \dfrac{4a^2b^2 - \left(a^2+b^2-c^2\right)^2}{4b^2}$

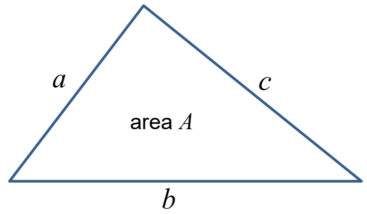
$h^2 = \dfrac{(2ab)^2 - \left(a^2+b^2-c^2\right)^2}{4b^2}$ Difference of two squares

$h^2 = \dfrac{\left(2ab+a^2+b^2-c^2\right)\left(2ab-a^2-b^2+c^2\right)}{4b^2}$

$h^2 = \dfrac{\left((a+b)^2-c^2\right)\left(c^2-(a-b)^2\right)}{4b^2}$

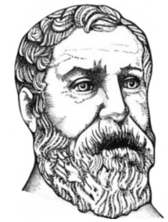
$h^2 = \dfrac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4b^2}$

Difference of two squares again!



$A = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \tfrac{1}{2}(a+b+c)$



Hero of Alexandria
10-70 AD

Define *semi-perimeter s*

$2s = a+b+c$

$\therefore h^2 = \dfrac{2s(2s-2c)(2s-2b)(2s-2a)}{4b^2}$

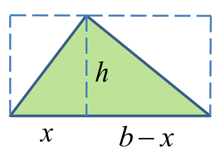
$h^2 = \dfrac{4s(s-a)(s-b)(s-c)}{b^2}$

Area of triangle A is:

$A = \tfrac{1}{2}bh = \sqrt{\tfrac{1}{4}b^2h^2}$

$\therefore A = \sqrt{s(s-a)(s-b)(s-c)}$

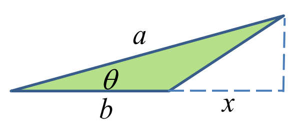
Triangle area basics



Area is clearly:

$A = \tfrac{1}{2}xh + \tfrac{1}{2}(b-x)h$

$A = \tfrac{1}{2}bh$



$h = a\sin\theta$

The triangle area formula "half base x perpendicular height" works in general since

$A = \tfrac{1}{2}(b+x)h - \tfrac{1}{2}xh = \tfrac{1}{2}bh = \tfrac{1}{2}ab\sin\theta$

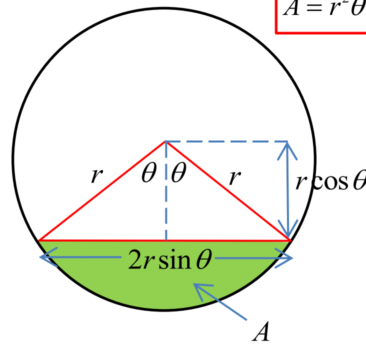
Area of a segment

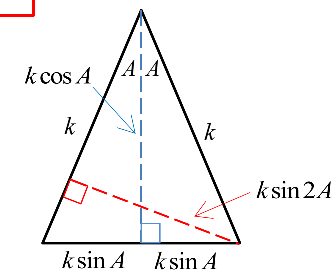
$A = \dfrac{2\theta}{2\pi}\pi r^2 - \tfrac{1}{2}2r\sin\theta \times r\cos\theta$

$A = r^2\theta - r^2\sin\theta\cos\theta$

Angle θ in radians

π radians $= 180^\circ$





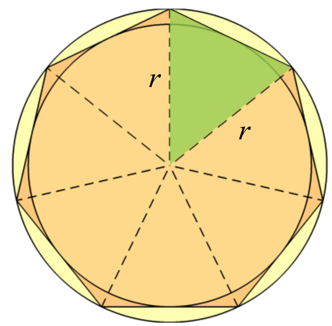
Double angle sine identity

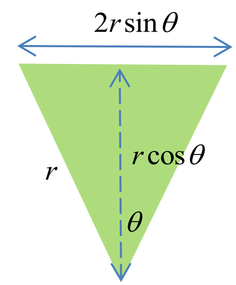
Area of an isosceles triangle can be calculated in two ways:

