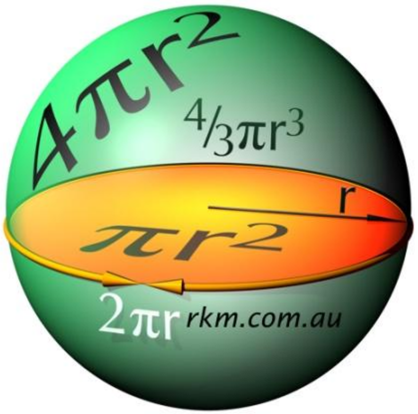
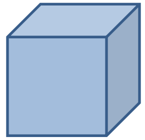
Volumes of basic solids

Sphere of radius r

$\boxed{V=\frac{4}{3}\pi r^3}$

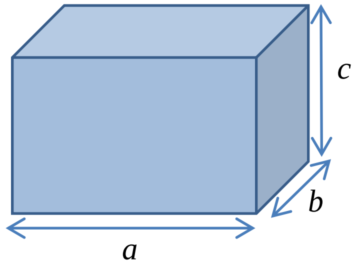


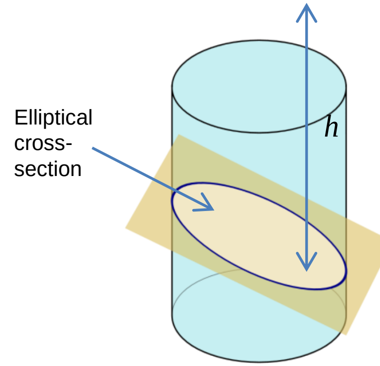
Cube of side a



$\boxed{V=a^3}$

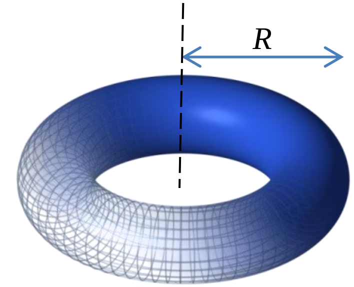
Cuboid $\boxed{V=abc}$





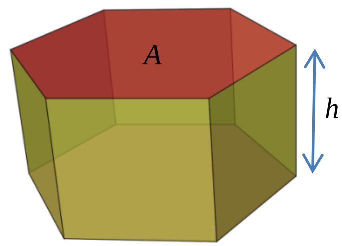
Cylinder, with radius r and height h

$\boxed{V=\pi r^2 h}$



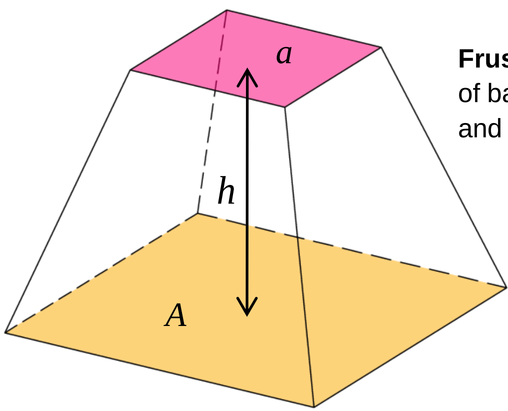
Torus, with a circular cross section of radius r

$\boxed{V=\pi r^2\times 2\pi R=2\pi^2 r^2 R}$



$\boxed{V=Ah}$

Prism, with uniform cross sectional area A and length h



Frustum, (a truncated pyramid) of base area A, top area a and perpendicular height h

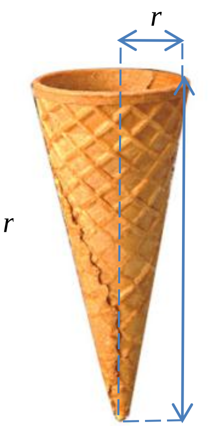
$\boxed{V=\frac{1}{3}h\left(A+\sqrt{aA}+a\right)}$

