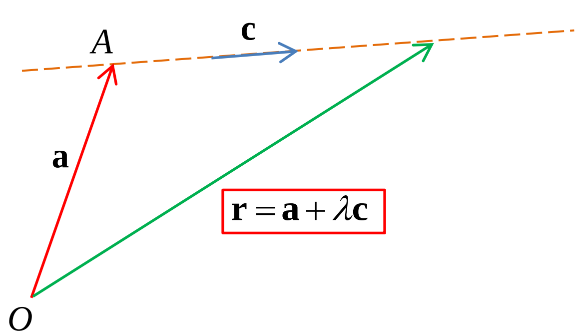
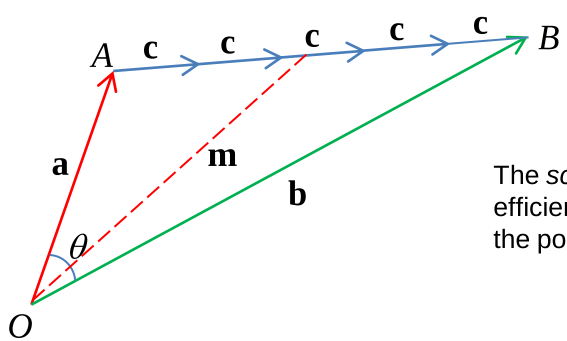
The vector equation of a line



To describe the position vector **r** of a point on a line, in two, three (or more!) dimensions, all we need is a position vector of a point on the line **a**, a vector parallel to the line **c**, and a scalar λ which tells us how many of these vectors makes up the difference between the known point on the line **a** and desired position vector **r**

Example: $\mathbf{b} = \mathbf{a} + 5\mathbf{c}$



Note the *mid-point* of the line between A and B is:

$$\boxed{\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})}$$

The *scalar product* can be used to efficiently find the angle θ between the position vectors **a** and **b**

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$$

Note **a** and **b** are *mutually perpendicular* if

$$\boxed{\mathbf{a} \cdot \mathbf{b} = 0}$$

Example:

$$\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

$$\mathbf{b} = \mathbf{a} + 5\mathbf{c}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -9 \end{pmatrix}$$

$$|\mathbf{a}| = \sqrt{5^2 + 0^2 + 1^2} = \sqrt{26}$$

$$|\mathbf{b}| = \sqrt{15^2 + 5^2 + 9^2} = \sqrt{331}$$

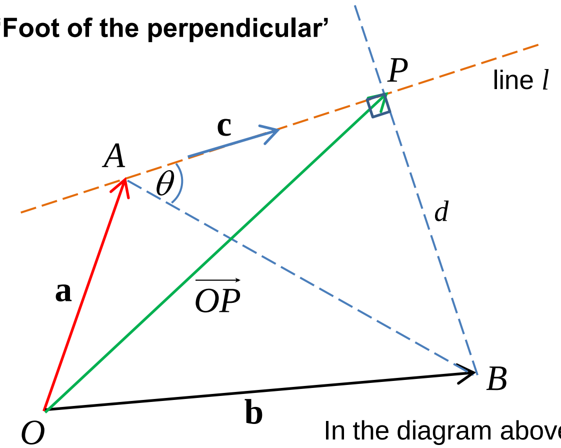
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\therefore \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ 5 \\ -9 \end{pmatrix} = \sqrt{26}\sqrt{331}\cos\theta$$

$$(5)(15) + (0)(5) + (1)(-9) = \sqrt{26}\sqrt{331}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{66}{\sqrt{26}\sqrt{331}}\right) = \boxed{44.6^\circ}$$

'Foot of the perpendicular'



It is often useful to be able to find the position vector of the *foot of the perpendicular*, that is the point on a line which is closest from a point off the line. This is useful is mechanics problems such as orbital calculations when the closest approach of an astronomical body might be desired.

In the diagram above, assume **a,b,c** are known and the position vector of the foot of the perpendicular connecting *B* to line *l* is desired.

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OP}$$

$$\overrightarrow{OP} = \mathbf{a} + \lambda\mathbf{c}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$|\overrightarrow{AB}|\cos\theta = |\overrightarrow{AP}|$$

$$|\mathbf{b} - \mathbf{a}|\cos\theta = \lambda|\mathbf{c}|$$

$$\therefore \lambda = \frac{|\mathbf{b} - \mathbf{a}|\cos\theta}{|\mathbf{c}|}$$

$$\overrightarrow{AB} \cdot \mathbf{c} = |\overrightarrow{AB}||\mathbf{c}|\cos\theta \qquad \therefore \cos\theta = \frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c}}{|\mathbf{b} - \mathbf{a}||\mathbf{c}|}$$

$$\therefore \lambda = \frac{|\mathbf{b} - \mathbf{a}|}{|\mathbf{c}|} \frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c}}{|\mathbf{b} - \mathbf{a}||\mathbf{c}|} = \frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c}}{|\mathbf{c}|^2}$$

$$\therefore \boxed{\overrightarrow{OP} = \mathbf{a} + \frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}}$$

