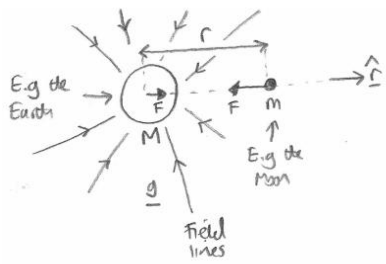
The *Newtonian* model of **gravity** is that of a *force* which permeates all space, and whose magnitude and direction is computable from the spatial distribution of *mass*, the *source of gravity*. Gravity is therefore a ***field of vectors.*** At any point in space we can draw an arrow pointing the direction of the gravitational force, and with a length proportional to the strength of the force. Force $\mathbf{f}$ on a mass m due to the gravity of mass M, which has a separation of r is**:** $\mathbf{f} = m\mathbf{g}$

where $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$ **.** $G = 6.67\times10^{-11}\,\mathrm{Nm^2kg^{-2}}$ is the *Universal Gravitational Constant* and $\hat{\mathbf{r}}$ is a radial unit vector. Note $\mathbf{g} = -g\hat{\mathbf{r}}$. The work done to move mass m from $r = a$ to $r = b$ where $b > a$ is $E = \int_a^b \mathbf{f}\cdot d\mathbf{r} = GMm\left(\frac{1}{a}-\frac{1}{b}\right)$, so we can assign the **Gravitational Potential Energy (GPE)**: $V = -\frac{GMm}{r}$ i.e. by doing



work $\int_r^\infty \mathbf{f}\cdot d\mathbf{r} = GMm\frac{1}{a}$ we have zero potential energy at $b = \infty$. If we define the *gravitational potential energy per unit mass* $\phi = -\frac{GM}{r}$, then the ***field strength*** $\mathbf{g}$ ***is - the potential gradient:*** $\mathbf{g} = -\frac{d\phi}{dr}\hat{\mathbf{r}}$. The total **energy** E of an object in a two-body gravitationally interacting system is: $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$. To escape from the gravitational field of mass M , mass m must have velocity v such that $E > 0$. Therefore *escape velocity* from a planet of radius R is: $v_{escape} = \sqrt{2GM/R}$

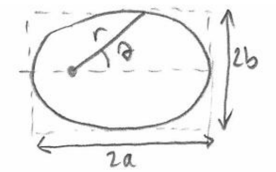
Gauss'a law of gravity: $\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi GM$. M is the mass *enclosed* within surface S . $d\mathbf{S}$ is an element of surface area multiplied by the local normal unit vector to the surface. Note mass outside S doesn't contribute to $\mathbf{g}$.

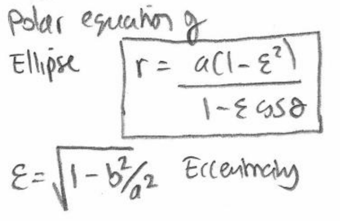
Kepler's laws of (two body) gravitationally bound orbits:

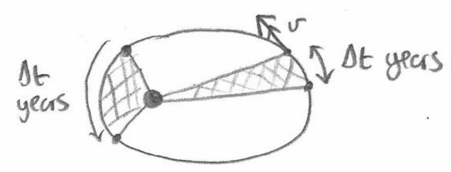
(K1): A planet or star of mass m will orbit a star or planet of mass M with radius r such that $r(\theta)$ describes an **ellipse** with the centre of mass of the system (the *barycenter*) being one *focus* of the ellipse.

(K2) : The radius of the elliptical orbit sweeps out equal areas in equal times.

(K3): $P^2 = \frac{4\pi^2}{G(M+m)}a^3$ where P is the orbital period and a is the semi-major axis of the elliptical orbit.







Question 1

(i) Evaluate the gravitational field strength g on the surface of a (spherical) Earth. $M_\oplus = 5.972\times10^{24}\,\mathrm{kg},\ R_\oplus = 6{,}371\,\mathrm{km}$.

(ii) The gravitational field strength on the surface of Mars is about 3.72N/kg. If the radius of Mars is 3,390km, calculate the mass of Mars in kg.

(iii) The mass of the sun is about $M_\odot \approx 332{,}837 M_\oplus$ and the mass of the moon is about $M_m = 7.35\times10^{22}\,\mathrm{kg}$. The Earth-Sun distance is about one Astronomical Unit (AU) = $1.496\times10^{11}\,\mathrm{m}$ and the Earth-Moon distance varies between $57.0R_\oplus$ and $63.8R_\oplus$. Compare the gravitational force on a kg of seawater on the Earth due to (i) the Sun and (ii) the Moon (find the maximum in this case). What has this got to do with tides?

