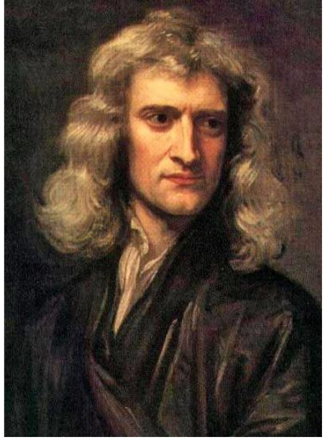
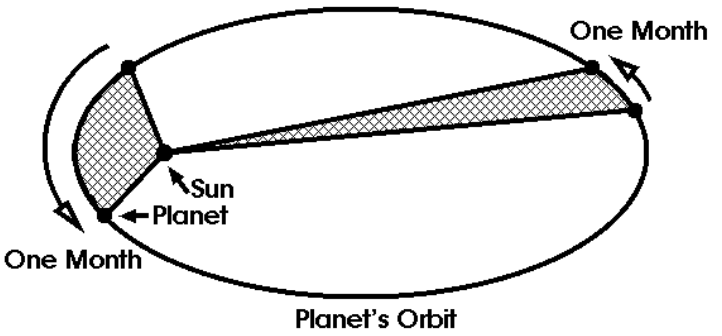
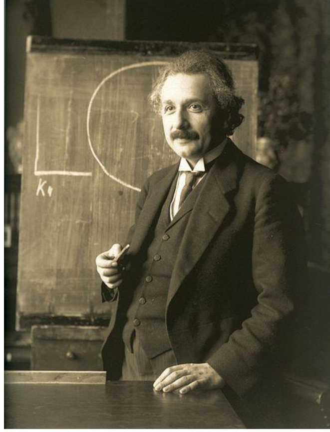


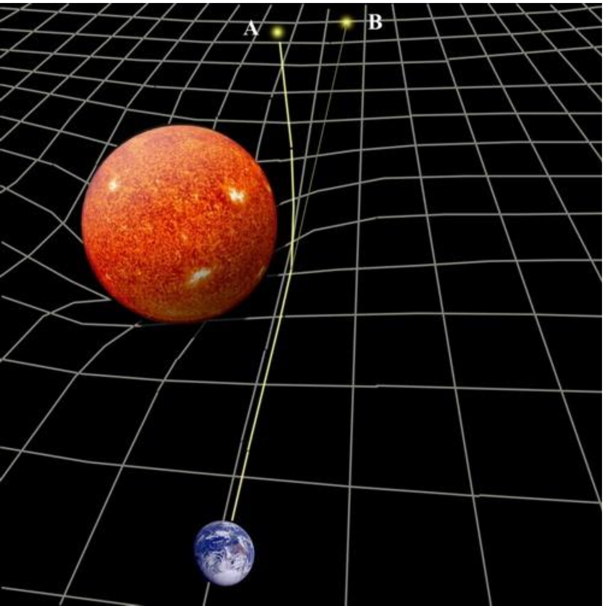


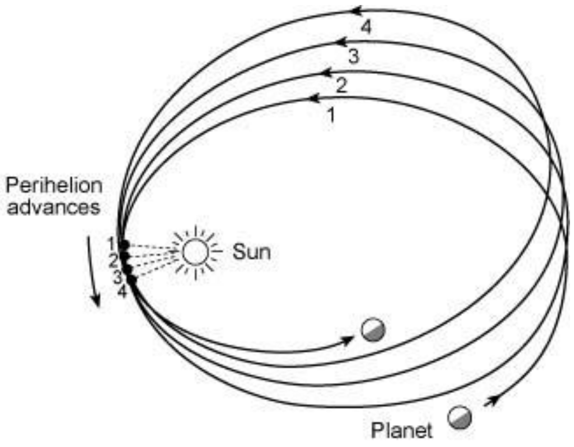


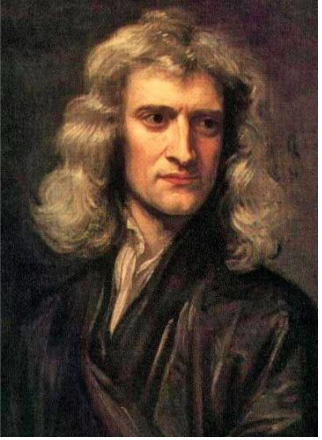
$$F = \frac{GMM_\odot}{r^2}$$

Gravity & orbits

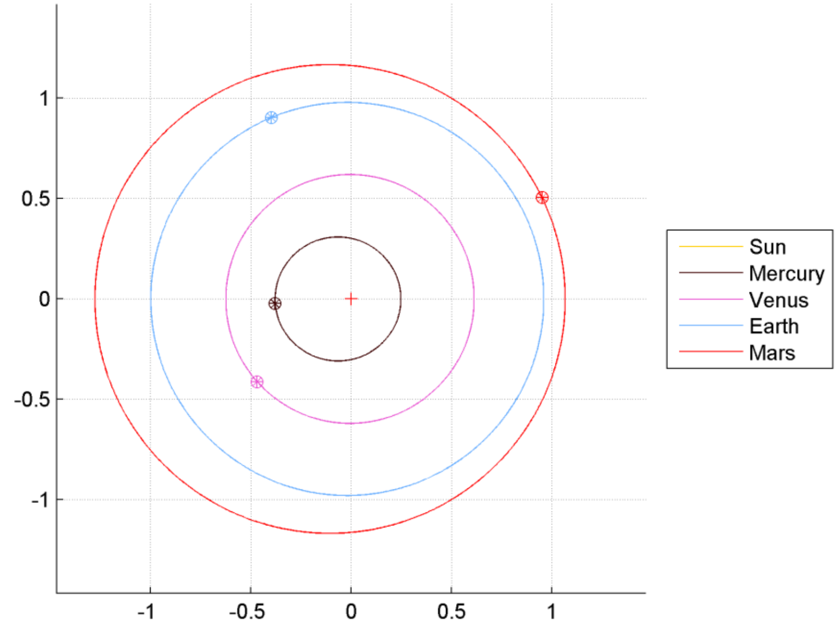
$$P^2 = \frac{4\pi^2}{G\left(M + M_\odot\right)} a^3$$







Isaac Newton (1642-1727) developed a mathematical model of Gravity which predicted the elliptical orbits proposed by Kepler



