GRAVITATIONAL FIELDS AND ORBITS

Q1/ (i) $g_\oplus = GM_\oplus / R_\oplus^2 = \dfrac{6.67\times10^{-11}\times 5.972\times10^{24}}{(6371\times10^3)^2}$

$= \boxed{9.81\ \text{N/kg}}$

(ii) $g_M = 3.72\ \text{N/kg}$.

$M = g_M R_M^2 / G$ $\quad\therefore\quad M = \dfrac{3.72\times(3390\times10^3)^2}{6.67\times10^{-11}}$

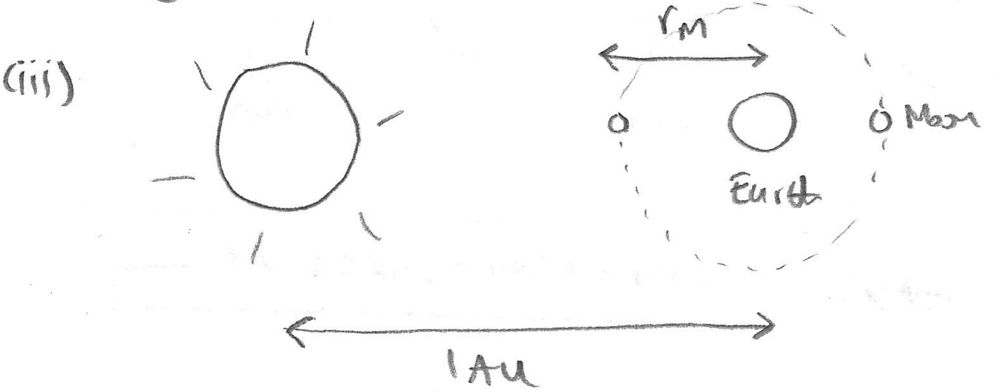
(mass of Mars)

$\boxed{M = 6.41\times10^{23}\ \text{kg}}$

ie, $0.107\ M_\oplus$

[Actually 6.39×10^{23} kg since $R = 3389.5$ km]

(iii)



$57.0 R_\oplus < r_M < 63.8 R_\oplus$

$1\ \text{AU} = 1.496\times10^{11}\ \text{m}$.

Force / kg of sea water is

(i) Due to the Sun:

$g_\odot = \dfrac{GM_\odot}{AU^2} = \dfrac{6.67\times10^{-11}\times 332\,837\times 5.972\times10^{24}}{(1.496\times10^{11})^2}$

$= \boxed{5.92\times10^{-3}\ \text{N/kg}}$ [0.06% of $g_\oplus$]

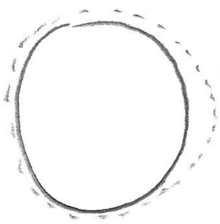
(ii) Due to the Moon:

$g_M = \dfrac{GM_M}{(57R_\oplus)^2} = \dfrac{6.67\times10^{-11}\times 7.35\times10^{22}}{(57\times 6371\times10^3)^2} = \boxed{3.72\times10^{-5}\ \text{N/kg}}$

↑ ie for max g_M $\qquad \simeq 90/159$

(if $r_M = 63.8 R_\oplus$, $g_M = g_\odot/200$; if $r_M = 60 R_\oplus$
$g_M = g_\odot/177$ ← which is what is stated in the NOAA tides educational website)

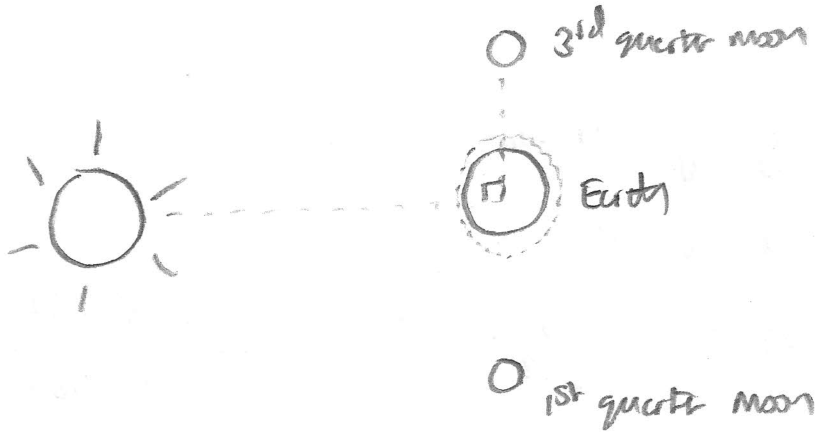
Tides are formed by the difference* in net gravitational force across the Earth, and here cause a 'tidal bulge' in the (movable) mass of liquid water in the oceans.



When the Sun and moon act together they can enhance the tidal bulge → Spring tides.

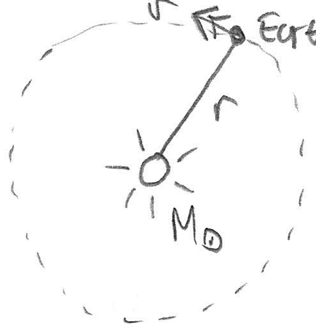
SPRING: SUN - MOON - EARTH in a line

NEAP: SUN - MOON - EARTH are $\perp$



* Since across the Earth, tidal forces go as $\boxed{\frac{1}{r^3}}$ (ie $\frac{d}{dr}\left(\frac{1}{r^2}\right)$) rather than $\frac{1}{r^2}$ as per gravitational fields.

(iv)



NII radially inwards: (and circular orbit assumed).

$$\frac{M_\oplus v^2}{r} = \frac{G M_\odot M_\oplus}{r^2}$$

$r = 1\,AU$

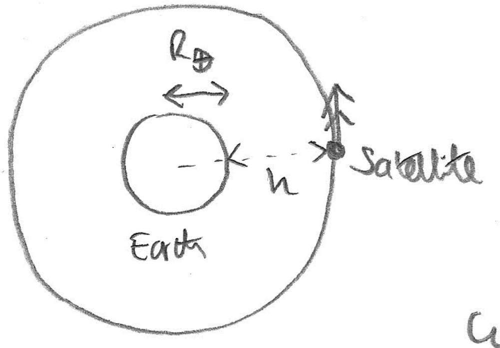
$$\therefore \quad v = \sqrt{\frac{G M_\odot}{r}}$$

$$\therefore \quad v = \sqrt{\frac{6.67 \times 10^{-11} \times 332837 \times 5.972 \times 10^{24}}{1.496 \times 10^{11}}}$$

$$\therefore \quad v = 2.98 \times 10^4 \ m/s$$

$$\boxed{v \simeq 29.8 \ km/s}$$

(v)



NII radially inwards:

$$(R_\oplus + h)\,\omega^2 = \frac{G M_\oplus}{(R_\oplus + h)^2}$$

$\omega = \frac{2\pi}{P}$ where $P = 24.0 \times 3600\,s$ (ie spin period of the Earth).

$$\therefore \quad (R_\oplus + h)^3 = \frac{G M_\oplus}{\omega^2}$$

$$\therefore \quad h = \sqrt[3]{\frac{G M_\oplus}{(2\pi/P)^2}} - R_\oplus$$

$$\therefore \quad h = \sqrt[3]{\frac{6.67 \times 10^{-11} \times 5.972 \times 10^{24}}{(2\pi/24 \times 3600)^2}} - 6371 \times 10^3 \quad (m)$$

$$h = 3.59 \times 10^7 m \simeq \boxed{5.63\,R_\oplus}$$

Energy at equator (per unit mass)

$$\boxed{E_0 = -\frac{GM_\oplus}{R_\oplus}}$$

Energy at geostationary orbit:

