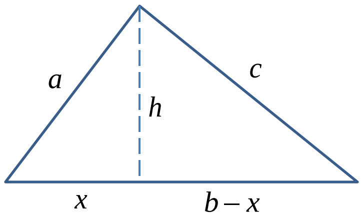
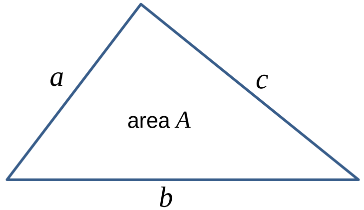
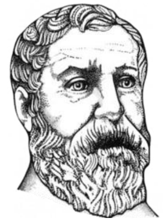
Proof of Hero's Formula* for the area of a triangle





$$A=\sqrt{s(s-a)(s-b)(s-c)}$$
$$s=\tfrac{1}{2}(a+b+c)$$



Hero of Alexandria
10-70 AD

$a^2=x^2+h^2$ Pythagoras' Theorem

$c^2=(b-x)^2+h^2$

$\therefore c^2-a^2=b^2-2bx+x^2+h^2-x^2-h^2$

$\therefore c^2-a^2=b^2-2bx$

$$\therefore x=\frac{a^2+b^2-c^2}{2b}$$

$a^2=x^2+h^2$

$$\therefore h^2=a^2-\frac{\left(a^2+b^2-c^2\right)^2}{4b^2}$$

$$h^2=\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4b^2}$$

$$h^2=\frac{(2ab)^2-\left(a^2+b^2-c^2\right)^2}{4b^2}$$ Difference of two squares

$$h^2=\frac{\left(2ab+a^2+b^2-c^2\right)\left(2ab-a^2-b^2+c^2\right)}{4b^2}$$

$$h^2=\frac{\left((a+b)^2-c^2\right)\left(c^2-(a-b)^2\right)}{4b^2}$$

$$h^2=\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4b^2}$$

Difference of two squares again!

Define *semi-perimeter s*

$$2s=a+b+c$$

$$\therefore h^2=\frac{2s(2s-2c)(2s-2b)(2s-2a)}{4b^2}$$

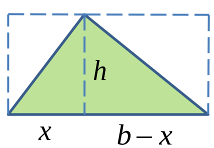
$$h^2=\frac{4s(s-a)(s-b)(s-c)}{b^2}$$

Area of triangle A is:

$$A=\tfrac{1}{2}bh=\sqrt{\tfrac{1}{4}b^2h^2}$$

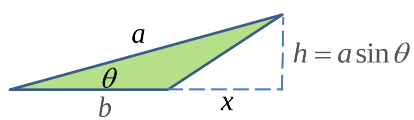
$$\therefore A=\sqrt{s(s-a)(s-b)(s-c)}$$

Triangle area basics



Area is clearly:

$$A=\tfrac{1}{2}xh+\tfrac{1}{2}(b-x)h$$
$$A=\tfrac{1}{2}bh$$



The triangle area formula "half base x perpendicular height" works in general since

$$A=\tfrac{1}{2}(b+x)h-\tfrac{1}{2}xh=\tfrac{1}{2}bh=\tfrac{1}{2}ab\sin\theta$$

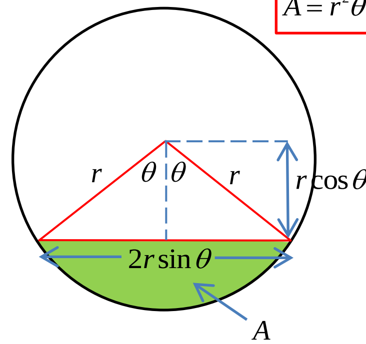
Area of a segment

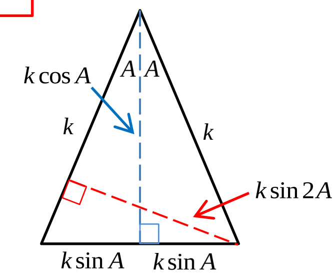
$$A=\frac{2\theta}{2\pi}\pi r^2-\tfrac{1}{2}2r\sin\theta\times r\cos\theta$$