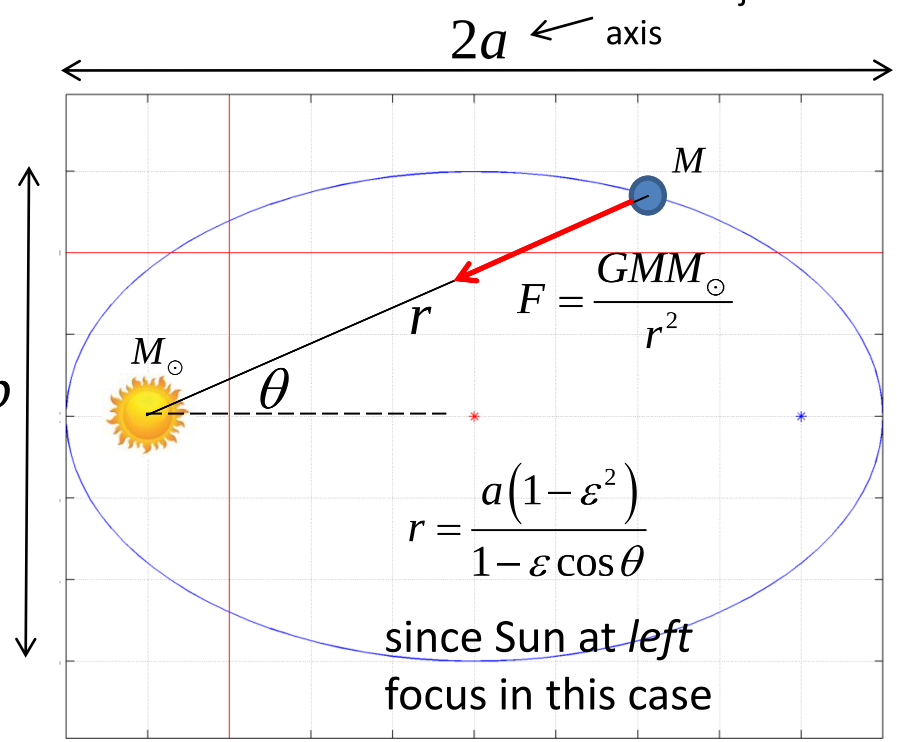
Planet and Solar masses

Force of gravity

$$\boxed{F = \frac{GMM_{\odot}}{r^2} \qquad G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}}$$

Semi-major axis

Semi-minor axis



$$r = \frac{a\left(1-\varepsilon^2\right)}{1 \pm \varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(M+M_{\odot})}a^3$$

Orbital period *P*

Kepler's three laws are:

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1 \pm \varepsilon \cos\theta}$$ *Polar* equation of ellipse

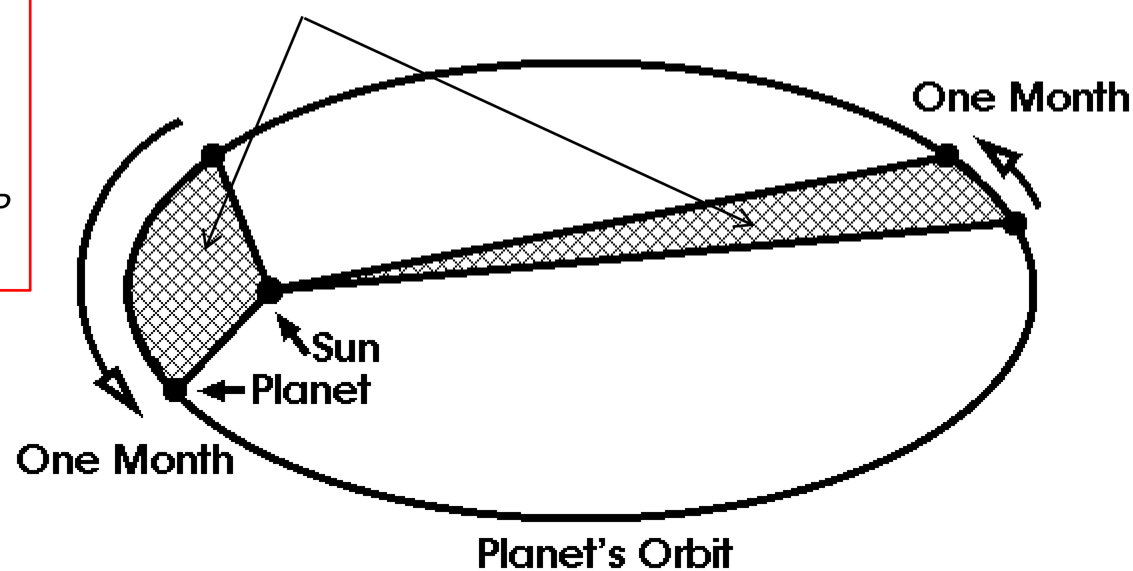
$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$$ *Eccentricity* of ellipse