Pyramid, with base area A and perpendicular height h

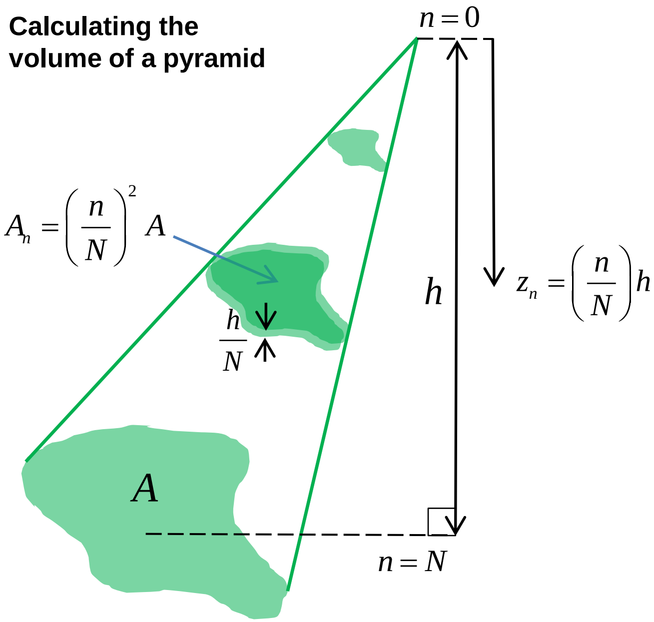
$\boxed{V=\frac{1}{3}Ah}$

Cone, with base radius r and height h

$\boxed{V=\frac{1}{3}\pi r^2 h}$



Calculating the volume of a pyramid



A pyramid of perpendicular height h and base area A can be thought of being composed of an (infinitely) large number of **similar** *laminae*.

The *enlargement factor* of each lamina *is in direct proportion to the perpendicular distance from the apex of the pyramid.* This is true since the sides of the pyramid are straight lines with a constant gradient.

Let there be N laminae of thickness *h/N.* Since all laminae are *similar*, the areas scale as $(n/N)^2$ where n is the lamina number from the apex. The volume of each lamina is:

$$V_n = A_n \times \frac{h}{N}$$

$$V_n = \left(\frac{n}{N}\right)^2 A \times \frac{h}{N}$$

$$V_n = \frac{Ah}{N^3} n^2$$

Therefore the total volume of the pyramid is the sum of the volumes of the N laminae

$$V_N = \frac{Ah}{N^3}1^2 + \frac{Ah}{N^3}2^2 + ... + \frac{Ah}{N^3}N^2$$

$$V_N = \frac{Ah}{N^3}\left(1^2 + 2^2 + ... + N^2\right)$$

Now the sum of the first N square numbers is

$$1^2 + 2^2 + ... + N^2 = \tfrac{1}{6}N(2N+1)(N+1)$$

Hence: $$V_N = \frac{Ah}{6N^3}N(2N+1)(N+1)$$

$$V_N = \frac{Ah}{6N^2}\left(2N^2 + 3N + 1\right)$$

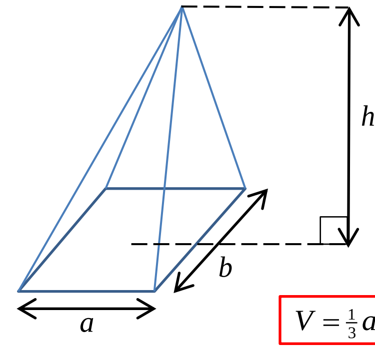
$$V_N = Ah\left(\tfrac{1}{3} + \tfrac{1}{2N} + \tfrac{1}{6N^2}\right)$$

Therefore as N becomes infinite, the volume tend towards

$$\boxed{V = \tfrac{1}{3}Ah}$$

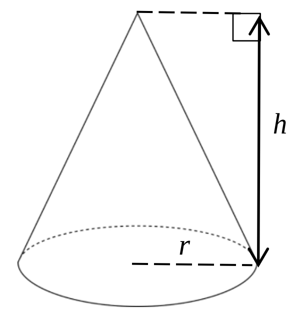
i.e. *"the volume of a pyramid is one third of the base area time the perpendicular height"*