$\tfrac{1}{2}k \times k\sin 2A = 2 \times \tfrac{1}{2}k\sin A \times k\cos A$

$\therefore \sin 2A = 2\sin A\cos A$

Area of a regular polygon





$A = n \times \tfrac{1}{2}2r\sin\theta \times r\cos\theta$

$A = \tfrac{1}{2}nr^2 2\sin\theta\cos\theta$

$A = \tfrac{1}{2}nr^2\sin 2\theta$

$A = \tfrac{1}{2}nr^2\sin\left(\dfrac{2\pi}{n}\right)$

$2\theta n = 2\pi$

$\therefore \theta = \dfrac{\pi}{n}$

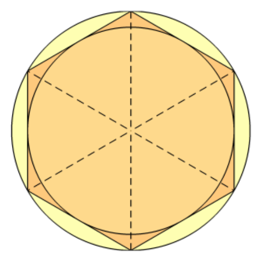
Area of a regular polygon of n sides

Radius of **inscribed circle** is: $r' = r\cos\theta = r\cos\left(\dfrac{\pi}{n}\right)$

Side length of polygon is: $a = 2r\sin\theta = 2r\sin\left(\dfrac{\pi}{n}\right)$

Hence: $A = \tfrac{1}{2}nr^2 2\sin\theta\cos\theta; \quad r = \dfrac{a}{2\sin\theta}$

$\therefore A = \tfrac{1}{2}n\dfrac{a^2}{4\sin^2\theta}2\sin\theta\cos\theta = \tfrac{1}{4}na^2\cot\left(\dfrac{\pi}{n}\right)$

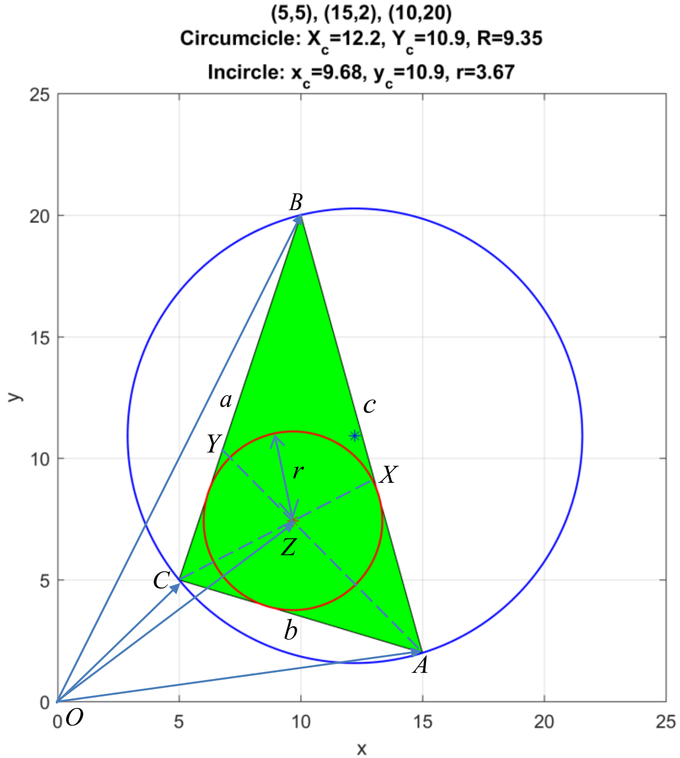


Example: area of a regular **hexagon** is:

$A = \tfrac{1}{2}6r^2\sin\left(\dfrac{2\pi}{6}\right) = 3r^2\sin\left(\tfrac{\pi}{3}\right) = \dfrac{3\sqrt{3}}{2}r^2$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

*also known as *Heron's Formula*



Inscribing a triangle with a circle – the 'incircle'

Inputs: $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$

Triangle *incircle* has a centre $\overrightarrow{OZ}$ at the intersections of the *angle bisectors* defined by the lines:

$\overrightarrow{OA}+\lambda\overrightarrow{AY}$ and $\overrightarrow{OC}+\mu\overrightarrow{CX}$ i.e $\overrightarrow{OA}+\lambda\overrightarrow{AY}=\overrightarrow{OC}+\mu\overrightarrow{CX}$