(iv) Assuming the Earth orbits the Sun in a perfect circle, calculate the orbital velocity of the Earth in km/s.

(v) How far above the surface of the Earth (in Earth radii) is the orbit of a geostationary satellite? Calculate the energy change (in J) per unit mass (kg) for an object at rest on the Equator, to the same object executing the geostationary orbit.

(vi) (a) Saturn has an orbital period of 29.629 years. How many AU is Saturn from the Sun? (b) Jupiter is 5.202AU from the Sun. How many years does it take to orbit? (c) Neptune is 30.246AU from the Sun, and Venus is 0.723AU from the Sun. How many Venusian years correspond to one Neptunian year?

(vii) A star of twice the mass of the Sun orbits, in a circular fashion, a star three times the mass of the Sun at an orbital separation of four astronomical units. Calculate the orbital period of the binary system in years.

(viii) Superman throws a bag of nuclear missiles from surface of the Earth. How fast must they be thrown it so that they escape the gravitational influence of the Earth?

(ix) Consider a binary star system of masses m and M. They execute circular orbits of radii r and R respectively about their common centre of mass. (a) Explain why there is no torque either star, and hence why the angular speed ω of each of the orbits is the same, and constant. (b) Show that the *total* energy of the system is $E = -\dfrac{\frac{1}{2}GMm}{R+r}$.

(x) Captain Marvel decides to improve upon Superman's feat and throw the missiles (perpendicular to Superman's radial throw) such that they escape the Sun's gravitational influence. Use the answer to (iv) to help you to work out how fast she must throw the missile.

(xi) Calculate the orbital speed v_m of the Moon about the Earth, using the average Earth-Moon separation. Hence calculate the angular momentum L of the Moon about the Earth. Compare this to the spin angular momentum S of the Earth, and the orbital angular momentum J of the Earth about the Sun. $L = M_m r_{\oplus m} v_m$, $L = M_\oplus r_{\oplus\odot} v_\oplus$, $S = \frac{2}{5} M_\oplus R_\oplus^2 \omega$ where ω is the angular rotation speed of the Earth. Assume the rotation period is exactly 24.0 hours.

Question 2 Taking the Earth-Moon distance to be $60R_\oplus$, show that the moon 'falls' about $\delta \approx d^2/120R_\oplus = 1.3\text{mm}$ towards the Earth every second, where d is one second of moon orbit. Using $\delta \approx \frac{1}{2} g_m t^2$, show that the ratio g_m/g is consistent with the inverse-square variation of Newton's law of Universal Gravitation. g is the gravitational field strength on the surface of the Earth.

Question 3 The planet *Cybertron*, home to the sentient Transformer robots, is a spherical planet made entirely of titanium (density 4.51g/cm^3) and with a radius of $1.5R_\oplus$. Inter-continental travel is efficiently achieved via a series of vacuum shafts drilled through the planet that connect one place on the surface to the diametric opposite, through the centre of Cybertron. *Optimus Prime* steps into one such shaft from rest. How long does it take him to reach the surface at the other end? The evil *Decepticons* decide that they have had enough of the do-good *Autobots*, and build an even bigger Cybertron nearby. It is also made from titanium, but the radius is now $2.0R_\oplus$. If *Megatron* attempts an equivalent journey to Optimus Prime (but on 'Cybertron II'), how long would this take?

Question 4 A *Black Hole* is perhaps the most extreme and exotic object in the Universe. Although its existence and properties are predicted by the *General Theory of Relativity,* we can gain some insight by defining it to be a spherical mass of radius R, density ρ and mass M such that the escape velocity is equal to the speed of light $c = 2.998 \times 10^8 \text{ms}^{-1}$.

(i) Show that $R < \dfrac{2GM}{c^2}$. (This is called the *Schwarzschild* radius). What is this R for our Sun?

(ii) If the maximum density of a Black hole is that of a proton (mass $m_p = 1.67 \times 10^{-27}\,\text{kg}$, radius $r_p = 0.855 \times 10^{-15}\,\text{m}$), determine the minimum mass (in solar masses) of a Black Hole.

(iii) It is thought that most galaxies (including our own, the Milky Way) have a *Super-massive* Black Hole at their centre. The recent (2017) image of M87* at the centre of the *Messier 87* galaxy has a mass of about 5.5 billion solar masses. Calculate the minimum density of M87*, and maximum radius (in AU). Comment on your answer!