$$P^2 = \frac{4\pi^2}{G(M + M_\odot)} a^3$$ Orbital period *P*

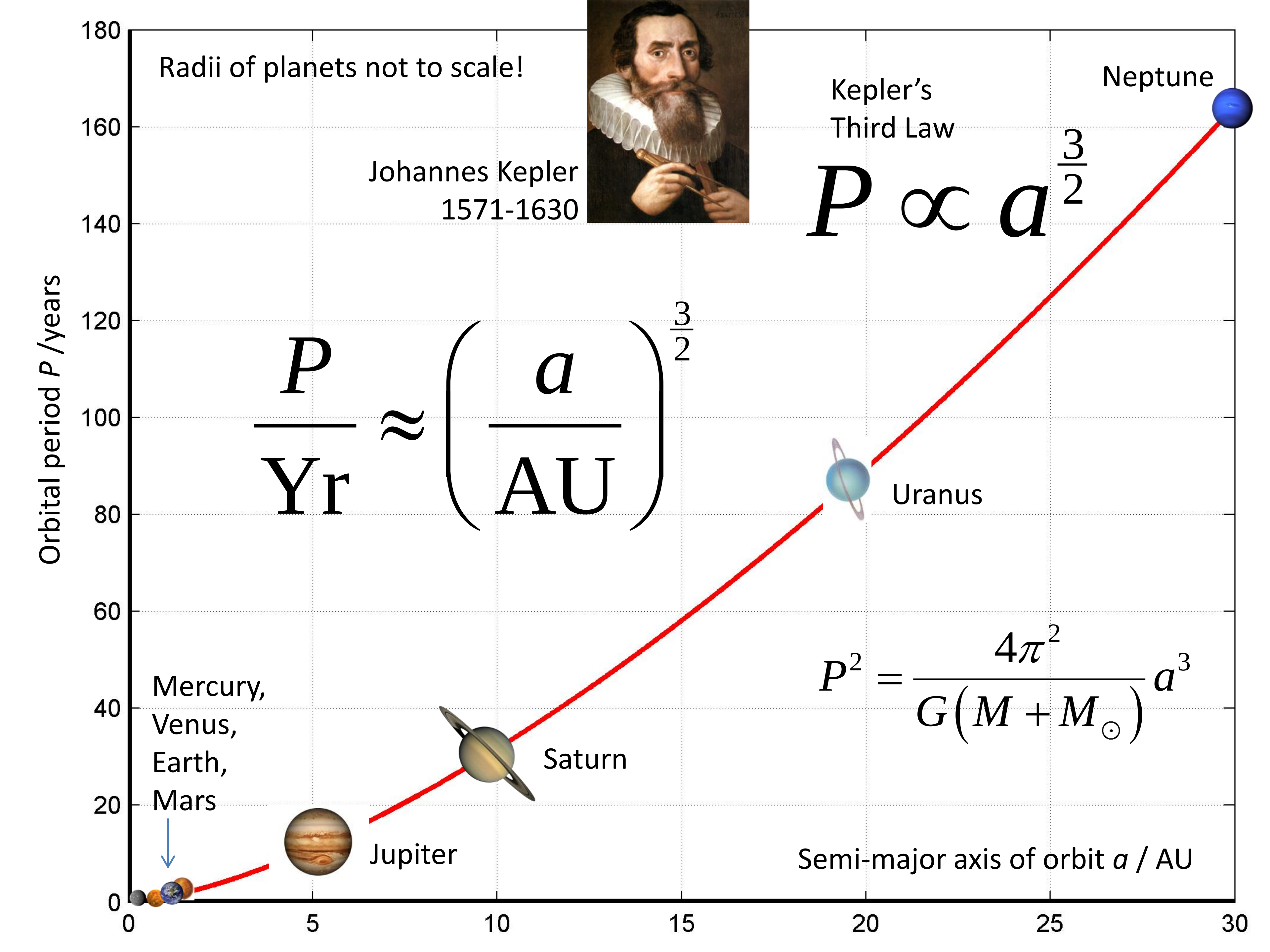
$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(M + M_\odot)\left(1-\varepsilon^2\right)a}$$

This is a constant

Equal areas swept out in equal times

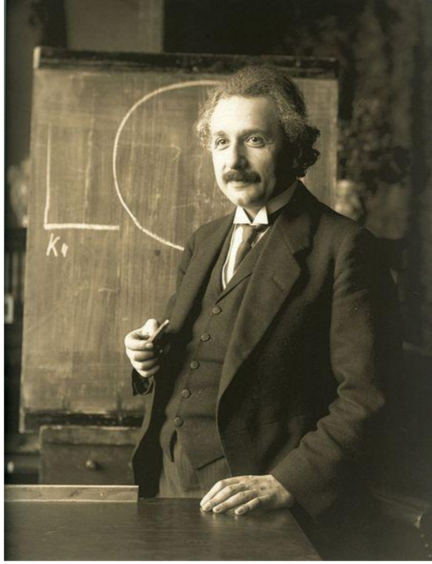


Johannes Kepler
1571-1630



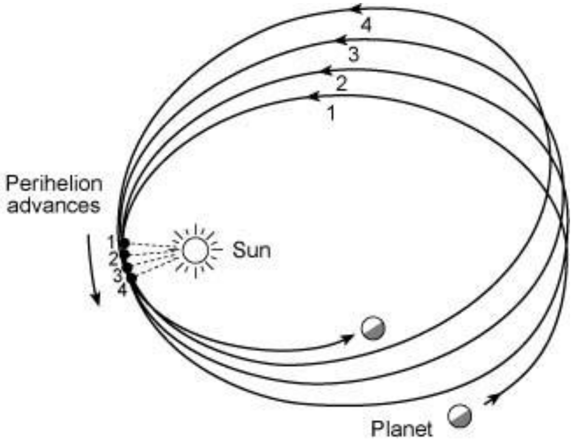
Radii of planets not to scale!
Johannes Kepler
1571-1630
Kepler's
Third Law
$P \propto a^{\frac{3}{2}}$
Neptune
$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$
Uranus
$P^2 = \frac{4\pi^2}{G\left(M + M_\odot\right)} a^3$
Saturn
Mercury,
Venus,
Earth,
Mars
Jupiter
Semi-major axis of orbit a / AU
Orbital period P /years
180
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30

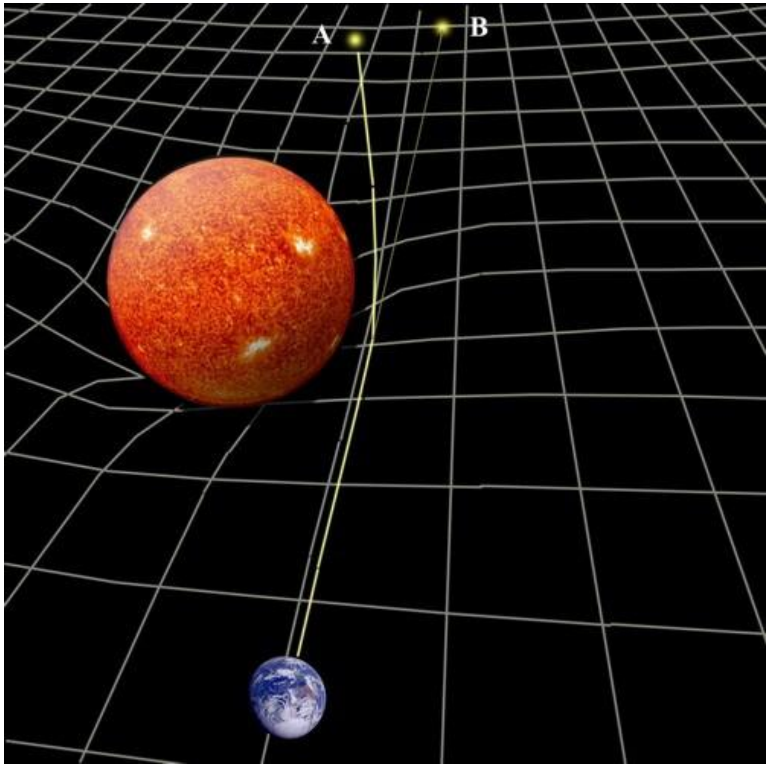
Albert Einstein (1879-1955) proposed a radical new theory of gravity, General Relativity, in which both space & time (‘*spacetime*’) are *curved* by the presence of mass. This helped to explain *anomalies* in the Newtonian model such as the *precession of the orbit Mercury* and the amount that light is bent by massive objects (*Gravitational lensing*). Note General Relativity predicts the *same* planetary dynamics as Newton’s model when gravity is fairly weak. i.e. Newton’s model can be thought of as an *approximation*.