Using angle bisector theorem

$$\overrightarrow{AY}=-\overrightarrow{OA}+\overrightarrow{OC}+\tfrac{b}{b+c}\overrightarrow{CB}$$
$$\overrightarrow{AY}=-\overrightarrow{OA}+\overrightarrow{OC}+\tfrac{b}{b+c}\left(-\overrightarrow{OC}+\overrightarrow{OB}\right)$$
$$\overrightarrow{AY}=-\overrightarrow{OA}+\overrightarrow{OC}\left(1-\tfrac{b}{b+c}\right)+\tfrac{b}{b+c}\overrightarrow{OB}$$
$$\overrightarrow{AY}=-\overrightarrow{OA}+\tfrac{b}{b+c}\overrightarrow{OC}+\tfrac{b}{b+c}\overrightarrow{OB}$$

$$\overrightarrow{CX}=-\overrightarrow{OC}+\overrightarrow{OA}+\tfrac{b}{a+b}\overrightarrow{AB}$$
$$\overrightarrow{CX}=-\overrightarrow{OC}+\overrightarrow{OA}+\tfrac{b}{a+b}\left(-\overrightarrow{OA}+\overrightarrow{OB}\right)$$
$$\overrightarrow{CX}=-\overrightarrow{OC}+\overrightarrow{OA}\left(1-\tfrac{b}{a+b}\right)+\tfrac{b}{a+b}\overrightarrow{OB}$$
$$\overrightarrow{CX}=-\overrightarrow{OC}+\tfrac{a}{a+b}\overrightarrow{OA}+\tfrac{b}{a+b}\overrightarrow{OB}$$

Hence:

$$\overrightarrow{OA}+\lambda\overrightarrow{AY}=\overrightarrow{OC}+\mu\overrightarrow{CX}$$
$$\overrightarrow{OA}+\lambda\left\{-\overrightarrow{OA}+\tfrac{b}{b+c}\overrightarrow{OC}+\tfrac{b}{b+c}\overrightarrow{OB}\right\}=\overrightarrow{OC}+\mu\left\{-\overrightarrow{OC}+\tfrac{a}{a+b}\overrightarrow{OA}+\tfrac{b}{a+b}\overrightarrow{OB}\right\}$$
$$\overrightarrow{OA}\left(1-\lambda-\mu\tfrac{a}{a+b}\right)+\overrightarrow{OB}\left(\lambda\tfrac{b}{b+c}-\mu\tfrac{b}{a+b}\right)+\overrightarrow{OC}\left(\lambda\tfrac{b}{b+c}-1+\mu\right)=0$$
$$\therefore 1-\lambda-\mu\tfrac{a}{a+b}=0 \qquad \therefore \lambda\tfrac{b}{b+c}-\mu\tfrac{b}{a+b}=0 \qquad \therefore \lambda\tfrac{b}{b+c}-1+\mu=0$$

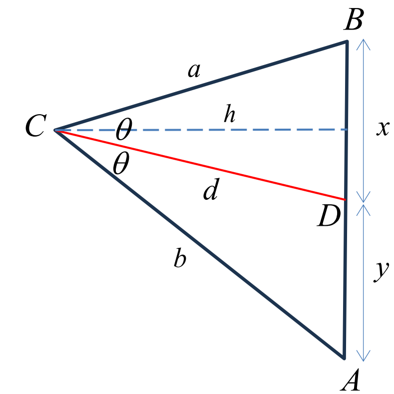
$$\therefore \lambda=\mu\frac{b+c}{a+b}$$
$$\Rightarrow 1-\mu\frac{b+c}{a+b}=\mu\frac{a}{a+b}$$
$$\Rightarrow a+b=\mu\left(a+b+c\right)$$
$$\Rightarrow \mu=\frac{a+b}{a+b+c}$$