Pythagoras' Theorem

$$|\overrightarrow{AB}|^2 = |\overrightarrow{AP}|^2 + d^2 \quad \therefore d = \sqrt{|\overrightarrow{AB}|^2 - |\overrightarrow{AP}|^2}$$

$$d = \sqrt{|\mathbf{b} - \mathbf{a}|^2 - \lambda^2|\mathbf{c}|^2}$$

$$d = \sqrt{|\mathbf{b} - \mathbf{a}|^2 - \left(\frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c}}{|\mathbf{c}|^2}\right)^2 |\mathbf{c}|^2}$$

$$\boxed{d = \sqrt{|\mathbf{b} - \mathbf{a}|^2 - \frac{((\mathbf{b} - \mathbf{a}) \cdot \mathbf{c})^2}{|\mathbf{c}|^2}}}$$

Example:

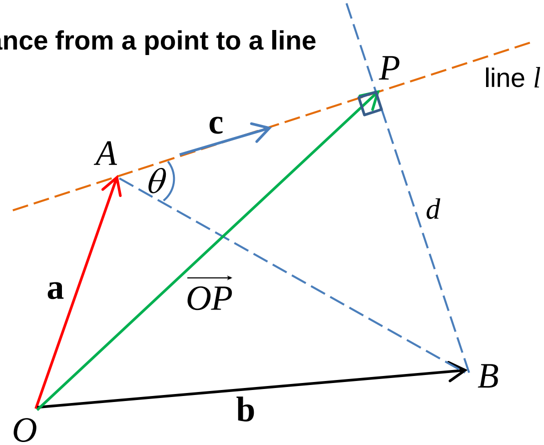
$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{\left(\begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{(\sqrt{2})^2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}$$

$$d = \sqrt{80 - \frac{8^2}{(\sqrt{2})^2}} = \boxed{\sqrt{48}}$$

Distance from a point to a line



On the previous page, the 'foot of the perpendicular' analysis gives the distance of point B from the line *l* as

$$\boxed{d = \sqrt{|\mathbf{b}-\mathbf{a}|^2 - \frac{\left((\mathbf{b}-\mathbf{a})\cdot\mathbf{c}\right)^2}{|\mathbf{c}|^2}}}$$

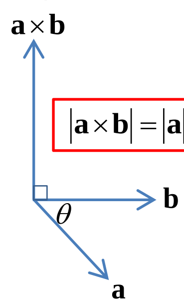
Also: $\boxed{\overrightarrow{OP} = \mathbf{a} + \frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}}$

Alternatively we can use the *vector-cross product*

$$d = \left|\overrightarrow{AB}\right|\sin\theta$$

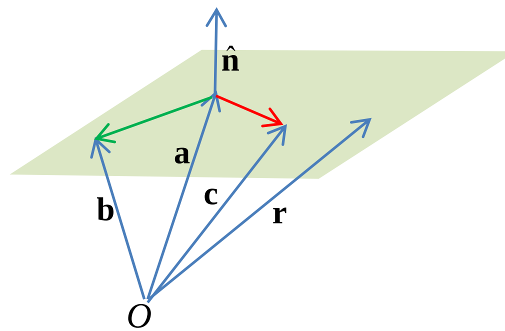
$$\left|\overrightarrow{AB}\times\mathbf{c}\right| = \left|\overrightarrow{AB}\right||\mathbf{c}|\sin\theta$$

$$\therefore \boxed{d = \frac{|(\mathbf{b}-\mathbf{a})\times\mathbf{c}|}{|\mathbf{c}|}}$$



$$\boxed{|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta}$$

The vector equation of a plane



To define a *plane* we need three points which lie on it given by position vectors **a,b,c**

The *unit normal* to the plane can be computed

$$\boxed{\hat{\mathbf{n}} = \frac{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})}{|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|}}$$

which means that **b** – **a** must not be parallel to **c** – **a** for this to be defined i.e.

$$\boxed{|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})| \neq 0}$$

The position vector **r** of another point on the plane must therefore satisfy the equation

$$\boxed{(\mathbf{r}-\mathbf{a})\cdot\hat{\mathbf{n}} = 0}$$

since **r** – **a** is parallel to the plane and hence must be perpendicular to the normal.

We can associate **b** – **a** and **c** – **a** as basis vectors for describing positions on the plane, hence

$$\boxed{\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) + \mu(\mathbf{c}-\mathbf{a})}$$

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

or

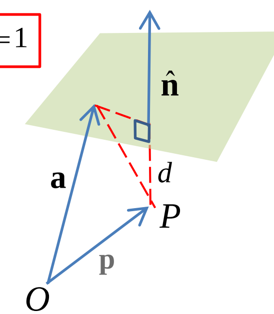
$$\boxed{\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}}$$

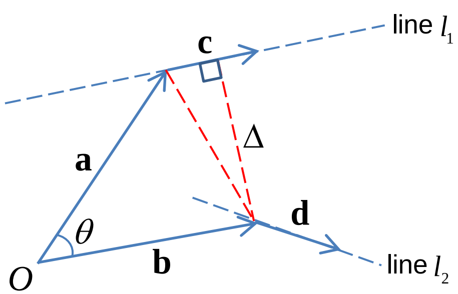
$$\alpha = 1-\beta-\gamma \Rightarrow \boxed{\alpha+\beta+\gamma = 1}$$

