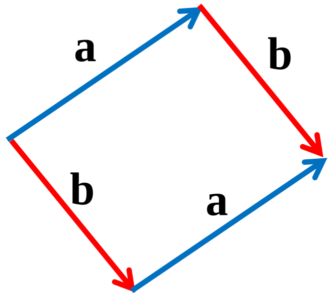
Vectors

A vector is a mathematical object which, in two or three dimensions, describes an *arrow*. i.e. it has a *magnitude* (i.e. length*)* and a *direction*. Many physical quantities such as displacement, velocity, acceleration, momentum, force etc are vectors, which means the laws of Physics are typically written as vector equations.

Vectors are *not* numbers ('scalars') but the algebra of vectors is very similar to the algebra of numbers.

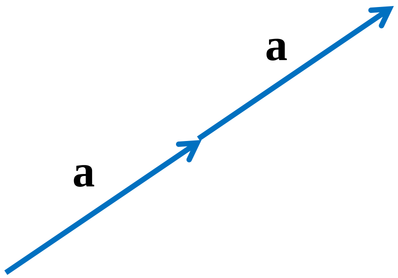


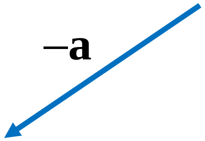
Vectors add 'tip to tail'

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$

Scalar multiplication
Stretches a vector

$$2\mathbf{a}=\mathbf{a}+\mathbf{a}$$





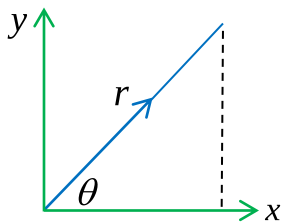
A negative vector (i.e. multiplying by-1) reverses the direction

A vector $\mathbf{r}$ has both *magnitude* and *direction*.

$$|\mathbf{r}|=r \qquad \theta$$

The magnitude (or 'modulus') of a vector can often be worked out using *Pythagoras' Theorem.*

$$r^2=x^2+y^2$$



If we use a two dimensional **Cartesian** (*x,y*), system, then the magnitude of a vector extending from the origin is the **range** *r* of the end of the vector. The direction is defined by the **polar angle** θ. This is typically measured *anticlockwise* from the horizontal *x* axis.

In the 2D Cartesian system it is *conventional* to express a vector in *x* and *y* components as a 2 x 1 matrix. This convention is very useful as it means *pre-multiplication* by a 2 x 2 matrix can *transform one vector into another*

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}10\\3\end{pmatrix}$$

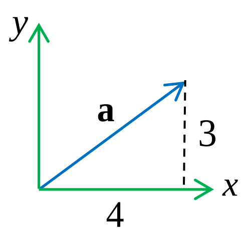
e.g. shear
x axis invariant, shear factor 2

x component

$$\mathbf{a}=\begin{pmatrix}4\\3\end{pmatrix}$$

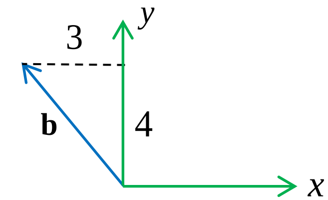
y component

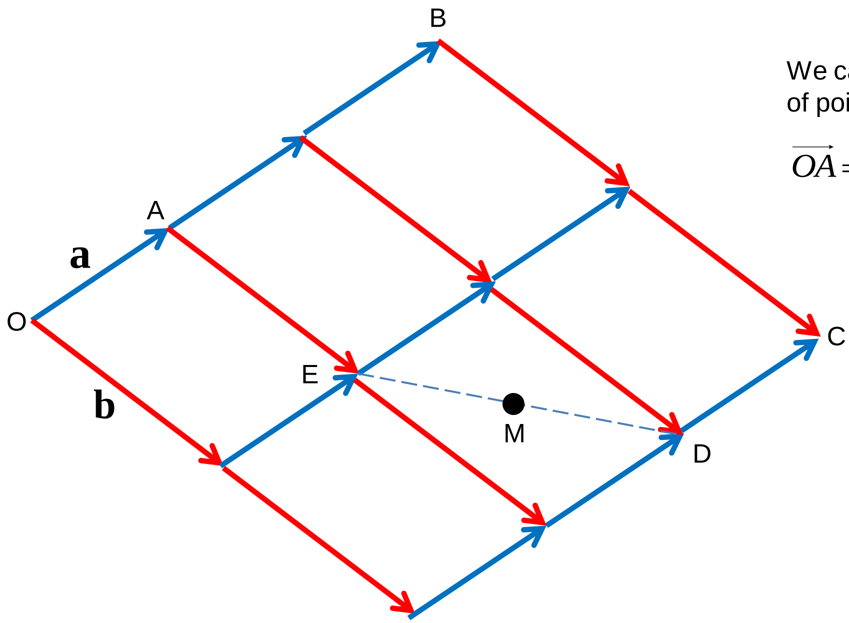
$$|\mathbf{a}|=5$$



$$\mathbf{b}=\begin{pmatrix}-3\\4\end{pmatrix}$$

$$|\mathbf{b}|=5$$





We can construct a *grid system* with vectors and use this to describe the location of points in space in terms of these **basis vectors**.