$$\overrightarrow{OZ}=\overrightarrow{OC}+\mu\overrightarrow{CX}$$
$$\mu=\frac{a+b}{a+b+c}$$
$$\overrightarrow{CX}=-\overrightarrow{OC}+\tfrac{a}{a+b}\overrightarrow{OA}+\tfrac{b}{a+b}\overrightarrow{OB}$$
$$\therefore \overrightarrow{OZ}=\overrightarrow{OC}+\frac{a+b}{a+b+c}\left\{-\overrightarrow{OC}+\tfrac{a}{a+b}\overrightarrow{OA}+\tfrac{b}{a+b}\overrightarrow{OB}\right\}$$
$$\therefore \overrightarrow{OZ}=\tfrac{a}{a+b+c}\overrightarrow{OA}+\tfrac{b}{a+b+c}\overrightarrow{OB}+\overrightarrow{OC}\left(1-\tfrac{a+b}{a+b+c}\right)$$

$$\therefore \overrightarrow{OZ}=\frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c}$$

Vector equation for **centre of inscribed circle** in terms of position vectors of triangle vertices.

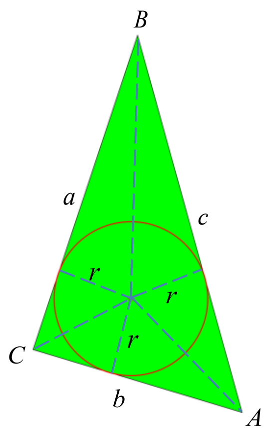
Angle bisector theorem



Area of triangle CDB $\tfrac{1}{2}xh=\tfrac{1}{2}ad\sin\theta$

Area of triangle CAD $\tfrac{1}{2}yh=\tfrac{1}{2}bd\sin\theta$

$$\therefore \frac{x}{y}=\frac{a}{b}$$
$$\Rightarrow \frac{x}{x+y}=\frac{x/y}{x/y+1}=\frac{a/b}{a/b+1}$$
$$\therefore \frac{x}{x+y}=\frac{a}{a+b}$$



Since the inscribed circle is tangent to the sides of the triangle, the triangle area is

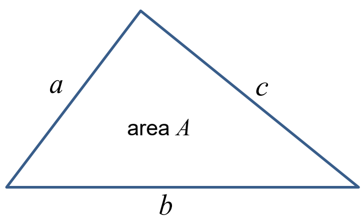
$$\tfrac{1}{2}ar+\tfrac{1}{2}br+\tfrac{1}{2}cr=\tfrac{1}{2}r\left(a+b+c\right)$$

Using Hero's formula:

$$rs=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Radius of inscribed circle in terms of the length of the triangle edges.



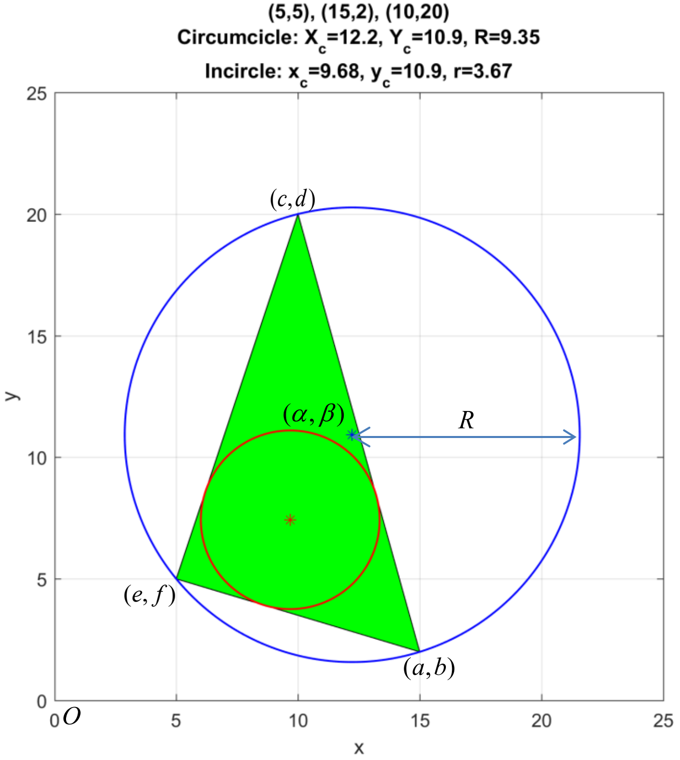
$$A=\sqrt{s(s-a)(s-b)(s-c)}$$
$$s=\tfrac{1}{2}\left(a+b+c\right)$$

Hero of Alexandria
10-70 AD

*also known as *Heron's Formula*

To compute the centre and radius of a **circumcircle** around a triangle define by the vertices $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ one can compute the intersection of the perpendicular bisectors of two of the edges. This will yield the centre of the circumcircle. From this point, calculate the distance to one of the vertices, and you have the radius of the circumcircle.