Sources of the precession of perihelion for Mercury

Amount (arcsec/Julian century)	Cause
531.63 ±0.69[4]	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun (quadrupole moment)
42.98 ±0.04[5]	General relativity
574.64±0.69	Total
574.10±0.65[4]	Observed





Escape velocity

To escape the gravity of a spherical astronomical body of mass *M* and radius *R* the total energy of the system must be positive at an infinite distance from the body.

In other words, it will have some kinetic energy and will never be gravitationally attracted back towards the body.

For a mass *m* blasting off with velocity *v*, it will escape the gravitational influence of *M* if:

$$\tfrac{1}{2}mv^2 - \frac{GMm}{R} > 0$$

$$\therefore \boxed{v > \sqrt{\frac{2GM}{R}}}$$

For Earth, the escape velocity is:

$$v_{escape} = \sqrt{\frac{2GM}{R}}$$

$$v_{escape} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.38 \times 10^{6}}} \approx \boxed{11.2\text{kms}^{-1}}$$

It is interesting to work out the radius of a star of mass *M* such that the escape velocity exceeds that of the speed of light. Since this is not possible, the star becomes a ***Black Hole***.

This inequality defines the maximum radius of a Black Hole, which is called the ***Schwarzschild radius***. This is the *event horizon*, or 'point of no return' from the centre of a Black Hole.

$$\sqrt{\frac{2GM}{R}} > c$$

$$\frac{2GM}{R} > c^2$$

$$\boxed{R < \frac{2GM}{c^2}}$$

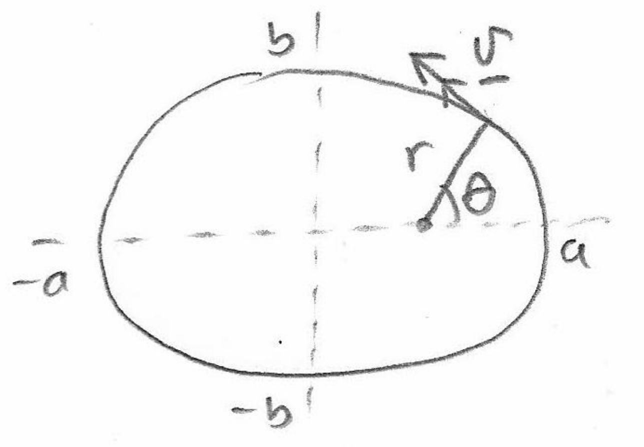
$$\rho_{\text{Black hole}} > \frac{M}{\tfrac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3}$$

$$\boxed{\rho_{\text{Black hole}} > \frac{3c^6}{32\pi G^3 M^2}}$$

For the Sun to become a Black Hole (*M* = 2 x 10^{30} kg, *R* = 6.96 x 10^{8} m) its radius would have to **shrink to less than 2.97 km.**

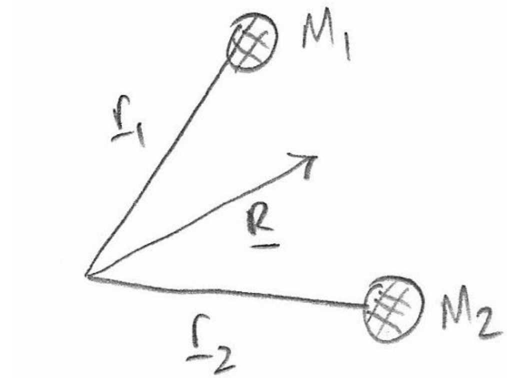
This is a mindblowing density of 1.8 x 10^{19} kgm^{-3} !

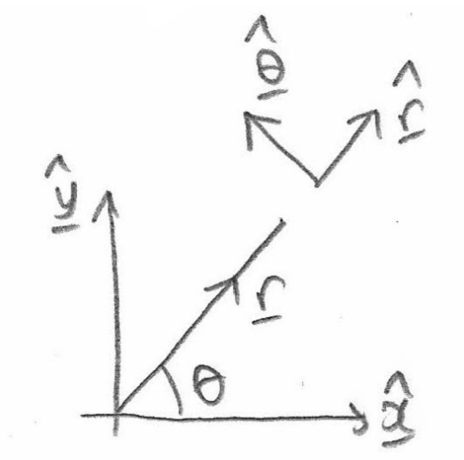
Two body Kepler problem summary



$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1+m_2}\mathbf{r}$$

$$\mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1+m_2}\mathbf{r}$$





$$\mathbf{r} = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}$$

$$\mathbf{v} = \sqrt{\frac{G(m_1+m_2)}{a\left(1-\varepsilon^2\right)}}\left(1+\varepsilon\cos\theta\right)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right)$$

$$b = a\sqrt{1-\varepsilon^2}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$$

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$$

$$\dot{\theta} = \sqrt{\frac{G(m_1 + m_2)}{a^3(1-\varepsilon^2)^3}}\left(1+\varepsilon\cos\theta\right)^2$$

Angular velocity

$$t = \sqrt{\frac{a^3(1-\varepsilon^2)^3}{G(m_1 + m_2)}}\int_0^{\theta}\frac{d\theta}{\left(1+\varepsilon\cos\theta\right)^2}$$

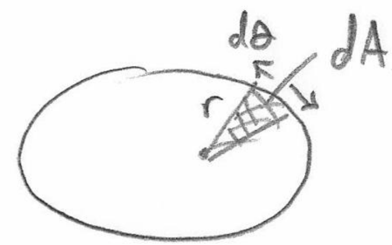
Orbital time

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)}a^3$$

Period vs semi-major axis (Kepler III)

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m_1 + m_2)\left(1-\varepsilon^2\right)a}$$

Equal areas swept out in equal times (Kepler II)



$$J^2 = \frac{Gm_1^2 m_2^2\left(1-\varepsilon^2\right)a}{m_1 + m_2}$$

Angular momentum (which is a constant)

$$E = -\frac{Gm_1m_2}{2a}$$

Total energy (which is a constant).