Volume of a rectangular based pyramid



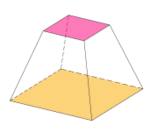
$$\boxed{V = \tfrac{1}{3}abh}$$

Volume of a cone



$$\boxed{V = \tfrac{1}{3}\pi r^2 h}$$

A *frustum* is a *truncated pyramid*



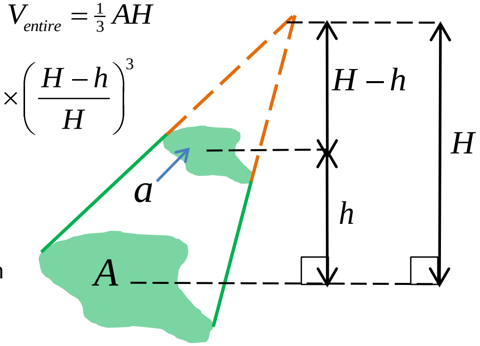
Volume of entire pyramid $V_{entire} = \tfrac{1}{3}AH$

Since cap is *similar* to entire pyramid, volume of removed 'cap' pyramid is: $V_{removed} = \tfrac{1}{3}AH \times \left(\frac{H-h}{H}\right)^3$

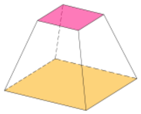
Therefore frustrum volume is $V = V_{enitre} - V_{removed}$

$$\boxed{V = \tfrac{1}{3}AH\left(1 - \left(\frac{H-h}{H}\right)^3\right) = \tfrac{1}{3}h\left(A + a + \sqrt{Aa}\right)}$$

This is proven on the next page



A *frustum* is a *truncated pyramid* Volume of entire pyramid $V_{entire} = \frac{1}{3}AH$



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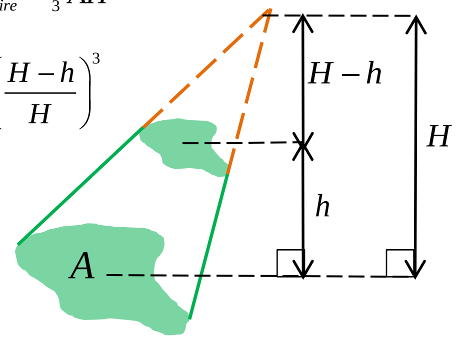
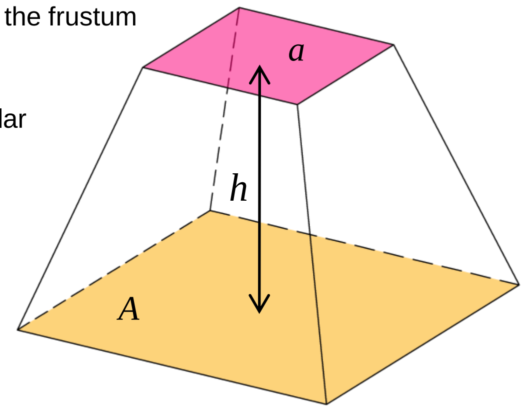


Therefore frustrum volume is $V = V_{enitre} - V_{removed}$

$$\boxed{V = \frac{1}{3}AH\left(1 - \left(\frac{H-h}{H}\right)^3\right)}$$

Now, while correct, this result is not in the most useful form. Typically we will know the base area (A) and top area (a) of the frustum and the perpendicular height *h.*



Since the removed 'cap' pyramid (which has base a) is similar to the entire pyramid

$$a = \left(\frac{H-h}{H}\right)^2 A \quad \therefore \boxed{\left(\frac{H-h}{H}\right)^3 = \left(\frac{a}{A}\right)^{\frac{3}{2}}}$$

Also:

$$a = \left(\frac{H-h}{H}\right)^2 A$$

$$\therefore \frac{H-h}{H} = \left(\frac{a}{A}\right)^{\frac{1}{2}}$$

$$H - h = H\left(\frac{a}{A}\right)^{\frac{1}{2}}$$

$$H - H\left(\frac{a}{A}\right)^{\frac{1}{2}} = h$$

$$\therefore \boxed{H = \frac{h}{1 - \left(\frac{a}{A}\right)^{\frac{1}{2}}}}$$

$$\therefore V = \frac{1}{3}AH\left(1 - \left(\frac{H-h}{H}\right)^3\right)$$

$$V = \frac{1}{3}\frac{Ah}{1 - \left(\frac{a}{A}\right)^{\frac{1}{2}}}\left(1 - \left(\frac{a}{A}\right)^{\frac{3}{2}}\right)$$