However, a perhaps more mathematically straightforward recipe is simply to use Pythagoras' theorem and solve directly for the centre and radius, using the (x,y) coordinates of the vertices.



$$(\alpha-a)^2+(\beta-b)^2=R^2 \Rightarrow \alpha^2-2a\alpha+a^2+\beta^2-2b\beta+b^2=R^2 \quad (1)$$
$$(\alpha-c)^2+(\beta-d)^2=R^2 \Rightarrow \alpha^2-2c\alpha+c^2+\beta^2-2d\beta+d^2=R^2 \quad (2)$$
$$(\alpha-e)^2+(\beta-f)^2=R^2 \Rightarrow \alpha^2-2e\alpha+e^2+\beta^2-2f\beta+f^2=R^2 \quad (3)$$

$$-2a\alpha+2c\alpha+a^2-c^2-2b\beta+2d\beta+b^2-d^2=0 \quad (1)-(2)$$
$$\therefore \alpha=\frac{\beta(2d-2b)+d^2+c^2-b^2-a^2}{2c-2a}$$

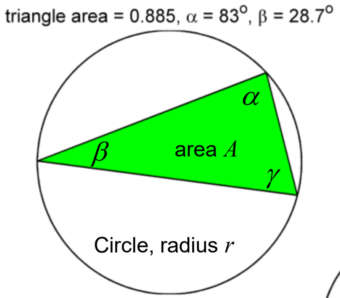
$$-2a\alpha+2e\alpha+a^2-e^2-2b\beta+2f\beta+b^2-f^2=0 \quad (1)-(3)$$
$$\therefore \alpha=\frac{\beta(2f-2b)+f^2+e^2-b^2-a^2}{2e-2a}$$

$$\therefore \frac{\beta(2d-2b)+d^2+c^2-b^2-a^2}{2c-2a}=\frac{\beta(2f-2b)+f^2+e^2-b^2-a^2}{2e-2a}$$
$$\therefore \beta\left\{\frac{d-b}{c-a}-\frac{f-b}{e-a}\right\}=\frac{f^2+e^2-b^2-a^2}{2e-2a}-\frac{d^2+c^2-b^2-a^2}{2c-2a}$$

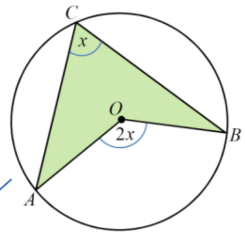
Equating expressions for α

$$\therefore \beta=\frac{\frac{f^2+e^2-b^2-a^2}{2e-2a}-\frac{d^2+c^2-b^2-a^2}{2c-2a}}{\frac{d-b}{c-a}-\frac{f-b}{e-a}}$$
$$\therefore \alpha=\frac{\beta(2d-2b)+d^2+c^2-b^2-a^2}{2c-2a}$$
$$\therefore R=\sqrt{(\alpha-a)^2+(\beta-b)^2}$$

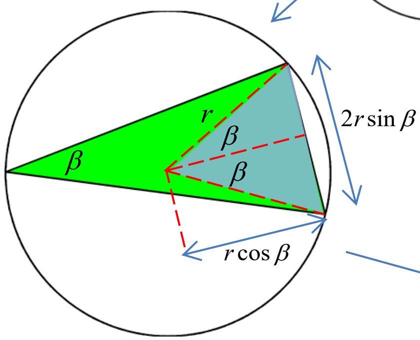
Centre (α,β) and **radius** R of the **circumcircle** of a triangle defined by vertices (a,b), (c,d) and (e,f).