Note this equation is only true for *bound* elliptical orbits. Other possibilities (parabolae, hyperbolae) are possible, but they are not *bound*.

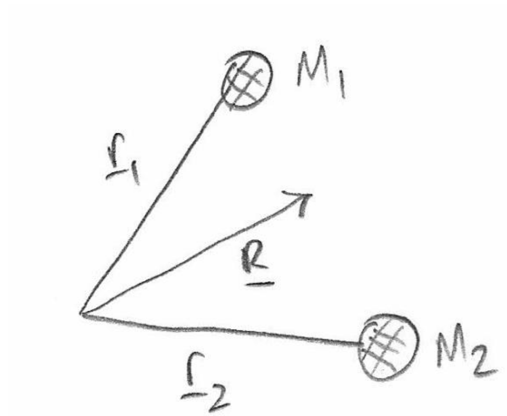
$$E = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_1v_1^2 - \frac{Gm_1m_2}{r}$$

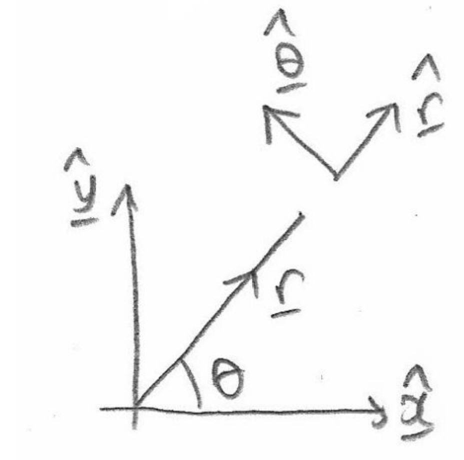
If $E > 0$ this means an orbit is not bound

Orbit initial condition for a two-body system

INPUTS $m_1, m_2, G, a, \varepsilon, \theta_0$

$$\mathbf{R} = 0,\ t = 0$$





$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{r}_0$$

$$\mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\mathbf{r}_0$$

$$\mathbf{r}_0 = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta_0}\hat{\mathbf{r}}$$

$$\mathbf{v}_0 = \sqrt{\frac{G(m_1+m_2)}{a\left(1-\varepsilon^2\right)}}\left(1+\varepsilon\cos\theta_0\right)\left(\frac{\varepsilon\sin\theta_0}{1+\varepsilon\cos\theta_0}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\right)$$

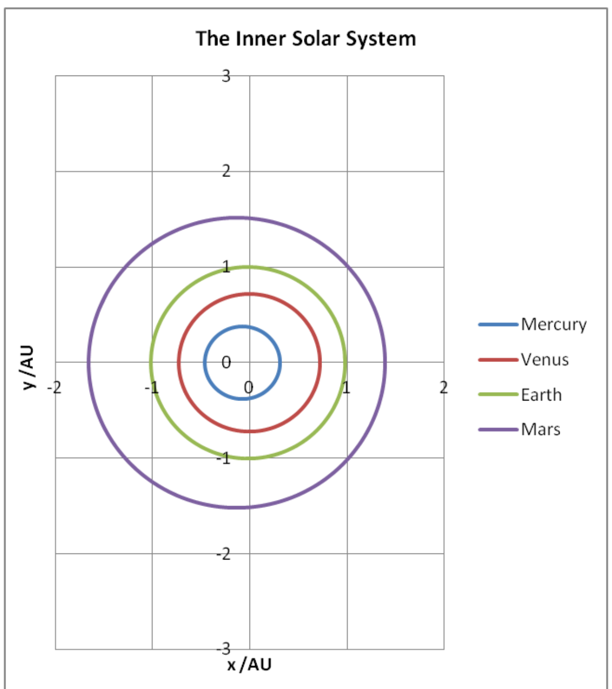
$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$$

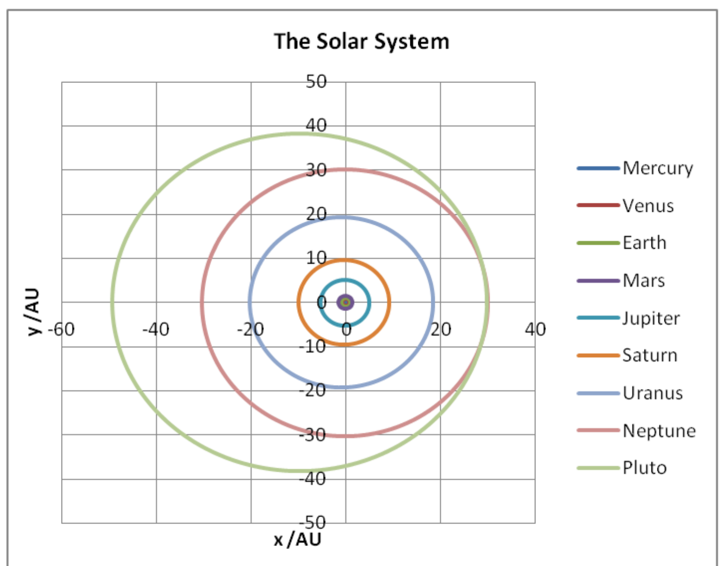
$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$$

Object	Mass in Earth masses	Distance from Sun in AU	Radius in Earth radii	Rotational period /days	Orbital period /years	Orbital eccentricity	Initial polar angle /degrees
Mercury	0.055	0.387	0.383	58.646	0.241	0.21	0
Venus	0.815	0.723	0.949	243.018	0.615	0.01	0
Earth	1	1	1	1	1	0.02	0
Mars	0.107	1.523	0.533	1.026	1.881	0.09	0
Jupiter	317.85	5.202	11.209	0.413	11.861	0.05	0
Saturn	95.159	9.576	9.449	0.444	29.628	0.06	0
Uranus	14.5	19.293	4.007	0.718	84.747	0.05	0
Neptune	17.204	30.246	3.883	0.671	166.344	0.01	0
Pluto	0.003	39.509	0.187	6.387	248.348	0.25	0
Sun	332,837	-	109.123	-	-	0	0

$$M_{\oplus} = 5.97 \times 10^{24}\,\text{kg} \qquad R_{\oplus} = 6.38 \times 10^{6}\,\text{m}$$

$$M_{\odot} = 1.99 \times 10^{30}\,\text{kg} \qquad 1\text{AU} = 1.496 \times 10^{11}\,\text{m}$$