Distance from a point to a plane

The shortest distance d from position vector **p** to the plane (characterized by vector **a** in the plane and the unit normal to the plane) is the projection of **a** – **p** upon the unit normal

$$\boxed{d = |(\mathbf{a}-\mathbf{p})\cdot\hat{\mathbf{n}}|}$$





Distance from a line to a line

The minimum distance Δ between two lines is the *projection* of the displacement between two known points on respective lines (**a** and **b**) and the *mutual perpendicular* to both lines, which can be found from the *cross product* of respective line direction vectors **c** and **d**.

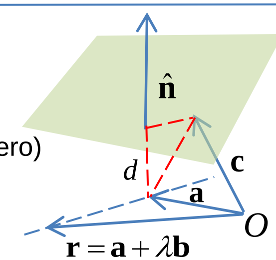
$$\boxed{\Delta = \frac{|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}\times\mathbf{d})|}{|\mathbf{c}\times\mathbf{d}|}}$$

Note if this distance is not zero then the lines are *askew* i.e. there will be no intersection.

Distance from a line to a plane

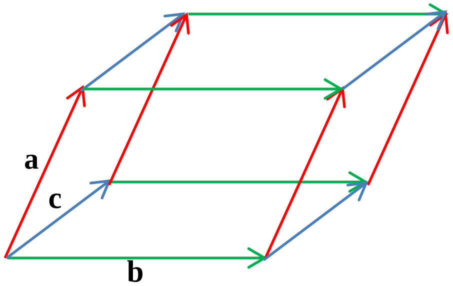
Line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ will intersect the plane at some point (i.e. minimum distance is zero) unless $\hat{\mathbf{n}}\cdot\mathbf{b} = 0$

In this case: $\boxed{d = |(\mathbf{c}-\mathbf{a})\cdot\hat{\mathbf{n}}|}$



NOTE: **c** is *any* position vector in the plane

Calculate the volume of a parallelepiped via a scalar triple product



Volume of a parallelepiped formed from vectors a, b, c is the scalar triple product of these vectors

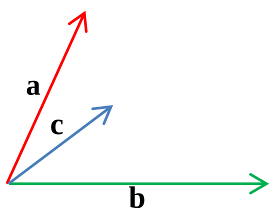
$$\boxed{V = [\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

Scalar triple product identities

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}]$$

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{c},\mathbf{b},\mathbf{a}]$$

Reciprocal vectors, which are useful in describing diffraction of electromagnetic waves from crystal lattice structures are a set of basis vectors {**a'**,**b'**,**c'**} which are related to another set of basis vectors {**a**,**b**,**c**} which describe positions within a crystal lattice.



$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

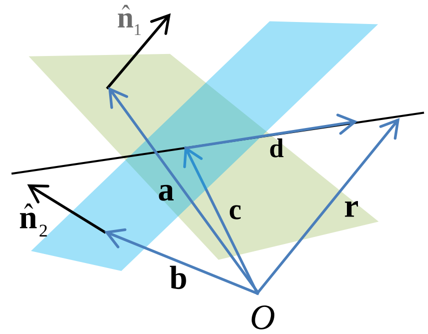
$$\mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

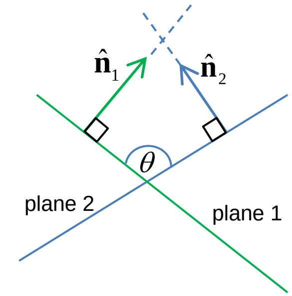
$$\mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

Cartesian equation of a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{c}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\boxed{\lambda = \frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}}$$





Intersection of two planes along line $\mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$

$$\boxed{\begin{aligned}(\mathbf{c}-\mathbf{a})\cdot\hat{\mathbf{n}}_1 &= 0 \\ (\mathbf{c}-\mathbf{b})\cdot\hat{\mathbf{n}}_2 &= 0\end{aligned}}$$

Plane equations

Note we can start anywhere along the intersection line, so there will be an *infinite* number of possible $\mathbf{c}$ vectors. Solving the plane equations for components of $\mathbf{c}$ will therefore yield an answer parameterized by a scalar.

$$\mathbf{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

$$\boxed{\mathbf{d} = \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2}$$

Vector $\mathbf{d}$ is perpendicular to the normals to each plane

$$\boxed{\begin{aligned}\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2 &= \cos\phi \\ |\hat{\mathbf{n}}_1| &= 1 \\ |\hat{\mathbf{n}}_2| &= 1 \\ \theta &= 180^\circ - \phi\end{aligned}}$$

Angle between planes

Cartesian equation of a plane

$$(\mathbf{r}-\mathbf{a})\cdot\hat{\mathbf{n}} = 0$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \hat{\mathbf{n}} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\boxed{\alpha(x-a) + \beta(y-b) + \gamma(z-c) = 0}$$