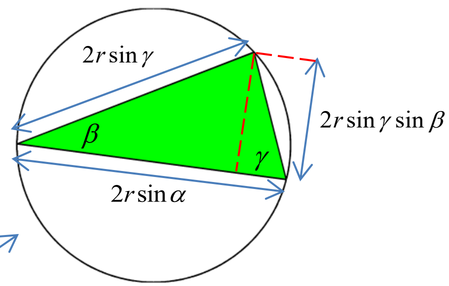


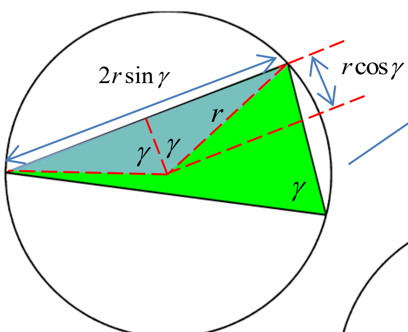
Area of a triangle given radius r of circumscribed circle and angles α, β

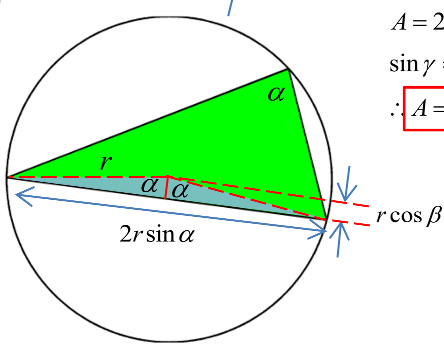


"Arrowhead" circle theorem









Hence:

$$A = \tfrac{1}{2}2r\sin\alpha \times 2r\sin\gamma\sin\phi$$
$$A = 2r^2\sin\alpha\sin\beta\sin\gamma$$
$$\sin\gamma = \sin\left(180^\circ - \alpha - \beta\right) = \sin\left(\alpha+\beta\right)$$
$$\therefore A = 2r^2\sin\alpha\sin\beta\sin\left(\alpha+\beta\right)$$

Green – **Median lines** meet at the **centroid**
This is the location of the *centre of mass* of a uniform triangular lamina.

$$\begin{pmatrix} x_{ct} \\ y_{ct} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}a\cos\theta + \frac{1}{3}b \\ \frac{1}{3}a\sin\theta \end{pmatrix}$$

Red – The **inscribed circle** or radius r.
The **incentre** is the intersection of **angle bisectors** of the elevated sides from the base

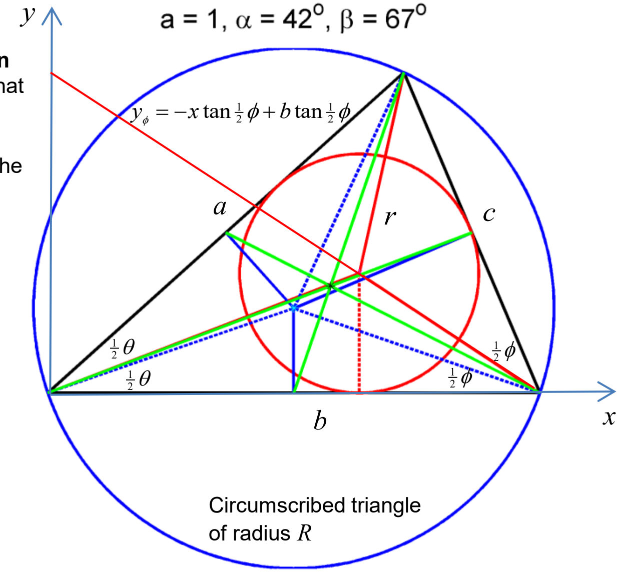
$$y_\theta = x\tan\tfrac{1}{2}\theta; \quad y_\phi = -x\tan\tfrac{1}{2}\phi + b\tan\tfrac{1}{2}\phi$$
$$\therefore x_i\tan\tfrac{1}{2}\theta = -x_i\tan\tfrac{1}{2}\phi + b\tan\tfrac{1}{2}\phi$$
$$\therefore x_i = \frac{b\tan\frac{1}{2}\phi}{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\phi}$$
$$\therefore y_i = \frac{b\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi}{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\phi}$$