$$A=r^2\theta-r^2\sin\theta\cos\theta$$

Angle θ in radians

π radians $=180^\circ$





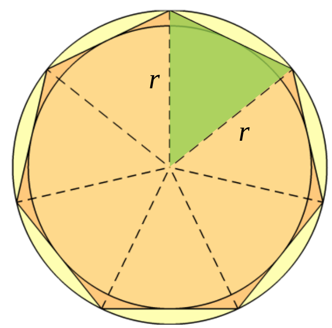
Double angle sine identity

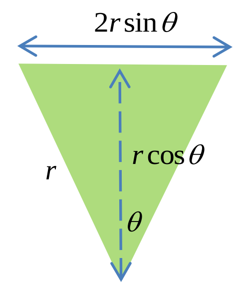
Area of an isosceles triangle can be calculated in two ways:

$$\tfrac{1}{2}k\times k\sin 2A=2\times\tfrac{1}{2}k\sin A\times k\cos A$$

$$\therefore \sin 2A=2\sin A\cos A$$

Area of a regular polygon





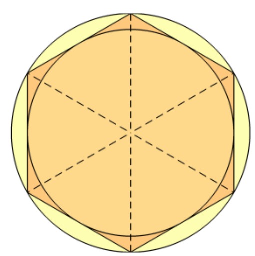
$$A=n\times\tfrac{1}{2}2r\sin\theta\times r\cos\theta$$
$$A=\tfrac{1}{2}nr^2\,2\sin\theta\cos\theta$$
$$A=\tfrac{1}{2}nr^2\sin 2\theta$$
$$A=\tfrac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$$

$$2\theta n=2\pi$$
$$\therefore\theta=\frac{\pi}{n}$$

Area of a regular polygon of n sides

Radius of **inscribed circle** is: $r'=r\cos\theta=r\cos\left(\frac{\pi}{n}\right)$

Side length of polygon is: $a=2r\sin\theta=2r\sin\left(\frac{\pi}{n}\right)$



Hence: $A=\tfrac{1}{2}nr^2\,2\sin\theta\cos\theta;\quad r=\dfrac{a}{2\sin\theta}$

$$\therefore A=\tfrac{1}{2}n\frac{a^2}{4\sin^2\theta}2\sin\theta\cos\theta=\tfrac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)$$

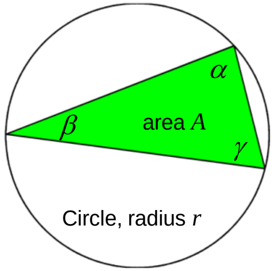
Example: area of a regular **hexagon** is:

$$A=\tfrac{1}{2}6r^2\sin\left(\frac{2\pi}{6}\right)=3r^2\sin\left(\tfrac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}r^2$$

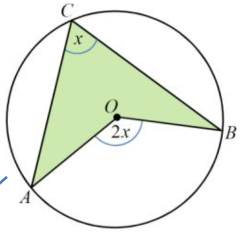
$$\cot\theta=\frac{1}{\tan\theta}=\frac{\cos\theta}{\sin\theta}$$

*also known as *Heron's Formula*

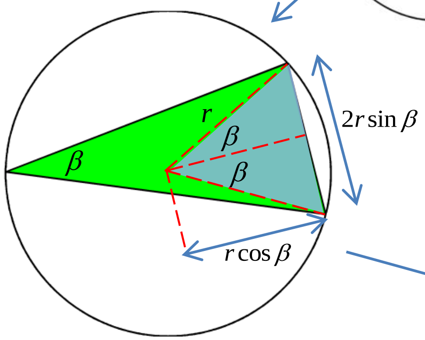
triangle area = 0.885, α = 83°, β = 28.7°