$\overrightarrow{OA} = \mathbf{a} \quad \overrightarrow{OB} = \mathbf{b}$

$\overrightarrow{OE} = \mathbf{a} + \mathbf{b} \qquad \overrightarrow{ED} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{OB} = 3\mathbf{a} \qquad \overrightarrow{OD} = 2\mathbf{a} + 2\mathbf{b}$

$\overrightarrow{OC} = 3\mathbf{a} + 2\mathbf{b}$

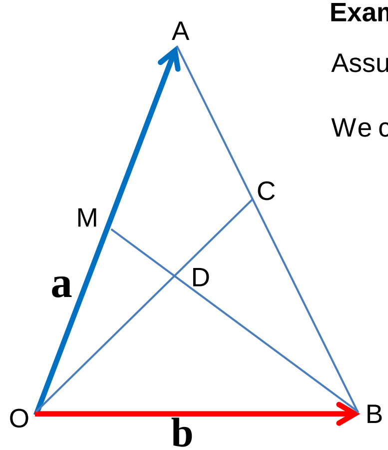
M is the mid-point of the line ED. The position vector of M can be easily Found in terms of vectors **a** and **b** via some simple algebra, and expressing The 'route' we want in terms of *what we already know*.

A nice feature of vector algebra, is that in terms of adding vectors, *all alternative routes are equivalent.*

$$\overrightarrow{OM} = \overrightarrow{OE} + \tfrac{1}{2}\overrightarrow{ED}$$

$$\overrightarrow{OM} = \mathbf{a} + \mathbf{b} + \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\boxed{\overrightarrow{OM} = \tfrac{3}{2}\mathbf{a} + \tfrac{3}{2}\mathbf{b}}$$



Example: Find the position vector $\overrightarrow{OD}$ in terms of basis vectors $\overrightarrow{OA} = \mathbf{a} \quad \overrightarrow{OB} = \mathbf{b}$

Assume ODC is a straight line and M is the mid-point of the line OA and $\overrightarrow{AC} = \tfrac{2}{5}\overrightarrow{AB}$

We can find $\overrightarrow{OD}$ in *two different ways*

$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD}$$

Since MD is parallel to MB

$$\overrightarrow{OD} = \tfrac{1}{2}\mathbf{a} + \alpha\overrightarrow{MB}$$

$$\overrightarrow{OD} = \tfrac{1}{2}\mathbf{a} + \alpha\left(\mathbf{b} - \tfrac{1}{2}\mathbf{a}\right)$$

$$\boxed{\overrightarrow{OD} = \tfrac{1}{2}\mathbf{a}(1-\alpha) + \alpha\mathbf{b}}$$

Since OD is parallel to OC

$$\overrightarrow{OD} = \beta\overrightarrow{OC}$$

$$\overrightarrow{OD} = \beta\left(\overrightarrow{OA} + \overrightarrow{AC}\right)$$

$$\overrightarrow{OD} = \beta\left(\mathbf{a} + \tfrac{2}{5}\overrightarrow{AB}\right)$$

$$\overrightarrow{OD} = \beta\left(\mathbf{a} + \tfrac{2}{5}(-\mathbf{a} + \mathbf{b})\right)$$

$$\boxed{\overrightarrow{OD} = \tfrac{3}{5}\beta\mathbf{a} + \tfrac{2}{5}\beta\mathbf{b}}$$

Comparing the scaling factors of **a** and **b** separately:

$$\overrightarrow{OD} = \tfrac{3}{5}\beta\mathbf{a} + \tfrac{2}{5}\beta\mathbf{b} = \tfrac{1}{2}\mathbf{a}(1-\alpha) + \alpha\mathbf{b}$$

$$\mathbf{a}: \ \tfrac{3}{5}\beta = \tfrac{1}{2}(1-\alpha) \qquad (1)$$

$$\mathbf{b}: \ \tfrac{2}{5}\beta = \alpha \qquad (2)$$

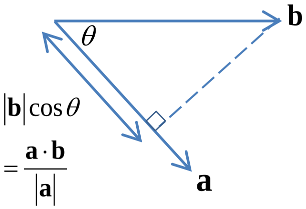
$$2(1)+(2): \ \tfrac{8}{5}\beta = 1 \Rightarrow \beta = \tfrac{5}{8}$$

$$\therefore \text{In}\,(2): \ \alpha = \tfrac{2}{5}\beta \quad \therefore \alpha = \tfrac{1}{4}$$

$$\therefore \ \boxed{\overrightarrow{OD} = \tfrac{3}{8}\mathbf{a} + \tfrac{1}{4}\mathbf{b}}$$