The inscribed circle is tangential to the base, so the radius is:

$$r = y_i = \frac{b\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi}{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\phi}$$

Euler's theorem in geometry states that the distance d between the circumcentre and the incentre is:

$$d = \sqrt{R(R-2r)}$$



Blue – The **circumcentre** of the triangle is the intersection of **perpendicular bisectors** of the sides of the triangle

$$x = \tfrac{1}{2}b \qquad y = -\frac{1}{\tan\theta}x + c$$
$$x_{cc} = \tfrac{1}{2}b$$
$$\therefore \tfrac{1}{2}a\sin\theta = -\frac{1}{\tan\theta}\tfrac{1}{2}a\cos\theta + c$$
$$\therefore \tfrac{1}{2}a\sin\theta + \tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$
$$\therefore y = -\frac{1}{\tan\theta}x + \tfrac{1}{2}a\sin\theta + \tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$
$$\therefore y_{cc} = -\frac{1}{\tan\theta}\tfrac{1}{2}b + \tfrac{1}{2}a\sin\theta + \tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

The input parameters for this situation are most naturally b, θ and ϕ
Therefore using the **sine rule**:

$$a = \frac{b\sin\phi}{\sin\left(\theta+\phi\right)} \qquad c = \frac{b\sin\theta}{\sin\left(\theta+\phi\right)}$$

Area of triangle is:

$$A = \tfrac{1}{2}ab\sin\theta$$

From analysis on the left hand side of this page**, the radius of the circumcircle** is given by:

$$2R\sin\left(180^\circ - \theta - \phi\right) = b$$
$$R\sin\left(\theta+\phi\right) = \tfrac{1}{2}b$$
$$R = \frac{\frac{1}{2}b}{\sin\left(\theta+\phi\right)}$$

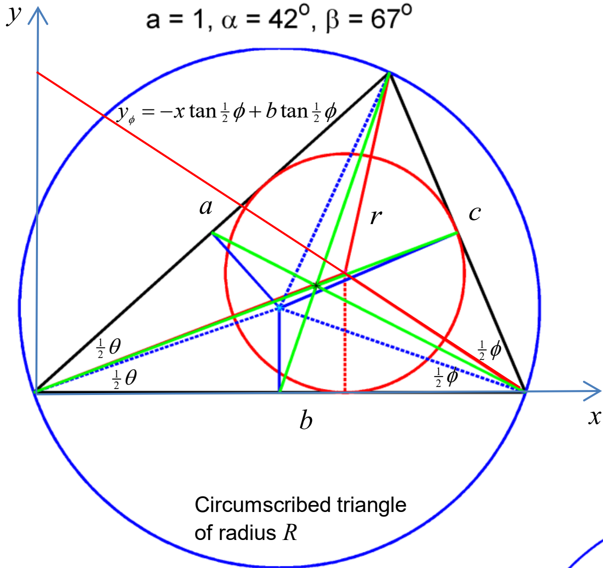
The **top vertex** of the triangle has coordinates

$$\begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}$$

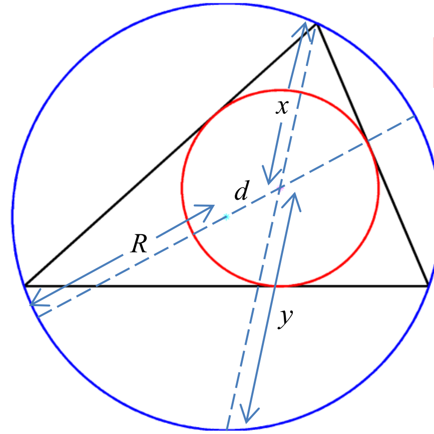
Euler's theorem in geometry states that the distance d between the circumcentre and the incentre is:

$$d=\sqrt{R(R-2r)}$$

a = 1, α = 42°, β = 67°

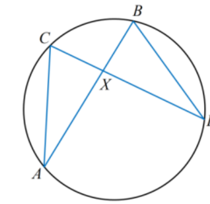


Circumscribed triangle of radius R



Intersecting chords circle theorem

$$(R-d)(R+d)=xy$$



$$AX\times BX = CX\times DX$$

Leonhard Euler
1707-1783

Incentre

$$x_i=\frac{b\tan\tfrac{1}{2}\phi}{\tan\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\phi}$$
$$y_i=x_i\tan\tfrac{1}{2}\theta$$
$$r=y_i$$