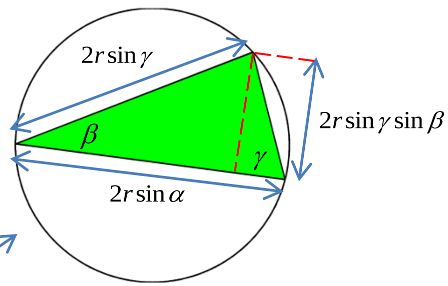


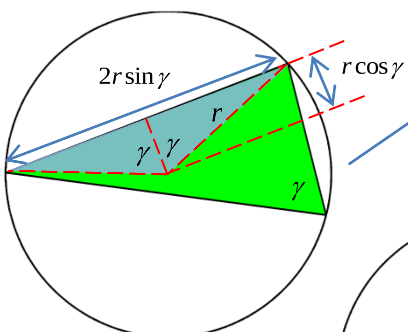
Area of a triangle given radius r of circumscribed circle and angles α, β

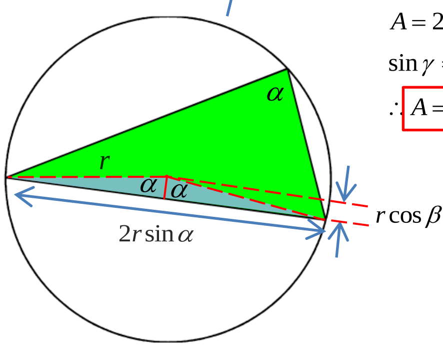


"Arrowhead" circle theorem









Hence:

$$A=\tfrac{1}{2}2r\sin\alpha\times 2r\sin\gamma\sin\phi$$

$$A=2r^2\sin\alpha\sin\beta\sin\gamma$$

$$\sin\gamma=\sin\left(180^\circ-\alpha-\beta\right)=\sin\left(\alpha+\beta\right)$$

$$\therefore A=2r^2\sin\alpha\sin\beta\sin\left(\alpha+\beta\right)$$

Green – **Median lines** meet at the **centroid**
This is the location of the *centre of mass* of a uniform triangular lamina.

$$\begin{pmatrix} x_{ct} \\ y_{ct} \end{pmatrix}=\begin{pmatrix} \tfrac{1}{3}a\cos\theta+\tfrac{1}{3}b \\ \tfrac{1}{3}a\sin\theta \end{pmatrix}$$

Red – The **inscribed circle** or radius r.
The **incentre** is the intersection of **angle bisectors** of the elevated sides from the base

$$y_\theta = x\tan\tfrac{1}{2}\theta;\quad y_\phi=-x\tan\tfrac{1}{2}\phi+b\tan\tfrac{1}{2}\phi$$

$$\therefore x_i\tan\tfrac{1}{2}\theta=-x_i\tan\tfrac{1}{2}\phi+b\tan\tfrac{1}{2}\phi$$

$$\therefore x_i=\frac{b\tan\tfrac{1}{2}\phi}{\tan\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\phi}$$

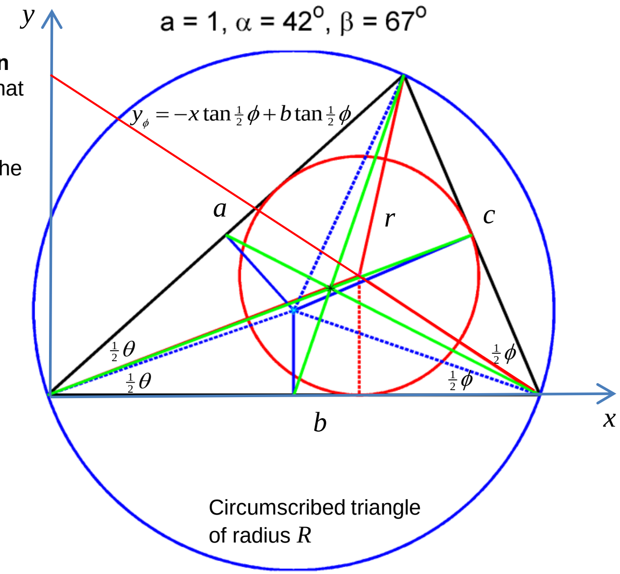
$$\therefore y_i=\frac{b\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi}{\tan\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\phi}$$

The inscribed circle is tangential to the base, so the radius is:

$$r=y_i=\frac{b\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi}{\tan\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\phi}$$