GPE / unit mass

$$E_h = -\frac{GM_\oplus}{R_\oplus + h} + \frac{1}{2}v^2$$

KE / unit mass

Now from NII: $\frac{v^2}{R_\oplus + h} = \frac{GM_\oplus}{(R_\oplus + h)^2}$

so $\frac{1}{2}v^2 = \frac{1}{2}\frac{GM_\oplus}{R_\oplus + h}$

$$\therefore \quad \boxed{E_h = -\frac{\frac{1}{2}GM_\oplus}{R_\oplus + h}}$$

$\therefore$ Energy change is: $\Delta E = E_h - E_0$

$$\Delta E = GM_\oplus\left(\frac{1}{R_\oplus} - \frac{1/2}{R_\oplus + h}\right)$$

$$\Delta E = \frac{GM_\oplus}{R_\oplus}\left(1 - \frac{1/2}{1 + h/R_\oplus}\right)$$

$$\Delta E = \frac{6.67\times10^{-11}\times5.972\times10^{24}}{6371\times10^{3}}\underbrace{\left(1 - \frac{1/2}{1+5.63}\right)}_{0.925}$$

$$= \boxed{5.78\times10^{7}\ \mathrm{J}}$$

(per kg)

(vi) Kepler III: $\boxed{P^2 = \frac{4\pi^2}{G(M+m)} a^3}$

if $m \ll M$ (eg Saturn compared to the Sun)
Earth ... $\rightarrow M+m \approx M$

$$Yr^2 = \frac{4\pi^2}{GM_\odot} Au^3$$

$$\therefore \boxed{\left(P/Yr\right)^2 \approx \left(\frac{a}{Au}\right)^3}$$

So (a) Saturn: $a/Au = \left(P/Yr\right)^{2/3}$

$$\therefore a = 29.629^{2/3} \text{ Au}$$
$$= \boxed{9.58 \text{ Au}}$$

b) Jupiter: $P/Yr = \left(\frac{a}{Au}\right)^{3/2}$

$$\therefore P = 5.202^{3/2} \text{ Yr}$$
$$\boxed{P = 11.86 \text{ Au}}$$

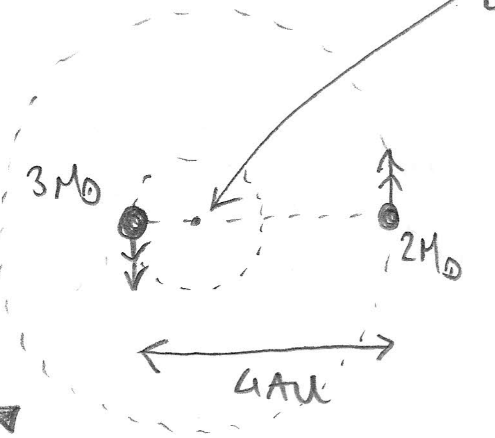
(c) $\frac{P_N}{P_V} = \left(\frac{a_N}{a_V}\right)^{3/2}$ N Neptune, V Venus

$$= \left(\frac{30.246}{0.723}\right)^{3/2} = 41.83.$$

So 1 Neptunian year $\approx$ 42 Venusian years.

(!) Here it is again ....

(vii)



Binary System (mutually circular orbits).

Kepler III:

$$P = \left[\frac{4\pi^2}{G(3M_\odot + 2M_\odot)} \times (4AU)^3\right]^{\frac{1}{2}}$$

$$P = \underbrace{\left(\frac{4\pi^2 AU^3}{GM_\odot}\right)^{\frac{1}{2}}}_{1\,Yr} \times \left(\frac{4^3}{5}\right)^{\frac{1}{2}}$$

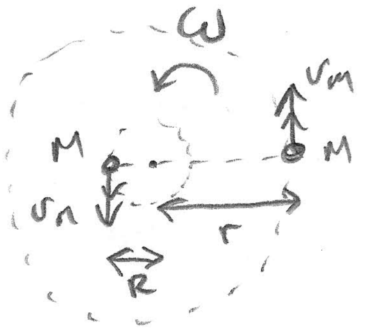
$$\boxed{P = 3.58\ yr}$$

(viii) Escape velocity of the Earth is $v = \sqrt{2GM_\oplus/R_\oplus}$

$$= \sqrt{\frac{2\times 6.67\times 10^{-11} \times 5.972\times 10^{24}}{6371\times 10^{3}}} = \boxed{11.2\ km/s}$$

Superman must throw the missiles at least this far. unfortunately at this speed they would probably overheat in the atmosphere and disperse the nuclear material ☹. So better to do in space, like in the movie Superman IV. Avoid creating Nuclear Man though!

(ix)



a) No net torque since force between stars is purely radial and orbital velocity is tangential.
∴ Angular momentum is conserved for each star. ∴
Since $L_M = Mr^2\omega$ and $L_M = MR^2\omega$
$\Rightarrow \omega$ is also constant.

b) Total energy of system

$$E = \frac{1}{2} m v_m^2 + \frac{1}{2} M v_M^2 - \frac{GmM}{r+R}$$

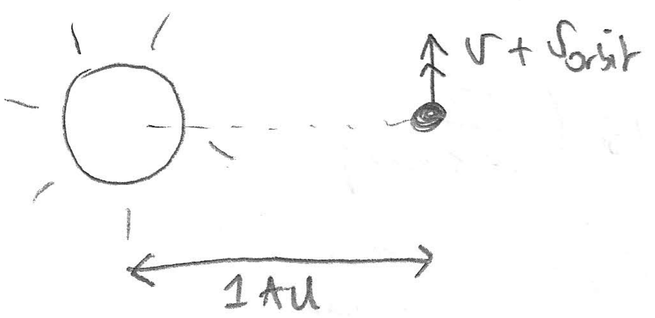
NII: $$\frac{v_m^2}{r} = \frac{GM}{(r+R)^2} \qquad \frac{v_M^2}{R} = \frac{GM}{(r+R)^2}$$

$$\therefore E = \frac{1}{2} GMm \left(\frac{r}{(r+R)^2} + \frac{R}{(r+R)^2} \right) - \frac{GmM}{r+R}$$

$$E = \frac{1}{2} GMm \left(\frac{r+R}{(r+R)^2} - \frac{2}{r+R} \right)$$

$$\boxed{E = \frac{1}{2} GMm / r+R}$$ as required.

(c)



Escape velocity for Sun is

$$v_{escape} = \sqrt{\frac{2GM_{\odot}}{1 AU}}$$

(Starting at R = 1 AU)

But note Capt. Marvel is already in orbit about the Sun at 29.8 km/s.

So $$v = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 332837 \times 5.972 \times 10^{24}}{1.496 \times 10^{11}}} - 29.8 \times 1000$$ (m/s)

$$= (42.1 - 29.8) \text{ km/s} = \boxed{12.3 \text{ km/s}}$$

(xi) orbital angular momentum is mrv

$= mr^2\omega = \boxed{\frac{2\pi}{P} mr^2}$

$\boxed{\text{Moon about Earth}}$ $P = \sqrt{\frac{4\pi^2}{GM_\oplus} \times (60 R_\oplus)^3}$ KIII

(Ignore deviation of Barycentre from centre of Earth, and Moon mass relative to Earth mass $M_\oplus / M_M \approx \frac{5.972 \times 10^{24}}{7.35 \times 10^{22}}$

$= \boxed{81.3}$, so not a brilliant approximation!).

$$P = \sqrt{\frac{4\pi^2}{6.67 \times 10^{-11} \times 5.972 \times 10^{24}} \times (60 \times 6371 \times 10^3)^3}$$

$= \boxed{2.35 \times 10^6 \, s} \approx \boxed{27.2 \text{ days}}$

so $L_M = \frac{2\pi}{2.35 \times 10^6} \times 7.35 \times 10^{22} \times (60 \times 6371 \times 10^3)^2$

$\boxed{L_M = 2.87 \times 10^{34} \text{ kg m}^2 s^{-1}}$

$\boxed{\text{Spin of Earth}}$ $S_\oplus = \frac{2}{5} M_\oplus R_\oplus^2 \left(\frac{2\pi}{24 \times 3600}\right)$

$S = \frac{2}{5} \times 5.972 \times 10^{24} \times (6371 \times 10^3)^2 \times \frac{2\pi}{24 \times 3600}$

$= \boxed{9.14 \times 10^{40} \text{ kg m}^2 s^{-1}}$

$\boxed{\text{Earth about Sun}}$ $L_\oplus = \frac{2\pi}{365 \times 24 \times 3600} \times 5.972 \times 10^{24} \times (1.496 \times 10^{11})^2$