Vector (dot or scalar) product

If the dot product is zero this means vectors are *perpendicular*



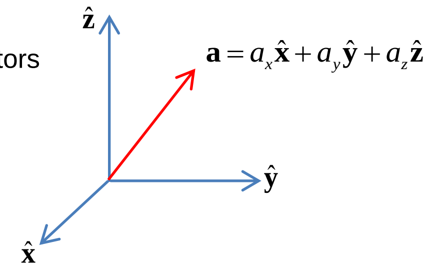
The *scalar product* is proportional to the *projection* or 'shadow' of one vector upon another

$$\boxed{\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta}$$

Cartesian coordinate unit vectors

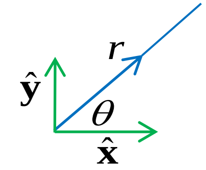
$$\boxed{\begin{array}{l}\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}=0\\ \hat{\mathbf{x}}\cdot\hat{\mathbf{x}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1\end{array}}$$

$$\boxed{\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z}$$



$$\boxed{\begin{array}{l}\mathbf{a}=a_x\hat{\mathbf{x}}+a_y\hat{\mathbf{y}}+a_z\hat{\mathbf{z}}=\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}\\ \mathbf{b}=b_x\hat{\mathbf{x}}+b_y\hat{\mathbf{y}}+b_z\hat{\mathbf{z}}=\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}\end{array}}$$

e.g. $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-2\end{pmatrix}=(1)(-1)+(2)(3)+(3)(-2)=-1$



$|\hat{\mathbf{y}}|=|\hat{\mathbf{x}}|=1$

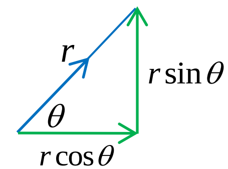
$$\boxed{\mathbf{r}=\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}}$$

x component

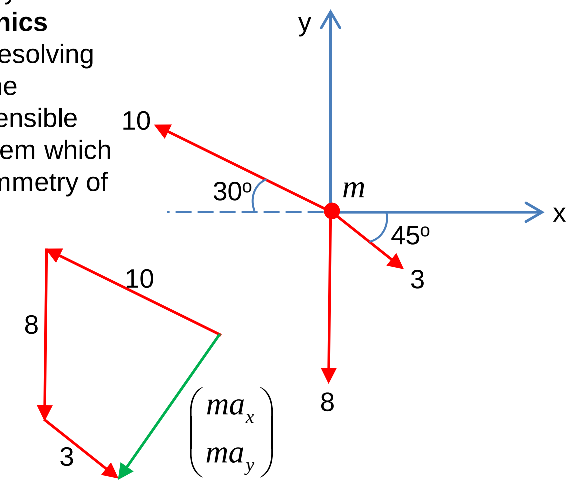
y component

We often speak of '**resolving**' a vector into **components**. Typically these will be with respect to *Cartesian basis vectors,* which are mutually perpendicular and are unit vectors, which means they have unit lengths. (i.e. they are '*orthonormal*').

$$\boxed{\mathbf{r}=r\cos\theta\,\hat{\mathbf{x}}+r\sin\theta\,\hat{\mathbf{y}}}$$

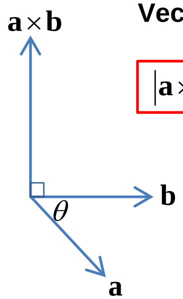


Resolving vectors into Cartesian components is used extensively when solving **Mechanics** problems, i.e. resolving the **forces** in the direction of a sensible coordinate system which exploits the symmetry of a problem.



Newton's Second Law
mass x acceleration = vector sum of forces

Vector (cross) product



$$\boxed{|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta}$$

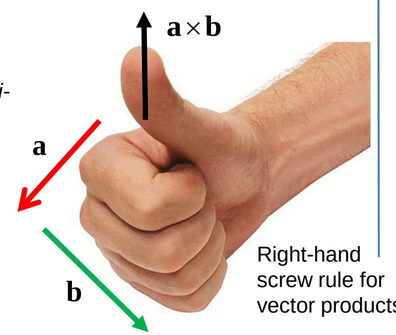
The *cross product* of two vectors creates a third vector which is *mutually perpendicular* to the two other vectors which form the product.

$$\boxed{\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}}$$

Note cross products are *anti-commutative*

$$\boxed{\begin{array}{l}\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}\\ \hat{\mathbf{y}}\times\hat{\mathbf{z}}=\hat{\mathbf{x}}\\ \hat{\mathbf{z}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}\end{array}}$$

Cartesian basis vectors form a *right handed set*



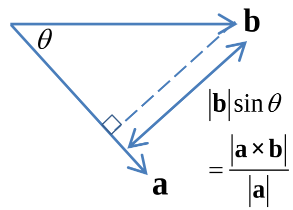
Right-hand screw rule for vector products

$$\boxed{\mathbf{a}\times\mathbf{b}=\hat{\mathbf{x}}\left(a_yb_z-a_zb_y\right)+\hat{\mathbf{y}}\left(a_zb_x-a_xb_z\right)+\hat{\mathbf{z}}\left(a_xb_y-a_yb_x\right)}$$

The magnitude of the cross product is proportional to the *perpendicular distance* of one vector from another. This explains the utility of vector products in Mechanics, as this is a defining property of a *moment of a force* **f** about a rotation axis.

i.e. a *torque* $\boldsymbol{\tau}=\mathbf{r}\times\mathbf{f}$

*(**r** is the displacement from the axis)*



Useful cross-product identities

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

Vector triple product. Note:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$$