Euler's theorem in geometry states that the distance d between the circumcentre and the incentre is:

$$d=\sqrt{R(R-2r)}$$



Blue – The **circumcentre** of the triangle is the intersection of **perpendicular bisectors** of the sides of the triangle

$$x=\tfrac{1}{2}b \qquad y=-\frac{1}{\tan\theta}x+c$$

$$x_{cc}=\tfrac{1}{2}b$$

$$\therefore \tfrac{1}{2}a\sin\theta=-\frac{1}{\tan\theta}\tfrac{1}{2}a\cos\theta+c$$

$$\therefore \tfrac{1}{2}a\sin\theta+\tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

$$\therefore y=-\frac{1}{\tan\theta}x+\tfrac{1}{2}a\sin\theta+\tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

$$\therefore y_{cc}=-\frac{1}{\tan\theta}\tfrac{1}{2}b+\tfrac{1}{2}a\sin\theta+\tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

Area of triangle is:

$$A=\tfrac{1}{2}ab\sin\theta$$

From analysis on the left hand side of this page**, the radius of the circumcircle** is given by:

$$2R\sin\left(180^\circ-\theta-\phi\right)=b$$

$$R\sin\left(\theta+\phi\right)=\tfrac{1}{2}b$$

$$R=\frac{\tfrac{1}{2}b}{\sin\left(\theta+\phi\right)}$$

The input parameters for this situation are most naturally b, θ and ϕ
Therefore using the **sine rule**:

$$a=\frac{b\sin\phi}{\sin\left(\theta+\phi\right)} \qquad c=\frac{b\sin\theta}{\sin\left(\theta+\phi\right)}$$

The **top vertex** of the triangle has coordinates

$$\begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}$$

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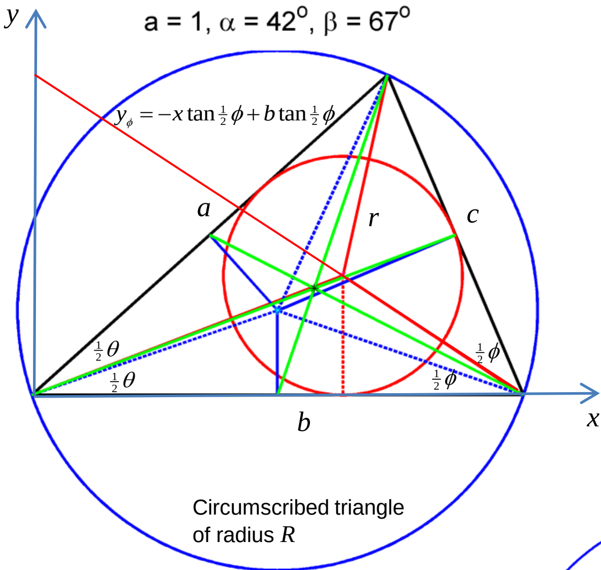
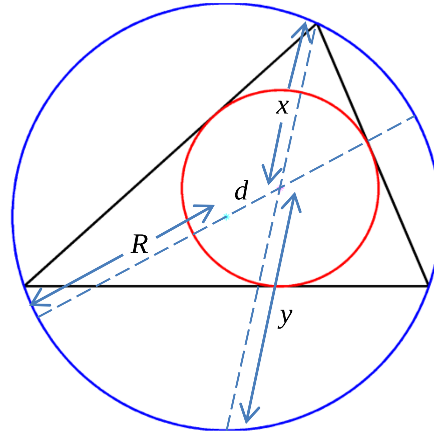
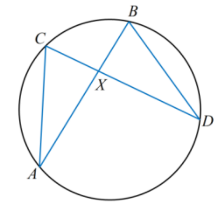