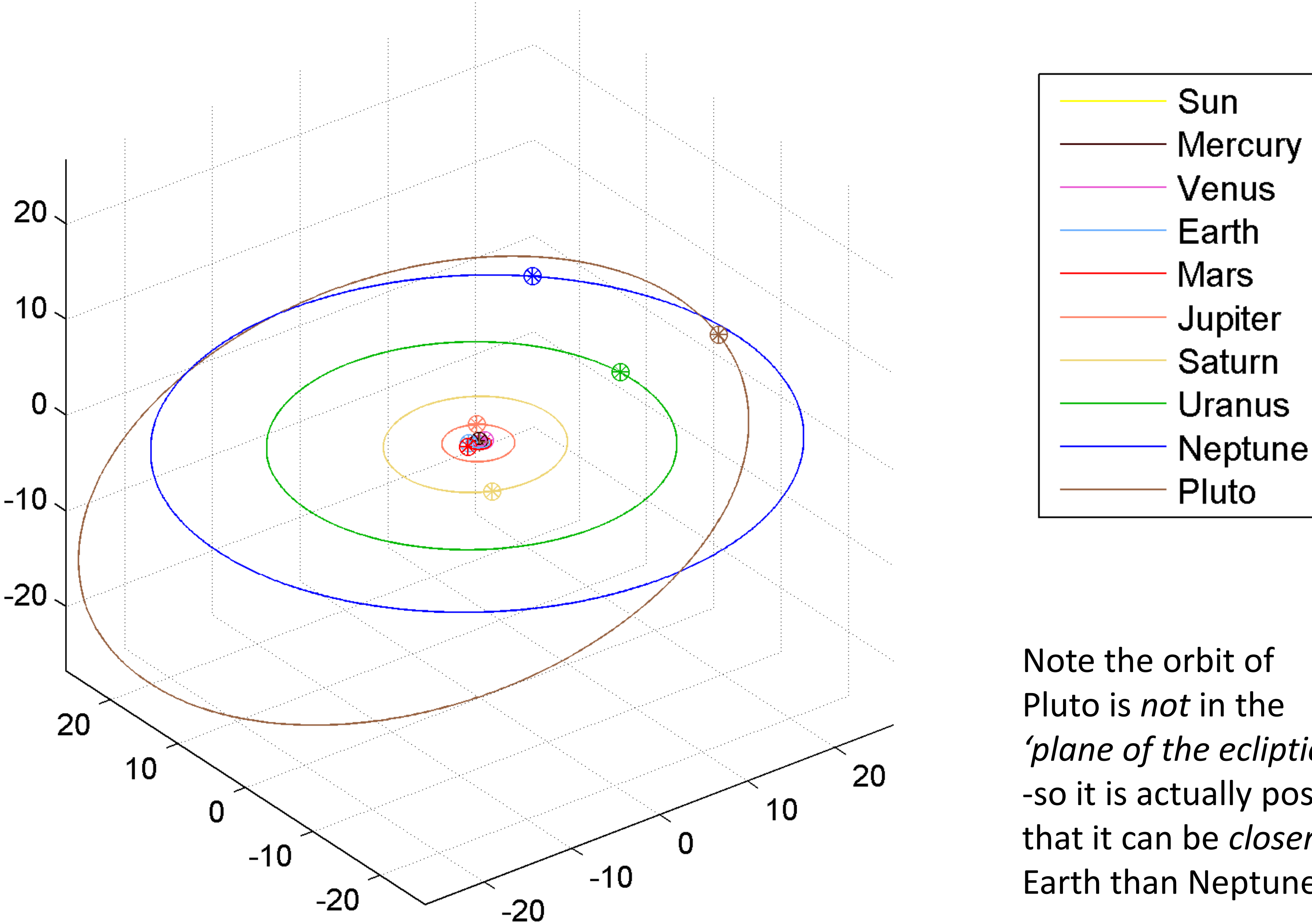




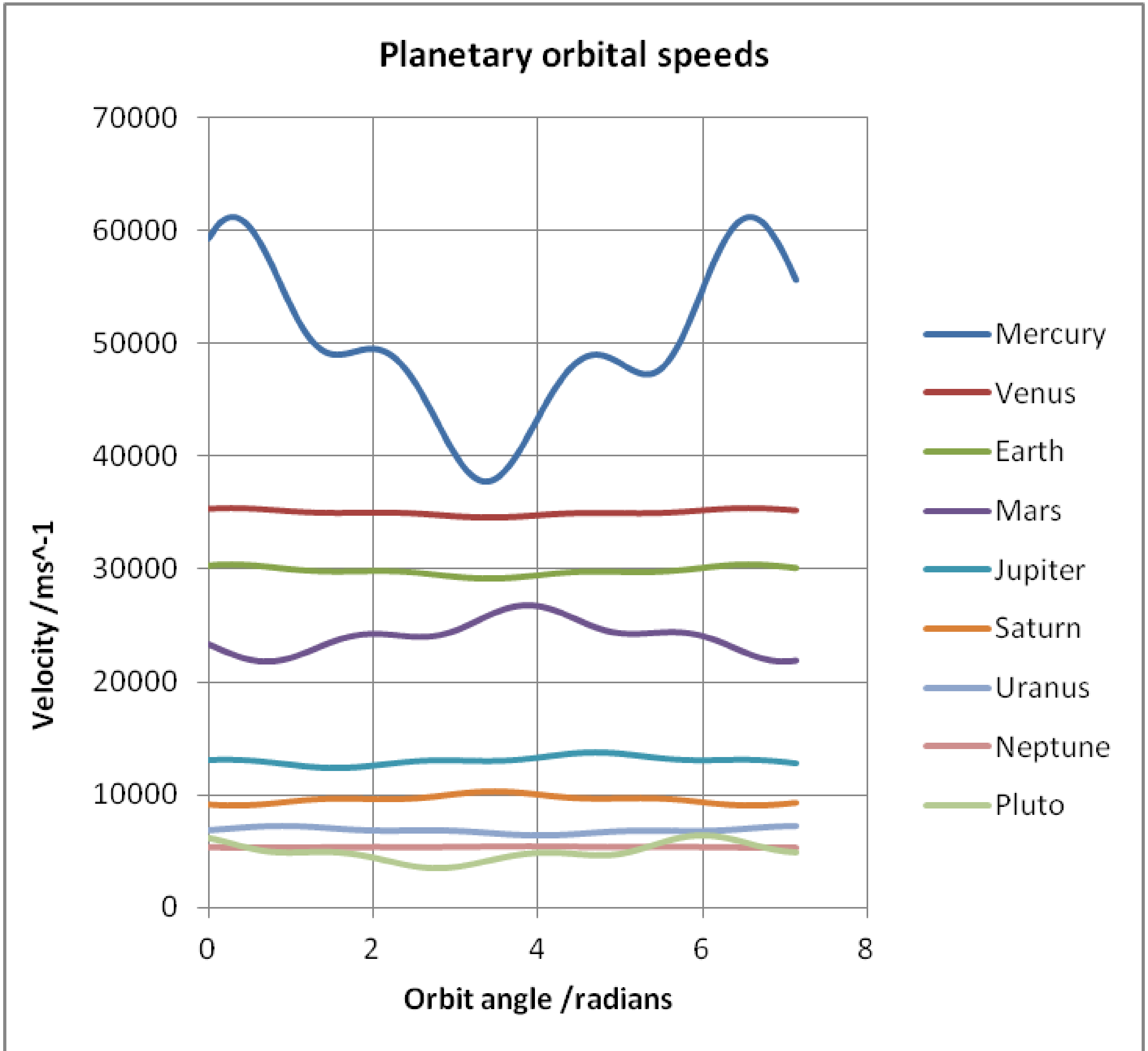
Assumes Pluto is in the same orbital plane as the other planets ....