$= \boxed{2.66 \times 10^{40} \text{ kg m}^2 s^{-1}}$

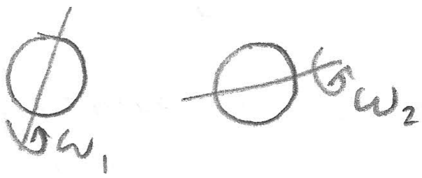
so $S_{\oplus}/L_M = 3.18 \times 10^6$

$L_{\oplus}/L_M = 9.27 \times 10^5$

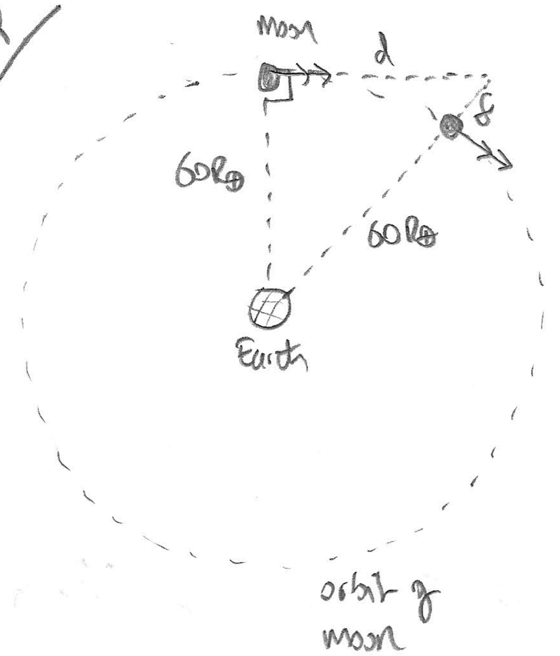
$\boxed{\text{conclusion}}$: tidal forces etc may cause angular momentum to be $\boxed{\text{exchanged}}$ between the Earth and Moon orbits and spins. (Note we haven't included the spin of the moon – which is tidally locked to the Earth, ie its spin has the same period as the orbit about the Earth).
↑ ie we always see the same 'face' of the moon.

However Since $L'_M \simeq 10^{-6}$ of the other angular momenta, we would expect this effect to be $\boxed{\text{small}}$. In fact the Moon is receding at $\approx$ $\boxed{3.8\,\text{cm/yr}}$ (from NASA laser rangefinding experiment).

↪ But it is thought the Moon helps stabilize the spin axis of the Earth.
If the Moon is too far away to influence significantly via tidal forces, the Earth may 'tumble' ie spin axis changes as well as rotation rate ω! seasons would be very different...

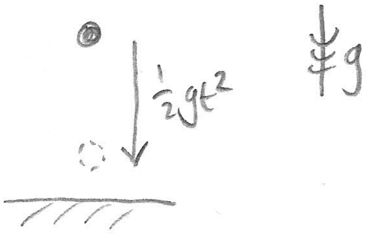


Q2/



Exaggerated scale!

In a uniform gravitational field of strength g, fall distance is $\frac{1}{2}gt^2$ if 'dropped' from rest.



$\therefore$ "Moon fall" distance of $\delta = \frac{1}{2} g_M t^2$

Now assuming an inverse-square law for $g(r)$

$$g_M = \frac{GM_\oplus}{(60R_\oplus)^2} \qquad \therefore \quad \boxed{\frac{g_M}{g} = \frac{1}{3600}}$$

$$g = \frac{GM_\oplus}{R_\oplus^2}$$

So predict moon falls $\delta = \frac{1}{2} \frac{9.81}{3600} \times 1.0^2$

$= \boxed{1.36\text{ mm}}$ in 1.0s

Now from diagram above:

Pythagorus: $(60R_\oplus + \delta)^2 = d^2 + (60R_\oplus)^2$

$120 R_\oplus \delta + \delta^2 = d^2$ since $\delta \ll R_\oplus$

$\therefore \boxed{\delta \simeq \frac{d^2}{120 R_\oplus}}$ let d be '1s of moon orbit'.

Kepler III: $P = \sqrt{\frac{4\pi^2}{GM_\oplus}(60R_\oplus)^3} = \sqrt{\frac{4\pi^2}{6.67\times10^{-11} \times 5.972\times10^{24}}(60\times6371\times10^3)^3}$

$= \boxed{2.353 \times 10^6 s} \simeq 27.2$ days

This would have been known to Newton, but not G, $M_\oplus$

Now 'Moon distance' $d = 60 R_\oplus \tan\theta \simeq \boxed{60 R_\oplus \theta}$

where θ is the orbit angle in 1 s.

$$\theta = \frac{1.0}{P} \times 2\pi$$

But Newton would have known $R_\oplus$

Eratosthenes

... Al Biruni....

$$\therefore \quad \delta \simeq \frac{\left(60 R_\oplus \times 2\pi/P\right)^2}{120 R_\oplus}$$

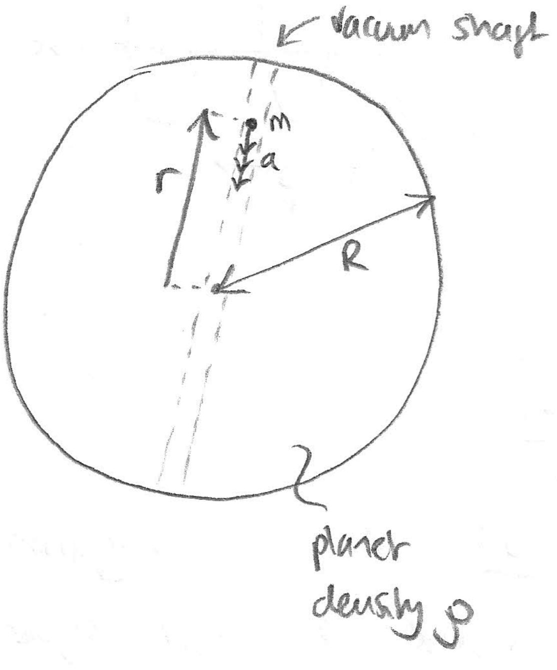
$$\delta \simeq 120\pi^2 R_\oplus / P^2 \simeq \frac{120\pi^2 \times 6371 \times 10^3}{(2.353 \times 10^6)^2}$$

$$= \boxed{1.36 \text{ mm}}$$

it works!

So overall logic is: "A 1.0 s Moon fall of 1.36 mm is consistent with an inverse square law of gravitation".

3/



if a mass m (eg "Optimus Prime") starts from rest at the top of the shaft

NII radially inwards:

$$ma = \frac{GM(r)m}{r^2}$$

[Gauss: only mass enclosed $M(r)$ contributes to $g(r)$.]

$$\therefore \quad a = \frac{G}{r^2}\left(\frac{4}{3}\pi r^3 \rho\right)$$

ie spherical mass of density ρ

$$\therefore \quad a = \frac{4}{3} G \pi \rho \, r$$

Now $\quad a = -\ddot{r} \quad$ since radially inward

$$\therefore \quad \ddot{r} = -\frac{4}{3} G\pi\rho \, r \qquad \text{SHM } \ddot{r} = -\omega^2 r$$

ie $\quad r(t) = R\cos\omega t \quad$ where $\quad \omega^2 = \frac{4}{3} G\pi\rho$

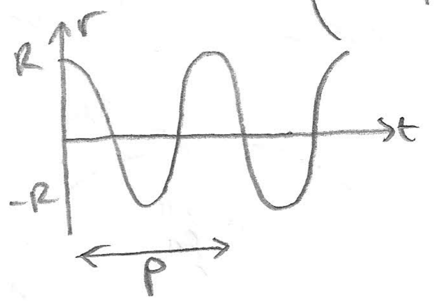
$\omega = 2\pi/P \qquad$ P is period of 'shaft oscillation'

$$\text{So} \quad \frac{2\pi}{P} = \frac{2}{\sqrt{3}}\sqrt{G\rho\pi}$$

$$\therefore \quad \frac{\pi\sqrt{3}}{\sqrt{G\rho\pi}} = P$$

$$\therefore \quad \boxed{P = \sqrt{\frac{3\pi}{G\rho}}}$$

ie $\quad r(t) = R\cos\left(2\pi t/P\right)$



So Ophimus Prime takes $P/2$ seconds to traverse a diameter of cybertron. $\rho = 4510 \text{ kg/m}^3$

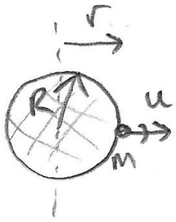
$$\frac{P}{2} = \sqrt{\frac{3\pi}{6.67\times10^{-11} \times 4510}} \times \frac{1}{2}$$

$$\therefore \quad \frac{P}{2} = 2799\text{s} \simeq \boxed{46 \text{ mins } 39\text{s}}$$

Interestingly, P only depends on ρ, NOT R. $\therefore$ Megatron also takes 46mins 39s to traverse the larger 'Cybertron 2" since this is also made from titanium.