Question 5 While wearing full space-walking kit in the *Baikonur Cosmodrome* in southern Kazakhstan, Svetlana can leap $h = 0.15\text{m}$ vertically. Local gravity is $g = 9.81\text{N/kg}$. Having maintained similar muscle strength during the journey to a spherical iron asteroid *Fe-Fi-Fo-Fum* (density 7.87g/cm^3), she considers repeating the same jump. Calculate the maximum radius r of the asteroid such that Svetlana will eventually escape its gravity, if she leaps with the same kinetic energy as in Baikonur.

Question 6 A planet orbits a star of mean density ρ and radius R. The orbit is assumed to be circular with radius r

(i) Determine how the period of the orbit varies with these parameters.

(ii) The density of our Sun is about 1406 kgm^{-3}, whereas Earth has an average density of 5488 kgm^{-3}. If a star was half as large as our sun and had the density of the Earth, calculate the period of a circular orbit of radius 5AU about this star in years.

Question 7 Take the Earth-Moon distance to be fixed at $60R_\oplus$ for this question, and assume all orbits are circular.

A satellite of mass m (negligible compared to the mass of the Moon and Earth) is to placed at the L1 *Lagrangian Point* between the Earth and the Moon. At this point the gravitational forces combine in such a fashion that the mass × centripetal acceleration of the satellite towards the Earth is such that the satellite orbits the Earth at the same angular speed that the Moon does. In other words, the satellite occupies a fixed point on a line joining the Earth and Moon.

(i) Assuming the satellite and the Moon both orbit the Earth about its centre (and therefore $M_m/M_\oplus \ll 1$), calculate the radius r_1 of the satellite at the L1 point. Show that you will need to solve the equation $\frac{x^2}{(1-x)^2} = \frac{M_\oplus}{M_m}\left(1-x^3\right)$ where $x = r_1/60R_\oplus$, and then use a graphical method or numerical method to find x and hence r_1. A sensible range of x is $0.4 \le x \le 0.92$.

(ii) Determine an expression for the potential energy per unit mass ϕ of the satellite vs radius from Earth r and hence an expression for the total energy E per unit mass of the satellite. Assume the satellite is moving at the angular speed of the moon at the instant the satellite crosses the Earth-moon separation vector.

(iii) Find $g = -d\phi/dr$ for the satellite. Plot for $30 \le r/R_\oplus \le 59$, $\phi(r), E(r)$ and $g(r)$ graphs and mark L1.

(iv) Calculate the *barycenter* of the Earth-moon system. Comment on the assumptions made in (i).

(v) Look up *Lagrangian Point* in the library and/or online. There are actually five! https://en.wikipedia.org/wiki/Lagrangian_point is a pretty good start.

Question 8 A cloud of anti-matter passes through the solar system and causes $\frac{1}{3}$ of the mass of the Sun to be annihilated. (i.e. the mass after the annihilation is $M = \mu M_\odot$ i.e. $\mu = \frac{2}{3}$). Assuming the resulting burst of gamma rays does not destroy the Earth, show the *eccentricity* $\varepsilon = \frac{1-\mu}{\mu}$ of the resulting orbit and hence, via Kepler III, the new period $\frac{P}{\text{Yr}} = \frac{\mu}{(2\mu-1)^{\frac{3}{2}}}$. You may assume the Earth was orbiting at 1AU in a circular orbit prior to the event. What will the new seasons be like?

Question 9 In the film *Interstellar*, a spacecraft flies directly into the super-massive Black Hole *Gargantua*, which has a mass of 100 million solar masses. If a spacecraft is stationary at radius r from a Black Hole, and uses gravity alone to pull it in, determine formulae for: (a) The velocity it reaches range r (in m/s) ; (b) How long this takes t (in s)

Use Newtonian (i.e. non-relativistic) mechanics throughout. You may wish to quote the following integral without proof:

$$\int \sqrt{\frac{x}{a-x}}\,dx = a\tan^{-1}\left(\sqrt{\frac{x}{a-x}}\right) + \sqrt{\frac{x}{a-x}}\,(x-a) + k \quad \text{where } k \text{ is an integration constant.}$$

In Interstellar, *Miller's Planet* orbits Gargantua at about twice the Schwarzschild radius ('Event horizon') of the Black Hole. If a spacecraft falls into Gargantua from rest from this radius (but nowhere near Miller's Planet so unaffected by its gravity), show that it will reach the event horizon in just under an hour.