Also that semi-major axis of all planets are aligned.

This is not true in general!



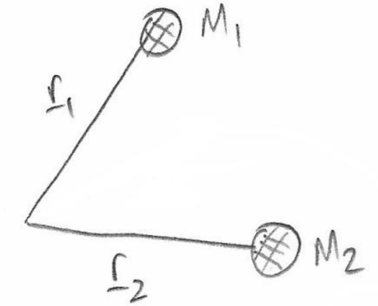
Note the orbit of Pluto is *not* in the *'plane of the ecliptic'* -so it is actually possible that it can be *closer* to Earth than Neptune


Planetary orbital speeds
70000
60000
50000
40000
30000
20000
10000
0
0
2
4
6
8
Orbit angle /radians
Velocity /ms^-1
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune
Pluto

Numerical orbit solver (Verlet method)

$$\mathbf{F}_{\mathbf{i}}(t) = \sum_{j \neq i} \frac{G m_i m_j}{\left| \mathbf{r}_i(t) - \mathbf{r}_j(t) \right|^2}$$

Newton's Law of universal gravitation



$$\mathbf{a}_i(t) = \frac{\mathbf{F}_i(t)}{m_i}$$

Newton's Second Law

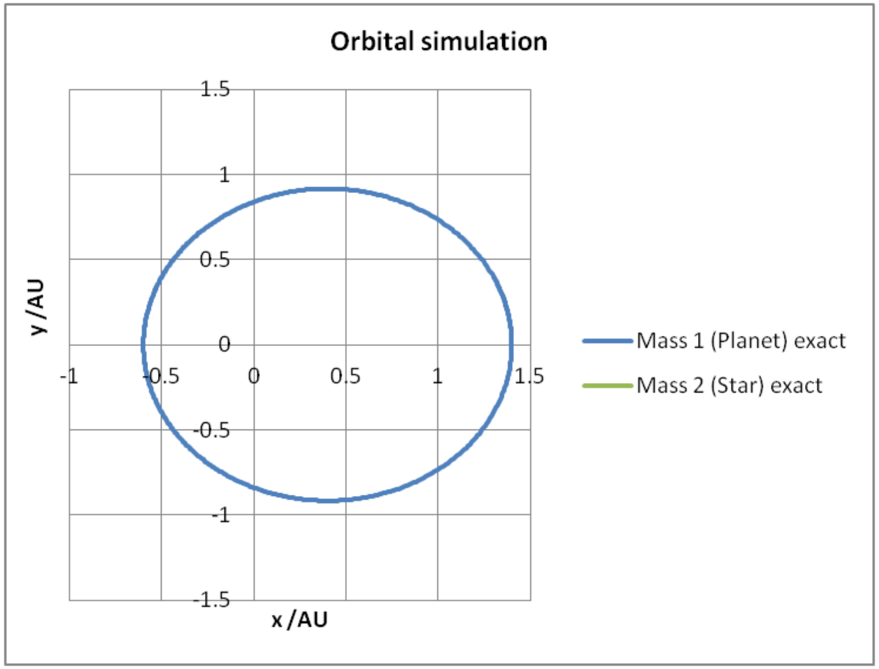
Update position and velocity vectors

$$\mathbf{r}_i(t+\Delta t) = \mathbf{r}_{\mathbf{i}}(t) + \mathbf{v}_i(t)\Delta t + \tfrac{1}{2}\mathbf{a}_i(t)\left(\Delta t\right)^2$$

$$\mathbf{v}_i(t+\Delta t) = \mathbf{v}_i(t) + \tfrac{1}{2}\left\{\mathbf{a}_i(t) + \mathbf{a}_i(t+\Delta t)\right\}\Delta t$$

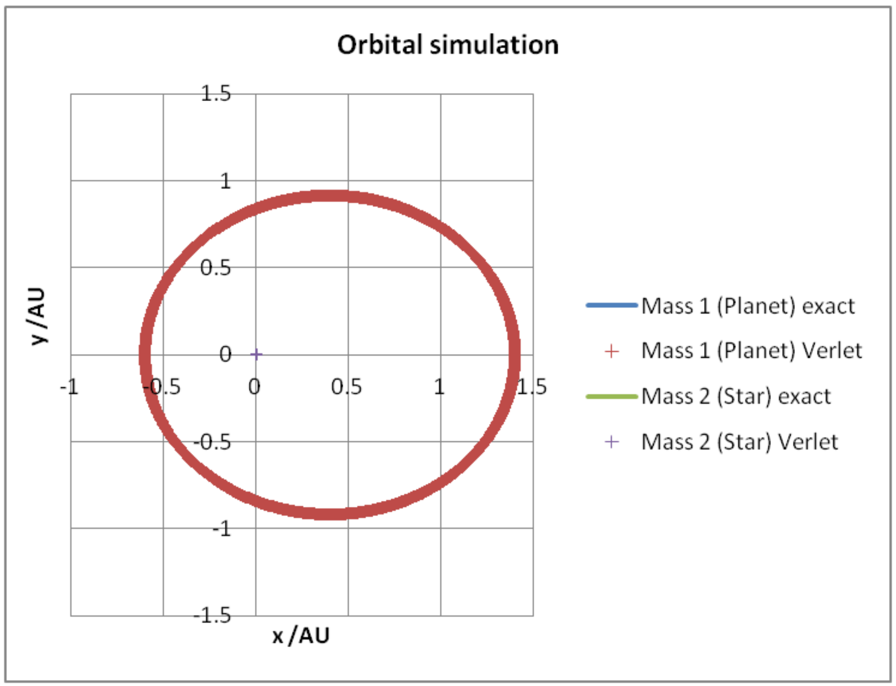
i.e. *constant acceleration motion* between fixed time steps Δt