4/

(i)



Black hole, radius R

Mass M

To escape from a Black Hole

$E > 0$ at the surface ($\Rightarrow KE > 0$ at $r = \infty$)

$$\boxed{E = \frac{1}{2} m u^2 - \frac{GMm}{R}}$$

if $E > 0 \Rightarrow \frac{1}{2} u^2 > \frac{GM}{R}$

$$\Rightarrow \boxed{u > \sqrt{\frac{2GM}{R}}}$$

Now let $u \to c$ the speed of light. ie escape velocity.

$\therefore \frac{1}{2} c^2 > \frac{GM}{R} \Rightarrow R > \frac{2GM}{c^2}$ for 'an object moving at c to escape'

$\therefore$ for light itself to not escape mass M:

$$\boxed{R < \frac{2GM}{c^2}}$$

"Schwarzschild radius" or "Event horizon"

[For $M = M_\odot$
$R < \boxed{2.95 \text{ km}}$]

(ii) let Black Hole density $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ (ie assume spherical)

Assume maximum density is that of a proton

ie $\rho_p = \frac{m_p}{\frac{4}{3}\pi r_p^3}$. Now for a Black Hole of mass M

$R = \left(\frac{M}{\frac{4}{3}\pi\rho}\right)^{1/3}$ so $\left(\frac{M}{\frac{4}{3}\pi\rho}\right)^{1/3} < \frac{2GM}{c^2}$ from $R < \frac{2GM}{c^2}$

$$\therefore \frac{M}{\frac{4}{3}\pi\rho} < \frac{8G^3M^3}{c^6}$$

$$\Rightarrow \boxed{\rho > \frac{c^6}{\frac{32}{3}\pi G^3} \frac{1}{M^2}}$$

{ interesting... minimum density $\downarrow$ as $M \uparrow$!! }

Now $\rho < \rho_p$ $\therefore$ $\dfrac{c^6}{\frac{32}{3}\pi G^3 M^2} < \rho_p$

$$\therefore \quad M^2 > \frac{3c^6}{32\pi G^3 \rho_p}$$

$$\therefore \quad M > \sqrt{\frac{\cancel{3}c^6}{32\cancel{\pi} G^3 \frac{m_p}{\frac{4}{\cancel{3}}\cancel{\pi} r_p^3}}}$$

$$\boxed{M > c^3 \sqrt{\frac{r_p^3}{8G^3 M_p}}}$$

So $$\frac{M}{M_\odot} > \frac{(2.998\times10^8)^3 \sqrt{\dfrac{(0.855\times10^{-15})^3}{8\times(6.67\times10^{-11})^3 \times 1.67\times10^{-27}}}}{332837 \times 5.972\times10^{24}}$$

$$\boxed{\frac{M}{M_\odot} > 5.4}$$

$2GM_\odot/c^2$ calculated above

Also $R < \dfrac{2GM}{c^2}$ so $R < 5.4 \times 2.95$ km

$\therefore$ $\boxed{R < 15.8 \text{ km}}$

(iii) From above: $\rho > \dfrac{3c^6}{32\pi G^3} \dfrac{1}{M^2}$ if $M = 5.5\times10^9 M_\odot$

$$\rho > \frac{3\times(2.998\times10^8)^6}{32\pi\times(6.67\times10^{-11})^3} \frac{1}{(5.5\times10^9)^2} \frac{1}{(332837\times5.972\times10^{24})^2} \text{ kg/m}^3$$

$\boxed{\rho > 0.61 \text{ kg/m}^3}$ i.e. less than the density of air!! (1.225 kg/m^3)

Numerically, this is: $\rho > 1.848 \times 10^{19} \left(\frac{M}{M_\odot}\right)^{-2}$

ie $\rho > 1.848 \times 10^{19} \left(5.5 \times 10^{9}\right)^{-2}$ kg/m3

$\boxed{\rho > 0.61 \text{ kg/m3}}$

if $\rho_{mat} = 1000$ kg/m3 ie density of water

$$1000 > 1.848 \times 10^{19} / (M/M_\odot)^2$$

$$M/M_\odot > \sqrt{1.848 \times 10^{16}}$$

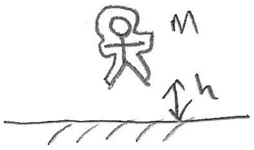
$$\boxed{M/M_\odot > 1.36 \times 10^{8}}$$

So a Black Hole of ≈ billion solar masses has a density < water.

So could you move through a Black Hole in a spacecraft? à la Interstellar? Possibly, as long as the tidal forces on the spacecraft were not too extreme that it becomes 'spaghettified'.

5/ Baikonur

Svetlana $g = 9.81$ N/kg



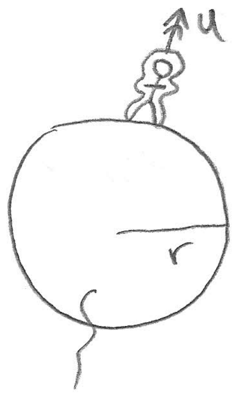
If Svetlana leapt with velocity u

$\frac{1}{2}mu^2 = mgh \Rightarrow \boxed{u = \sqrt{2gh}}$

if $h = 0.15$ m $\Rightarrow u = \sqrt{2 \times 9.81 \times 0.15}$

$\boxed{u = 1.72 \text{ m/s}}$

Assume same leap speed on the asteroid (since KE derives from her muscle power).



Fe-Fi-Fo-Fum
iron asteroid of
density 7870 kg/m3

To escape $$r > \frac{2G \frac{4}{3}\pi r^3 \rho}{u^2}$$

(Analysis from previous question, or rearrange escape velocity $u > \sqrt{\frac{2GM}{R}}$)

$$\therefore \frac{3u^2}{8\pi G\rho} > r^2$$

$$\therefore r < u\sqrt{\frac{3}{8\pi G\rho}}$$

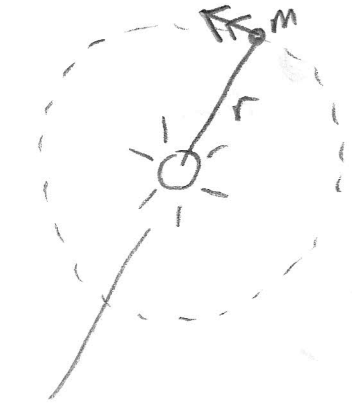
$$r < \sqrt{\frac{2gh \times 3}{8\pi G\rho}}$$

$$\boxed{r < \sqrt{\frac{3gh}{4\pi G\rho}}}$$

$$\Rightarrow r < \sqrt{\frac{3 \times 9.81 \times 0.15}{4\pi \times 6.67\times 10^{-11} \times 7870}}$$

$\boxed{r < 818 \text{ m}}$ for Svetlana to escape just by leaping from the surface.