To simplify this note the useful result

$$\boxed{y^3 - x^3 = (y-x)(x^2 + xy + y^2)}$$

Hence using the substitution $x = \left(\frac{a}{A}\right)^{\frac{1}{2}}$

$$V = \frac{1}{3}Ah\frac{(1-x^3)}{1-x}$$

$$V = \frac{1}{3}Ah\frac{(1-x)(1+x+x^2)}{1-x}$$

$$V = \frac{1}{3}Ah(1+x+x^2)$$

Which means:

$$V = \frac{1}{3}Ah\left(1 + \left(\frac{a}{A}\right)^{\frac{1}{2}} + \frac{a}{A}\right)$$

This simplifies to a satisfying result in terms of the parameters we area interested in:

$$\boxed{V = \frac{1}{3}h\left(A + \sqrt{aA} + a\right)}$$

And finally ... Consider constructing a *cube* of side length *a* from *six identical square based pyramids*.

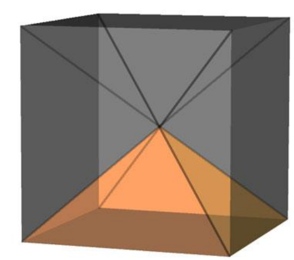
The pyramid height is $h = \frac{1}{2}a$ and base area is a^2

The volume of each pyramid is $V = \frac{1}{6}a^3$

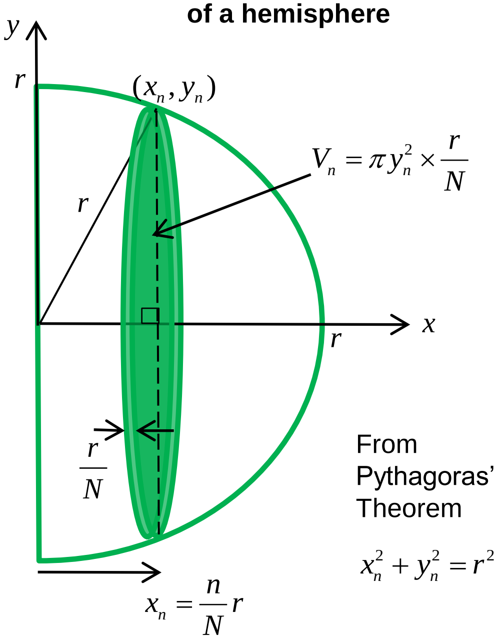
$$\therefore V = \frac{1}{3} \times \frac{1}{2}a \times a^2$$

$$\therefore \boxed{V = \frac{1}{3}Ah}$$

This is obviously a very special case but might help to suggest a formula for the volume of a pyramid.



Calculating the volume of a hemisphere



From Pythagoras' Theorem

$$x_n^2 + y_n^2 = r^2$$

A hemisphere of radius r can be comprised of an (infinitely) large number of discs of radius y.

If there are N discs, the volume of each disc is:

$$V_n = \pi y_n^2 \times \frac{r}{N}$$

Hence:

$$V_n = \frac{\pi r}{N}\left(r^2 - x_n^2\right)$$

$$V_n = \frac{\pi r}{N}\left(r^2 - \frac{n^2}{N^2} r^2\right)$$

$$V_n = \frac{\pi r^3}{N^3}\left(N^2 - n^2\right)$$

N discs therefore have volume

$$V_N = \frac{\pi r^3}{N^3}\left\{N^2 - 1^2 + N^2 - 2^2 + ... + N^2 - N^2\right\}$$

$$V_N = \frac{\pi r^3}{N^3}\left\{N \times N^2 - \left(1^2 + 2^2 + ... + N^2\right)\right\}$$