Circumcentre

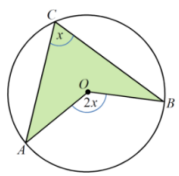
$$x_{cc}=\tfrac{1}{2}b$$

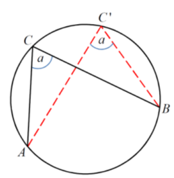
$$y_{cc}=-\frac{1}{\tan\theta}\tfrac{1}{2}b+\tfrac{1}{2}a\sin\theta+\tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

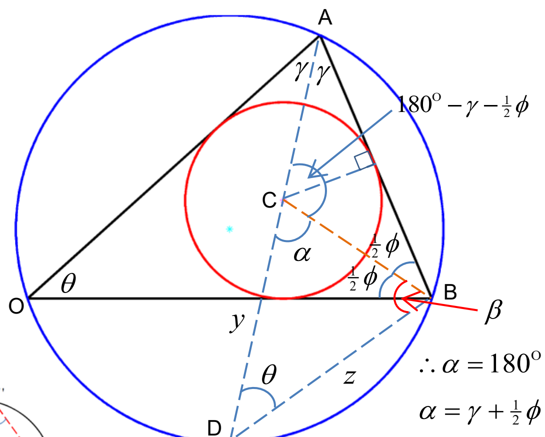
$$y_{cc}=\frac{-b\cos\theta+a\sin^2\theta+a\cos^2\theta}{2\sin\theta}$$

$$y_{cc}=\frac{a-b\cos\theta}{2\sin\theta}$$

$$R=\frac{\tfrac{1}{2}b}{\sin(\theta+\phi)}$$







$$\therefore \alpha=180^\circ-\left(180^\circ-\gamma-\tfrac{1}{2}\phi\right)$$
$$\alpha=\gamma+\tfrac{1}{2}\phi$$

$$\therefore \alpha=\beta$$
$$\therefore y=z$$

Isosceles triangle

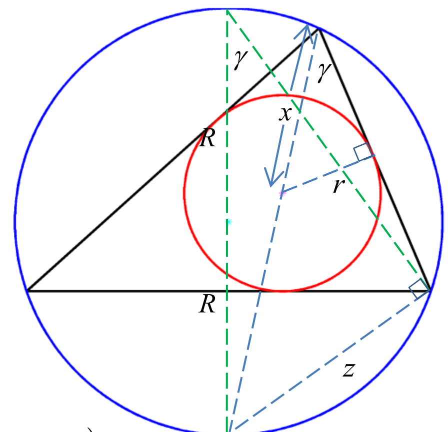
$\theta=180^\circ-2\gamma-\phi$ Black triangle OAB

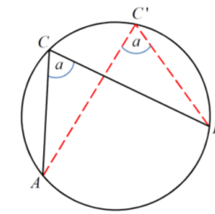
$180^\circ=\theta+\beta+\alpha$ Triangle from incentre DCB

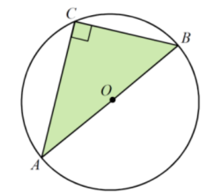
$$\therefore 180^\circ=\left(180^\circ-2\gamma-\phi\right)+\beta+\alpha$$
$$\therefore 2\gamma+\phi-\alpha=\beta$$
$$2\gamma+\phi-\left(\gamma+\tfrac{1}{2}\phi\right)=\beta \quad \leftarrow \quad \alpha=\gamma+\tfrac{1}{2}\phi$$
$$\gamma+\tfrac{1}{2}\phi=\beta$$
$$\therefore \alpha=\beta$$

Hence:

$$(R-d)(R+d)=xz=2Rr$$
$$\therefore R^2-d^2=2Rr$$
$$\therefore d=\sqrt{R(R-2r)}$$







$$\frac{2R}{z}=\frac{x}{r}\ \therefore 2Rr=xz$$

Similar triangles