6/



Star density ρ and radius R

Kepler III

$$P^2 = \frac{4\pi^2}{G\frac{4}{3}\pi R^3 \rho} r^3$$

$$P^2 = \frac{3\pi}{G\rho}\left(\frac{r}{R}\right)^3$$

$$\boxed{P = \sqrt{\frac{3\pi}{G\rho}}\left(\frac{r}{R}\right)^{3/2}}$$

(ii) $\rho_\odot = 1406\ \mathrm{kg/m^3}$ $\qquad \rho_\oplus = 5488\ \mathrm{kg/m^3}$

$$1\,\mathrm{Yr} = \sqrt{\frac{3\pi}{G\rho_\odot}}\left(\frac{1\,\mathrm{AU}}{R_\odot}\right)^{3/2}$$ (Earth about the sun)

$$P = \sqrt{\frac{3\pi}{G\rho_\oplus}}\left(\frac{5\,\mathrm{AU}}{\frac{1}{2}R_\odot}\right)^{3/2}$$ (Star in this problem)

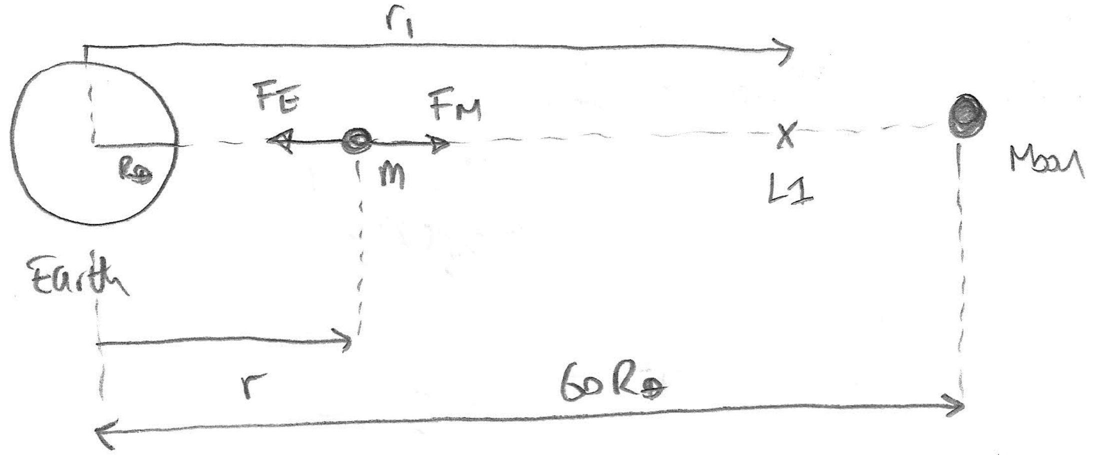
Dividing both expressions:

$$P/\mathrm{Yr} = \sqrt{\frac{1}{\rho_\oplus/\rho_\odot}}\left(10\right)^{3/2}$$

$$P/\mathrm{Yr} = \sqrt{\frac{\rho_\odot}{\rho_\oplus}} \times 10^{3/2}$$

$$P/\mathrm{Yr} = \sqrt{\frac{1406}{5488}} \times 10^{3/2} = \boxed{16.0 \text{ years}}$$

7/



(i) Newton II on satellite of mass M at r from centre of Earth (undergoing a circular orbit of angular speed ω where $\omega = \frac{2\pi}{P_{Moon}}$)

$$m r \omega^2 = F_E - F_M$$

$$m r \omega^2 = \frac{G M_\oplus M}{r^2} - \frac{G M_M M}{(60R_\oplus - r)^2}$$

$$\frac{r\omega^2}{G} = \frac{M_\oplus}{r^2} - \frac{M_M}{(60R_\oplus - r)^2} \qquad \Bigg| \qquad \frac{M_M r^2}{(60R_\oplus - r)^2} = M_\oplus - \frac{r^3\omega^2}{G}$$

$$\frac{r^3\omega^2}{G} = M_\oplus - \frac{M_m r^2}{(60R_\oplus - r)^2}$$

$$M_M r^2 = \left(M_\oplus - \frac{r^3\omega^2}{G}\right)\left(60R_\oplus - r\right)^2$$

A Quintic!!
Can't solve this
→ Numeric method

$$\frac{r^2}{R_\oplus^2\left(60 - r/R_\oplus\right)^2} = \frac{M_\oplus}{M_m} - \frac{r^3\omega^2}{M_M G}$$

Now $\sqrt{\frac{(60R_\oplus)^3\, 4\pi^2}{GM_\oplus}} = P$ and $\omega = \frac{2\pi}{P}$

so $\omega^2 = 4\pi^2/P^2$

$$\frac{1}{P^2} = \frac{GM_\oplus}{(60 R_\oplus)^3 4\pi^2} \quad \therefore \quad \omega^2 = \frac{4\pi^2}{P^2} = \frac{GM_\oplus}{(60 R_\oplus)^3}$$

So $$\left(\frac{r}{R_\oplus}\right)^2 \left(60 - \frac{r}{R_\oplus}\right)^{-2} = \frac{M_\oplus}{M_M} - \left(\frac{r}{R_\oplus}\right)^3 \frac{GM_\oplus}{60^3 M_M G}$$

$$\left(\frac{r}{R_\oplus}\right)^2 \left(60 - \frac{r}{R_\oplus}\right)^{-2} = \frac{M_\oplus}{M_M}\left(1 - \left(\frac{r}{R_\oplus}\right)^3 \frac{1}{60^3}\right)$$

let $x = \frac{r}{R_\oplus}$

$$\therefore \quad \frac{x^2}{(60-x)^2} = \frac{M_\oplus}{M_M}\left(1 - \left(\frac{x}{60}\right)^3\right)$$

$$\therefore \quad \frac{\left(\frac{x}{60}\right)^2}{\left(1 - \frac{x}{60}\right)^2} = \frac{M_\oplus}{M_M}\left(1 - \left(\frac{x}{60}\right)^3\right)$$

So let $\frac{x}{60} \to x$ ie $x = \frac{r}{60 R_\oplus}$

(fraction of Earth-Moon distance)

$$\Rightarrow \quad \boxed{\frac{x^2}{(1-x)^2} = \frac{M_\oplus}{M_M}(1-x^3)}$$

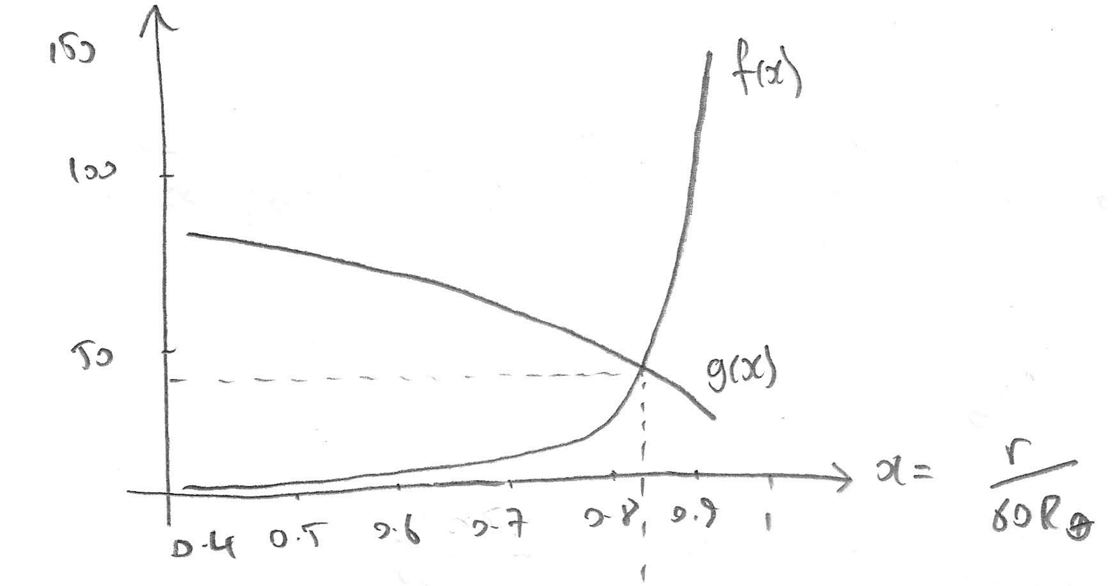
Solve for $f(x) = g(x)$ graphically

$$f(x) = \frac{x^2}{(1-x)^2} \qquad g(x) = \frac{M_\oplus}{M_M}(1-x^3)$$

$$\frac{x^2}{(1-x)^2} = \frac{M_\oplus}{M_M}(1-x^3) \qquad \frac{M_\oplus}{M_M} = \frac{5.972\times10^{24}}{7.35\times10^{22}}$$

$$\simeq 81.25$$

Sketch from MATLAB code output



$x_1 \simeq 0.849$

So the Earth - Moon L1 point is $\approx$ 85% of the Earth - Moon separation.

ie $r_1 = 0.849 \times 60 \times 6371$ km

$\boxed{r_1 = 324{,}500 \text{ km}}$ ie $50.9 R_\oplus$

(ii) ϕ for Satellite is: (GPE / unit mass)

$$\boxed{\phi = -\frac{GM_\oplus}{r} - \frac{GM_M}{60R_\oplus - r}}$$

E is $m\phi + \frac{1}{2}mr^2\omega^2$

where $\omega^2 = \frac{GM_\oplus}{(60R_\oplus)^3}$ from above.