$$V_N = \pi r^3 - \frac{\pi r^3}{N^3}\left(1^2 + 2^2 + ... + N^2\right)$$

$$V_N = \pi r^3 - \frac{\pi r^3}{N^3}\tfrac{1}{6}N(2N+1)(N+1)$$

Using formula for the sum of the first N square numbers

$$V_N = \pi r^3 - \frac{\pi r^3}{6N^2}\left(2N^2 + 3N + 1\right)$$

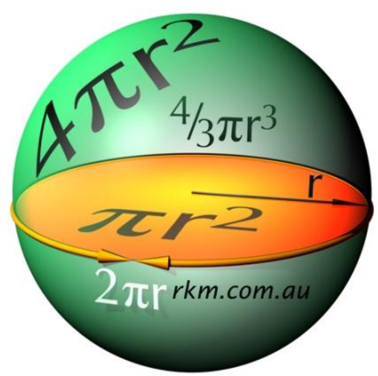
$$V_N = \pi r^3 - \pi r^3\left(\tfrac{1}{3} + \tfrac{1}{2N} + \tfrac{1}{6N^2}\right)$$

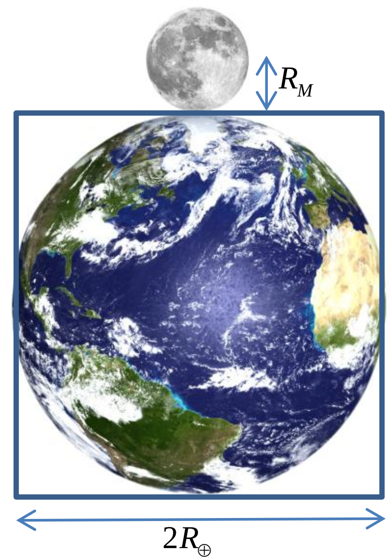
Hence as N tends to infinity, the **hemisphere volume** tends towards

$$\boxed{V = \tfrac{2}{3}\pi r^3}$$

Which means the **volume of a sphere** is

$$\boxed{V = \tfrac{4}{3}\pi r^3}$$





Assuming the Earth and Moon are spheres ....

$$R_\oplus = 6371\text{km}$$

$$R_M = 1737\text{km}$$

$$\therefore R_\oplus \approx \tfrac{11}{3} R_M$$

$$\frac{V_\oplus}{V_M} \approx \left(\frac{11}{3}\right)^3$$

$$\frac{V_\oplus}{V_M} \approx 49.3$$

Therefore the volume of the Earth is approximately *fifty times the volume of the Moon*

The circumference of the Earth plus the circumference of the Moon is given by:

$$C = 2\pi R_\oplus + 2\pi R_M$$

$$C = 2\pi R_M\left(\tfrac{11}{3} + 1\right)$$

$$C = 2\pi R_M\left(\tfrac{11}{3} + \tfrac{3}{3}\right) = 2\pi R_M \times \tfrac{14}{3}$$

$$C = \tfrac{4\times 7}{3}\pi R_M$$

The perimeter of a square bounding the Earth is

$$P = 4 \times 2R_\oplus = 4 \times \tfrac{22}{3} R_M$$

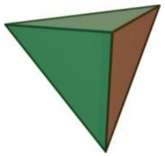
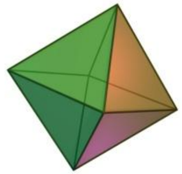
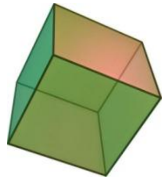
$$\therefore \quad \frac{P}{C} = \frac{4 \times \frac{22}{3} R_M}{\frac{4\times 7}{3}\pi R_M} = \tfrac{22}{7} \times \tfrac{1}{\pi} = 1.000402...$$

i.e. $P = C$ to a very good approximation!

So the Earth and the Moon 'square the circle' (!)

Convexity means *"every point on a line joining two points on the solid is also **within** the solid"*

A **Platonic Solid** is a regular, convex polyhedron with *congruent faces* of regular polygons and the same number of faces meeting at each vertex. *Only five* solids meet these criteria, and each is named after its number of faces.