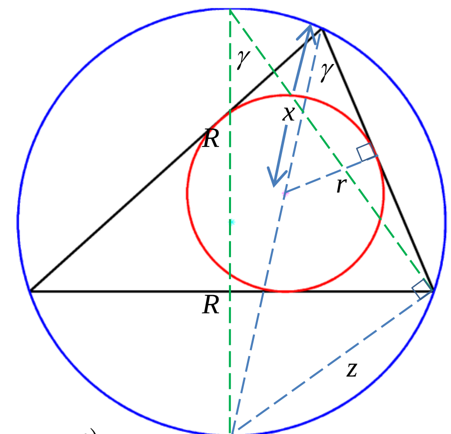
Intersecting chords circle theorem

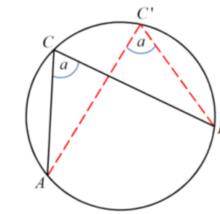
$$(R-d)(R+d)=xy$$

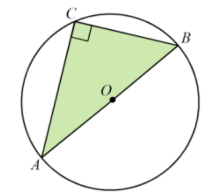


$$AX\times BX=CX\times DX$$

Leonhard Euler 1707-1783







$$\frac{2R}{z}=\frac{x}{r}\therefore 2Rr=xz$$

Similar triangles

Incentre

$$x_i=\frac{b\tan\frac{1}{2}\phi}{\tan\frac{1}{2}\theta+\tan\frac{1}{2}\phi}$$
$$y_i=x_i\tan\frac{1}{2}\theta$$
$$r=y_i$$

Circumcentre

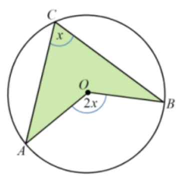
$$x_{cc}=\tfrac{1}{2}b$$

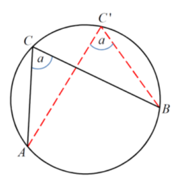
$$y_{cc}=-\frac{1}{\tan\theta}\tfrac{1}{2}b+\tfrac{1}{2}a\sin\theta+\tfrac{1}{2}a\frac{\cos^2\theta}{\sin\theta}$$

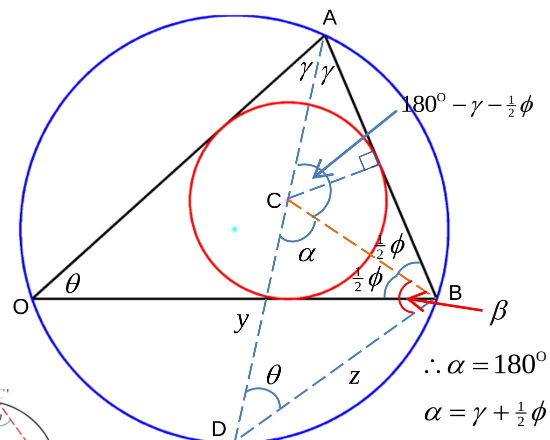
$$y_{cc}=\frac{-b\cos\theta+a\sin^2\theta+a\cos^2\theta}{2\sin\theta}$$

$$y_{cc}=\frac{a-b\cos\theta}{2\sin\theta}$$

$$R=\frac{\frac{1}{2}b}{\sin(\theta+\phi)}$$







$$\therefore\alpha=180^\circ-\left(180^\circ-\gamma-\tfrac{1}{2}\phi\right)$$
$$\alpha=\gamma+\tfrac{1}{2}\phi$$

$$\therefore\alpha=\beta$$
$$\therefore y=z$$

Isosceles triangle

$\theta=180^\circ-2\gamma-\phi$ Black triangle OAB

$180^\circ=\theta+\beta+\alpha$ Triangle from incentre DCB

$$\therefore 180^\circ=\left(180^\circ-2\gamma-\phi\right)+\beta+\alpha$$
$$\therefore 2\gamma+\phi-\alpha=\beta$$
$$2\gamma+\phi-\left(\gamma+\tfrac{1}{2}\phi\right)=\beta \longleftarrow \alpha=\gamma+\tfrac{1}{2}\phi$$
$$\gamma+\tfrac{1}{2}\phi=\beta$$
$$\therefore\alpha=\beta$$

Hence:

$$(R-d)(R+d)=xz=2Rr$$
$$\therefore R^2-d^2=2Rr$$
$$\therefore d=\sqrt{R(R-2r)}$$