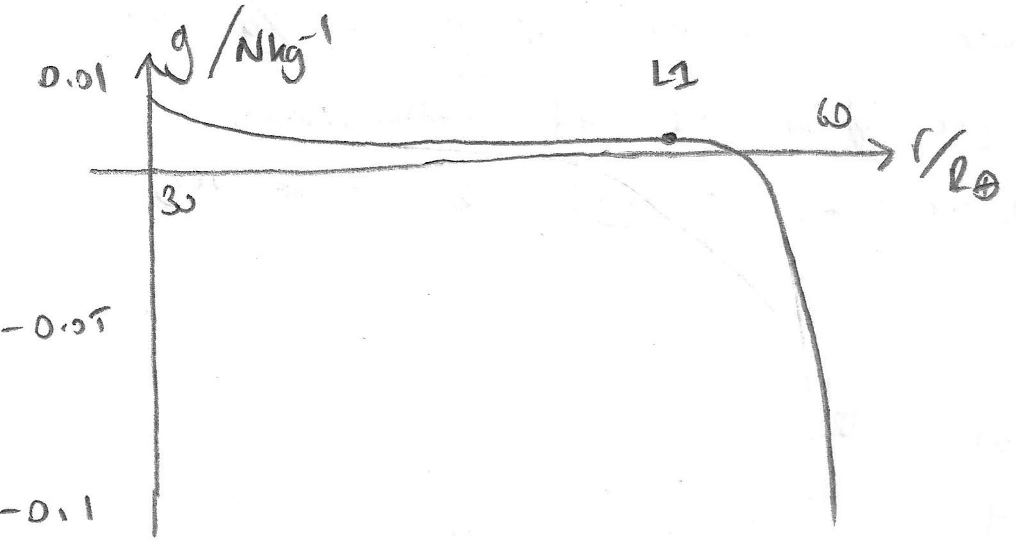
So $$\boxed{E/M = \frac{1}{2} r^2 \frac{GM_\oplus}{(60 R_\oplus)^3} - \frac{GM_\oplus}{r} - \frac{GM_M}{60R_\oplus - r}}$$

Now $g = -\frac{d\phi}{dr}$ should be the same as $\frac{F_E - F_M}{m}$

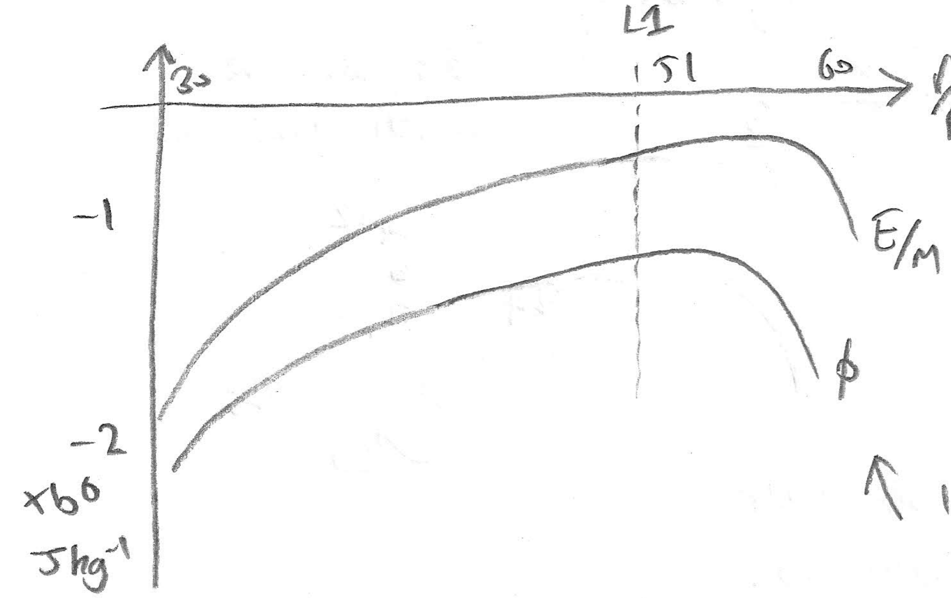
i.e. $$\boxed{g(r) = \frac{GM_\oplus}{r^2} - \frac{GM_M}{(60R_\oplus - r)^2}}$$ which it is!

{ Note this is acceleration towards the Earth }

→ see MATLAB graphs.



Note $g > 0$ at L1 since it is still accelerating in a circle i.e. ∴ must have a net force towards the Earth



↑ i.e. expect L1 to NOT be at $\frac{d\phi}{dr} = 0$

Note since orbit is bound, both ϕ and E/M < 0.

(iv) Barycentre for Earth Moon ie centre of mass from Earth

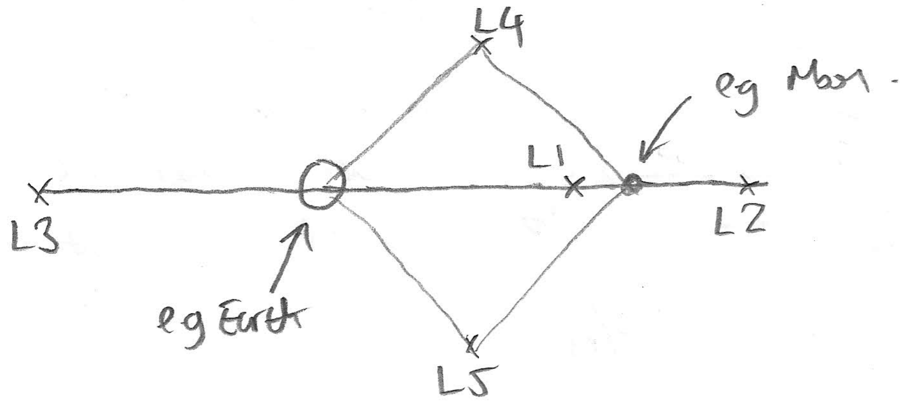
$$\bar{r} = \frac{M_\oplus (0) + M_M (60 R_\oplus)}{M_M + M_\oplus}$$

$$\bar{r} = \frac{7.35\times10^{22}}{7.35\times10^{22} + 5.972\times10^{24}} \times 60 \times R_\oplus$$

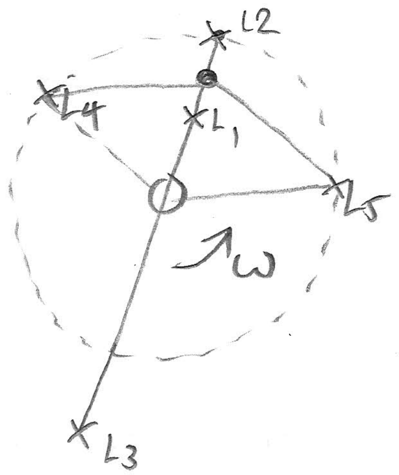
$$\boxed{\bar{r} = 0.73\ R_\oplus}$$

This should really be the centre of the orbit we take r about. However $\frac{0.73}{60} \simeq 1\%$ so we are not too far off for our L1 calculation. To place a satellite at L1, we would need to be very precise of course!

(v) Five Lagrangian points!