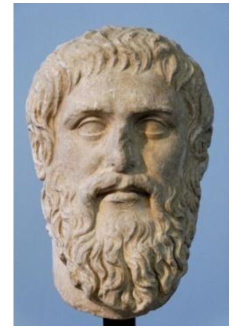




Platonic solid	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
Faces (F)	4	8	6	20	12
Edges (E)	6	12	12	30	30
Vertices (V)	4	6	8	12	20
$V-E+F=2$	4 – 6 + 4 = 2	6 – 12 + 8 = 2	8 -12 + 6 = 2	12 – 30 + 20 = 2	20 – 30 + 12 = 2
Vertex coordinates	$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}$	$\begin{pmatrix}\pm1\\0\\0\end{pmatrix},\begin{pmatrix}0\\\pm1\\0\end{pmatrix},\begin{pmatrix}0\\0\\\pm1\end{pmatrix}$	$\begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix}$	$\begin{pmatrix}0\\\pm1\\\pm\phi\end{pmatrix},\begin{pmatrix}\pm1\\\pm\phi\\0\end{pmatrix},\begin{pmatrix}\pm\phi\\0\\\pm1\end{pmatrix}$	$\begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix},\begin{pmatrix}0\\\pm\phi^{-1}\\\pm\phi\end{pmatrix},\begin{pmatrix}\pm\phi^{-1}\\\pm\phi\\0\end{pmatrix},\begin{pmatrix}\pm\phi\\0\\\pm\phi^{-1}\end{pmatrix}$
Surface area (A) and volume (V) given side a	$A=\sqrt{3}a^2$ $V=\dfrac{a^3}{6\sqrt{2}}$	$A=2\sqrt{3}a^2$ $V=\frac{1}{3}\sqrt{2}a^3$	$A=6a^2$ $V=a^3$	$A=5\sqrt{3}a^2$ $V=\frac{5}{12}\left(3+\sqrt{5}\right)a^3$	$A=3a^2\sqrt{25+10\sqrt{5}}$ $V=\frac{1}{4}\left(15+7\sqrt{5}\right)a^3$

Euler's Formula

$$\boxed{V-E+F=2}$$

Leonhard Euler 1707-1783

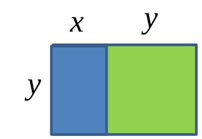



Plato 428-347BC   Euclid 400BC-300C

One of the major goals of Euclid's *Elements* was the *construction* of the platonic solids. i.e. via straight edge and compass.

Golden Ratio

$$\boxed{\phi=\tfrac{1}{2}\left(1+\sqrt{5}\right)}$$



$$\phi=\frac{y}{x}=\frac{x+y}{y}$$

$$\therefore \phi=\frac{1}{\phi}+1$$

$$\Rightarrow \phi^2=1+\phi$$

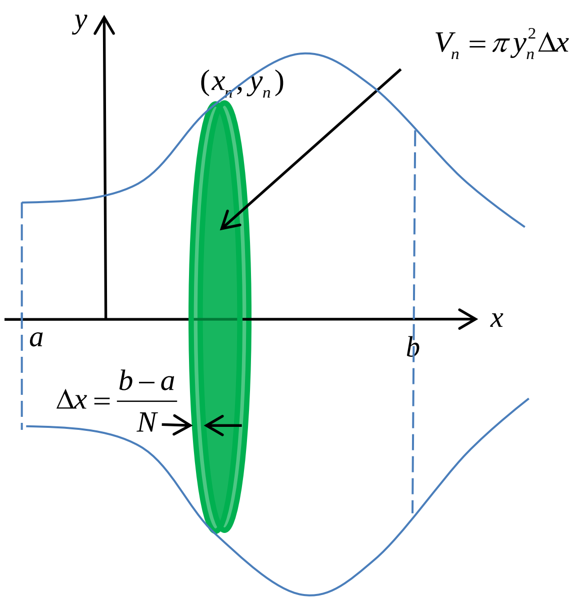
$$\Rightarrow \phi^2-\phi-1=0$$

$$\therefore \boxed{\phi=\tfrac{1}{2}\left(1\pm\sqrt{5}\right)}$$

Volumes of revolution

The *volume of revolution* of a curve about an axis (e.g. the x or y axis) can be computed from the sum of volumes of *infinitesimally thin discs*, whose radii vary according to the curve. The limit of N such discs, of width $\Delta x = (b-a)/N$ such that N tends to infinity can be written as (for volume of revolution about the x axis)*

$$V = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\pi y_n^2\left(\frac{b-a}{N}\right)\right\} = \lim_{\Delta x\to 0}\left\{\sum_{n=1}^{N}\pi y_n^2\Delta x\right\} = \boxed{\int_a^b \pi y^2 dx}$$



i.e. the volume can be written as an *integral* over the *square* of the function $y(x)$

Example1 : Find the volume of a solid formed by revolving $y = e^{-x}$ about the x axis between x limits of $[0,4]$

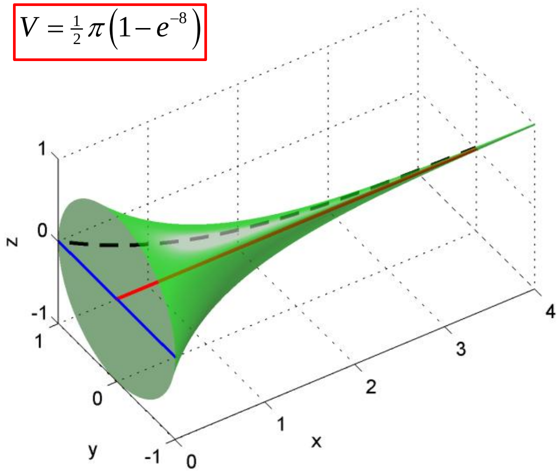
$$V = \int_0^4 \pi y^2 dx$$

$$V = \int_0^4 \pi e^{-2x} dx$$

$$V = \left[\frac{\pi e^{-2x}}{-2}\right]_0^4$$

$$V = \left(\frac{\pi e^{-8}}{-2}\right) - \left(\frac{\pi}{-2}\right)$$

$$\boxed{V = \tfrac{1}{2}\pi\left(1 - e^{-8}\right)}$$



Example2 : Find the volume of a solid formed by revolving $y = 1 + \sin x$ about the x axis between x limits of $[0,2\pi]$

$$V = \int_0^{2\pi} \pi y^2 dx$$

$$\therefore V = \pi\int_0^{2\pi} (1+\sin x)^2 dx$$

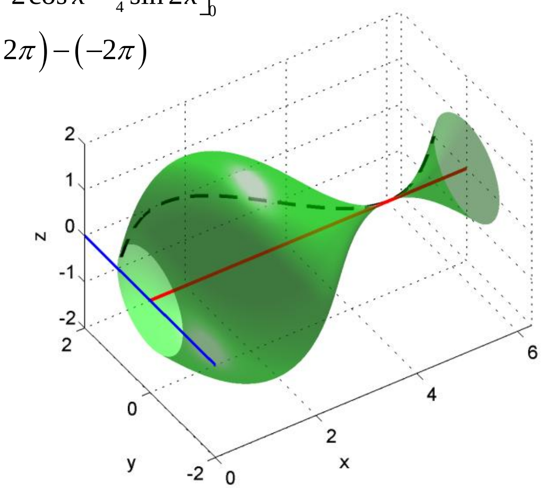
$$V = \pi\int_0^{2\pi} \left(1 + 2\sin x + \sin^2 x\right) dx$$

$$V = \pi\int_0^{2\pi} \left(1 + 2\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx$$

$$V = \pi\left[\tfrac{3}{2}x - 2\cos x - \tfrac{1}{4}\sin 2x\right]_0^{2\pi}$$

$$V = \left(3\pi^2 - 2\pi\right) - \left(-2\pi\right)$$

$$\boxed{V = 3\pi^2}$$



*The equivalent expression for a volume of revolution about the y axis from y limits of $[a,b]$ is

$$V_y = \int_{y=a}^{b} \pi x^2 dy$$

To evaluate this you will need to rearrange x in terms of y. i.e. find the *inverse* function of $y = f(x)$. Then hope the square of this can be integrated!