This structure rotates about O

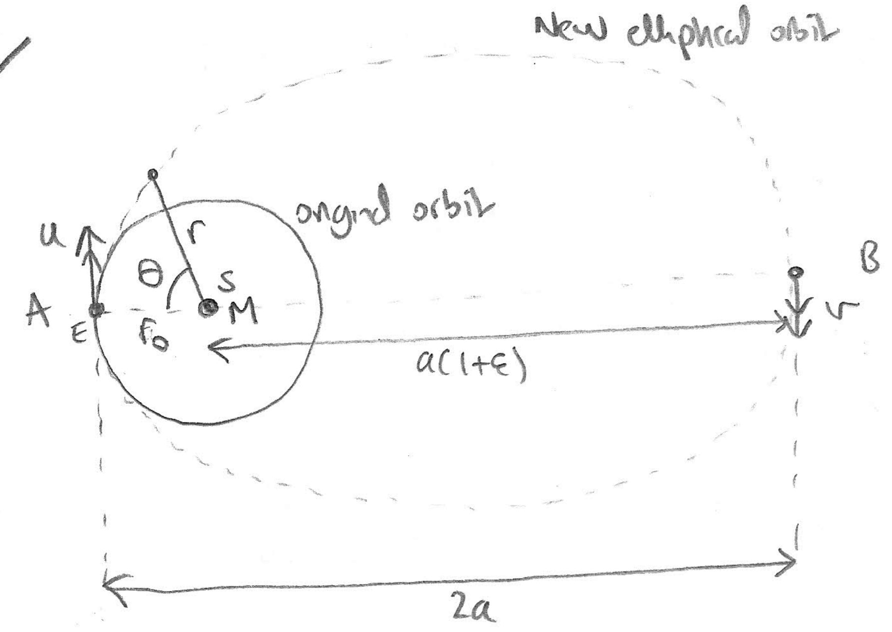


if $M_O \gg M_\bullet$ → All objects orbit about O at the same ω.

↑

(well all objects inc O orbit about barycentre of O and • in general)

Q8/



$r_0 = 1\,AU$

ie Earth orbiting Sun.

Assume circular

So $\dfrac{u^2}{r_0} = \dfrac{GM_\odot}{r_0^2}$

(NII radially inward)

$\therefore \boxed{u^2 = \dfrac{GM_\odot}{r_0}}$

Then Sun loses $\frac{1}{3}M_\odot$ of mass ie $M_\odot \to \mu M_\odot$ where $\mu = 2/3$. $\therefore$ u is too fast to maintain the circular orbit of radius r_0 $\Rightarrow$ it becomes elliptical.

[ε = eccentricity]

$$\boxed{r(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}}$$

{ if start θ from ... not ... }

So when $\theta = 0$ $\quad r = \dfrac{a(1-\varepsilon)(1+\varepsilon)}{1+\varepsilon}$

$\therefore r_0 = a(1-\varepsilon) \quad \therefore a = \dfrac{r_0}{1-\varepsilon}$

and when $\theta = 180°$ $\quad r = \dfrac{(1-\varepsilon)(1+\varepsilon)a}{1-\varepsilon} = a(1+\varepsilon)$

Conservation of $\boxed{\text{angular momentum}}$ at A, B
(for Earth)

$r_0 u = a(1+\varepsilon)v$

$r_0 u = r_0 \dfrac{1+\varepsilon}{1-\varepsilon} v$

$\Rightarrow \boxed{v = \dfrac{1-\varepsilon}{1+\varepsilon} u}$

Now by conservation of energy (½ mass of Earth) at A, B

$$\frac{1}{2}u^2 - \frac{GM}{r_0} = \frac{1}{2}v^2 - \frac{GM}{a(1+\varepsilon)}$$

$\therefore$ using $u^2 = \frac{GM_0}{r_0}$ and $M = \mu M_0$

and $v^2 = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 u^2 = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \frac{GM_0}{r_0}$

$$\frac{1}{2}\frac{GM_0}{r_0} - \mu\frac{GM_0}{r_0} = \frac{1}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\frac{GM_0}{r_0} - \frac{\mu GM_0(1-\varepsilon)}{r_0(1+\varepsilon)}$$

$$a(1+\varepsilon) = \frac{r_0(1+\varepsilon)}{1-\varepsilon}$$

$$\Rightarrow \quad 1 - 2\mu = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 - 2\mu\frac{(1-\varepsilon)}{1+\varepsilon}$$

So $\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 - 2\mu\left(\frac{1-\varepsilon}{1+\varepsilon}\right) - 1 + 2\mu = 0$

$$\left(\frac{1-\varepsilon}{1+\varepsilon} - \mu\right)^2 - \mu^2 - 1 + 2\mu = 0$$

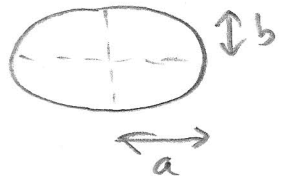
$$\left(\frac{1-\varepsilon}{1+\varepsilon} - \mu\right)^2 = \mu^2 - 2\mu + 1$$

$$\left(\frac{1-\varepsilon}{1+\varepsilon} - \mu\right)^2 = (\mu - 1)^2$$

$$\therefore \quad \frac{1-\varepsilon}{1+\varepsilon} - \mu = \pm(\mu - 1)$$

$$\frac{1-\varepsilon}{1+\varepsilon} = \mu \pm \mu \mp 1$$

Now $0 < \varepsilon < 1$



Since $\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$ and $b \geq a$.

So can't have $-$ve root $\Rightarrow$ $\varepsilon = 0$ ie circular orbit.

unless $\mu = 1$.

$$\therefore \quad \frac{1-\varepsilon}{1+\varepsilon} = 2\mu - 1 \quad \Rightarrow \quad 1 - \varepsilon = (2\mu - 1)(1+\varepsilon)$$

$$\Rightarrow \quad 1 - 2\mu + 1 = \varepsilon(2\mu - 1 + 1)$$

$$\Rightarrow \quad 2 - 2\mu = 2\mu\varepsilon$$

$$\Rightarrow \quad \boxed{\frac{1-\mu}{\mu} = \varepsilon}$$

Now from Kepler III: $P^2 = \frac{4\pi^2}{GM} a^3$

$$\therefore \quad P^2 = \frac{4\pi^2}{GM_\odot \mu} \frac{r_0^3}{(1-\varepsilon)^3}$$

Now $1\,Yr^2 = \frac{4\pi^2}{GM_\odot} r_0^3$

So $\left(\frac{P}{Yr}\right)^2 = \frac{1}{\mu(1-\varepsilon)^3}$

$$1 - \varepsilon = 1 - \frac{1-\mu}{\mu}$$

$$\therefore 1 - \varepsilon = \frac{\mu - 1 + \mu}{\mu}$$

$$1 - \varepsilon = \frac{2\mu - 1}{\mu}$$

$$\therefore \quad \left(\frac{P}{Yr}\right)^2 = \frac{1}{\mu} \frac{\mu^3}{(2\mu-1)^3}$$

$$\Rightarrow \quad \boxed{\frac{P}{Yr} = \frac{\mu}{(2\mu-1)^{3/2}}}$$

So if $\mu = 2/3$

$$P/\text{yr} = \frac{2/3}{(4/3 - 1)^{3/2}} = \frac{\frac{2}{3}}{\left(\frac{1}{3}\right)^{3/2}}$$

$$\boxed{P/\text{yr} = 2\sqrt{3}}$$ as required.

ie $\boxed{P \simeq 3.46 \text{ yr}}$

So $\varepsilon = \frac{1-\mu}{\mu} = \frac{1 - 2/3}{2/3} = \boxed{\frac{1}{2}}$

$\therefore$ At B: $r = \frac{r_0(1+\varepsilon)}{1-\varepsilon} = \frac{r_0 \, 3/2}{1/2} = \boxed{3r_0}$

Now from Kepler II, planet will move faster when it is nearer the star (ie max speed at A).

$\therefore$ Short summers similar to Earth (pre-annihilation!) followed by long, very cold winters.

Heat flux from Sun $\propto \frac{1}{r^2}$ so

at B, earth receives $\frac{1}{9}$ flux as at A.

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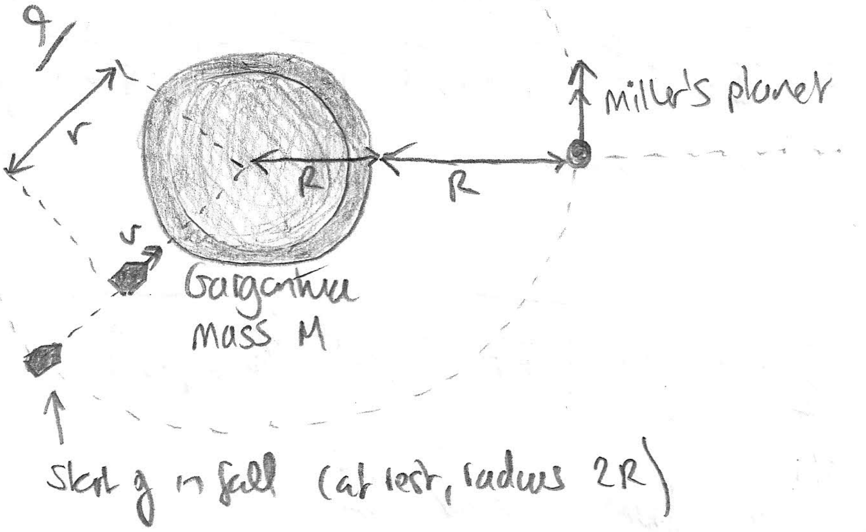


$R = 2GM/c^2$ Schwarzschild radius.

$M = 100 \times 10^6 M_\odot$

$M_\odot = 332{,}837 M_\oplus$

$M_\oplus = 5.972 \times 10^{24}$ kg

Conservation of energy (/unit mass of spacecraft):

$$-\frac{GM}{r_0} = -\frac{GM}{r} + \frac{1}{2}v^2$$

$$\therefore \quad 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) = v^2$$

$$\therefore \quad v = \sqrt{2GM}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{\frac{1}{2}}$$

Now $v = -\frac{dr}{dt}$ so $\int_0^t dt' = \int_{r_0}^{r} -dr'/v$

$$\Rightarrow \quad t(r) = \frac{1}{\sqrt{2GM}} \int_r^{r_0} \left(\frac{1}{r'} - \frac{1}{r_0}\right)^{-\frac{1}{2}} dr'$$

$$t(r) = \frac{1}{\sqrt{2GM}} \int_r^{r_0} \left(\frac{r_0 - r'}{r' r_0}\right)^{-\frac{1}{2}} dr'$$

$$t(r) = \frac{1}{\sqrt{2GM}} \int_r^{r_0} \sqrt{\frac{r' r_0}{r_0 - r'}}\, dr'$$

Now $\boxed{\int \sqrt{\frac{x}{a-x}}\, dx = a \tan^{-1}\sqrt{\frac{x}{a-x}} + \sqrt{\frac{x}{a-x}}\,(x-a) + k}$

So
$$t(r) = \sqrt{\frac{r_0}{2GM}}\left[r_0 \tan^{-1}\sqrt{\frac{r_0}{r_0 - r_0}} + \sqrt{\frac{r_0}{r_0 - r_0}}\,(r_0 - r_0) \leftarrow 0 \right.$$
$$\left. - r_0 \tan^{-1}\sqrt{\frac{r}{r_0 - r}} - (r - r_0)\sqrt{\frac{r}{r_0 - r}}\right]$$

"$\tan^{-1}\infty$" $= \frac{\pi}{2}$

so $$\boxed{t(r) = \sqrt{\frac{r_0}{2GM}}\left(\frac{r_0\pi}{2} - r_0\tan^{-1}\left(\sqrt{\frac{r}{r_0-r}}\right) + (r_0-r)\sqrt{\frac{r}{r_0-r}}\right)}.$$

$$\Rightarrow t(r) = \sqrt{\frac{r_0^3}{2GM}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{r}{r_0-r}} + \left(1-\frac{r}{r_0}\right)\sqrt{\frac{r}{r_0-r}}\right)$$

let $x = \frac{r}{r_0}$. Also note from Kepler III

$$P^2 = \frac{4\pi^2}{GM}r_0^3$$

(circular orbit) at $r = r_0$

↑ ie period of Miller's planet.

so $$\frac{P^2}{4\pi^2} = \frac{r_0^3}{GM}$$

$$\frac{P^2}{8\pi^2} = \frac{r_0^3}{2GM}$$

$8 = 2^2 \times 2$

$\sqrt{8} = 2\sqrt{2}$

$$\therefore \boxed{\frac{P}{2\sqrt{2}\pi} = \sqrt{\frac{r_0^3}{2GM}}}$$

so $$\boxed{t(x) = \frac{P}{2\sqrt{2}\pi}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{1}{1-x}} + (1-x)\sqrt{\frac{1}{1-x}}\right)}$$

let $r_0 = 2R$ and $r = R$ where $R = 2GM/c^2$

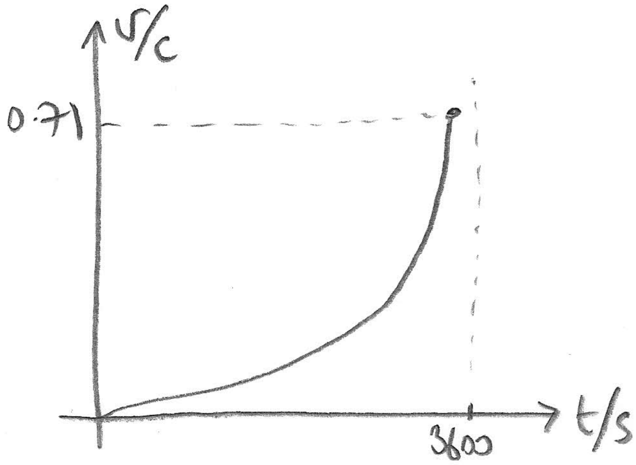
$$P = \frac{2\pi}{\sqrt{GM}}\left(\frac{2 \cdot 2GM}{c^2}\right)^{3/2} = 24{,}712\,s \approx \boxed{6 \text{ hours } 52 \text{ mins}}$$

$x = \frac{1}{2}$ $\therefore$ $$t\left(\tfrac{1}{2}\right) = \frac{24712}{2\sqrt{2}\pi}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt{\frac{1}{2}}}\right) + \left(1-\tfrac{1}{2}\right)\frac{1}{\sqrt{1-\frac{1}{2}}}\right)$$

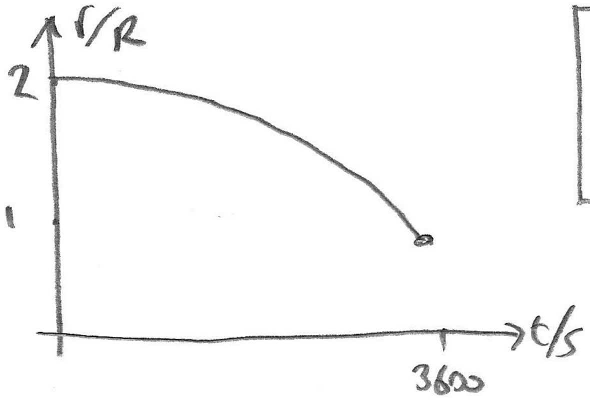
$$= \frac{24712}{2\sqrt{2}\pi}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\right) = \frac{24712}{2\sqrt{2}\pi}\left(\frac{\pi}{2} + \frac{1}{\sqrt{2}} - \tan^{-1}\sqrt{2}\right)$$

$$\approx \boxed{3575\,s} \approx \boxed{1\,hr}$$

From MATLAB code: $\frac{v_{max}}{c} \approx 0.707$ (at $r = R$)



$$\boxed{v = \sqrt{2GM}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{\frac{1}{2}}}$$



$$\boxed{t(r) = \sqrt{\frac{r_0^3}{2GM}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{r}{r_0 - r}} + \left(1 - \frac{r}{r_0}\right)\sqrt{\frac{r}{r_0 - r}}\right)}$$

$t(R) = 3575\,s$

$R = 2GM/c^2 = 1.971$ AU

(i.e. ≈ twice the orbit radius of the Earth).

AF. 26/11/20