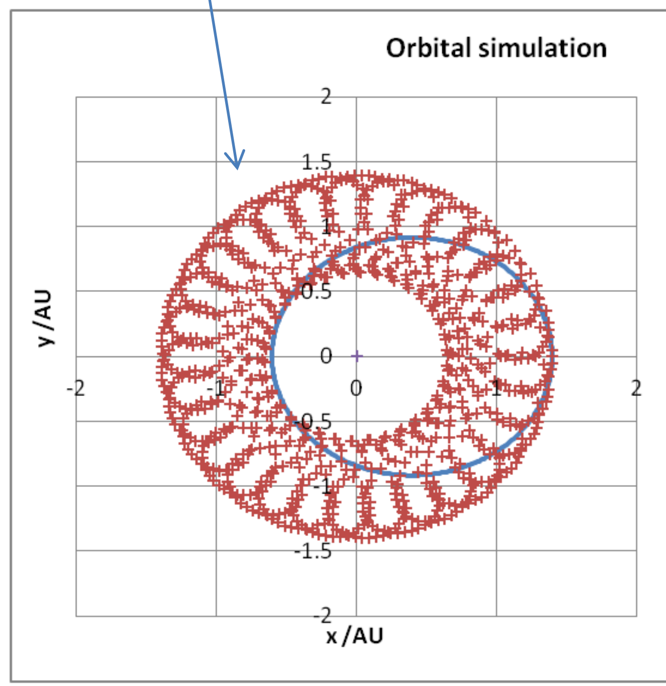
Note **velocity** calculation incorporates *updated acceleration*, determined using the updated position vector. This increases the accuracy of the method.



Mass of star (solar masses)	1
Mass of planet (Earth masses)	1
Semi-major axis of orbit /AU	1
Orbital eccentricity	0.4
Initial polar angle /degrees	180
timestep /years	0.001
period /years	1.0008977

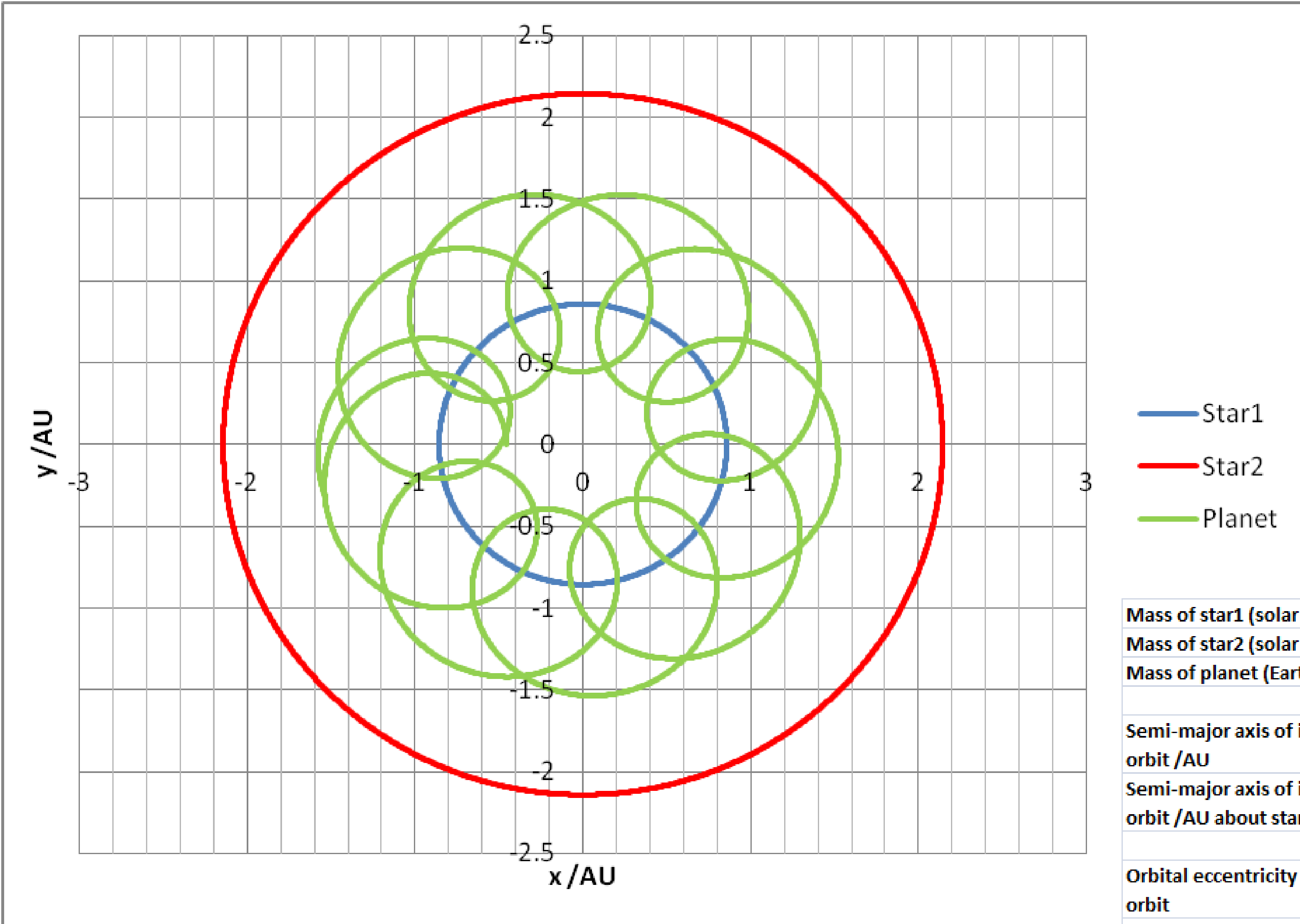
‘Numerical artefacts'
due to timestep being too large result in an *erroneous orbit*





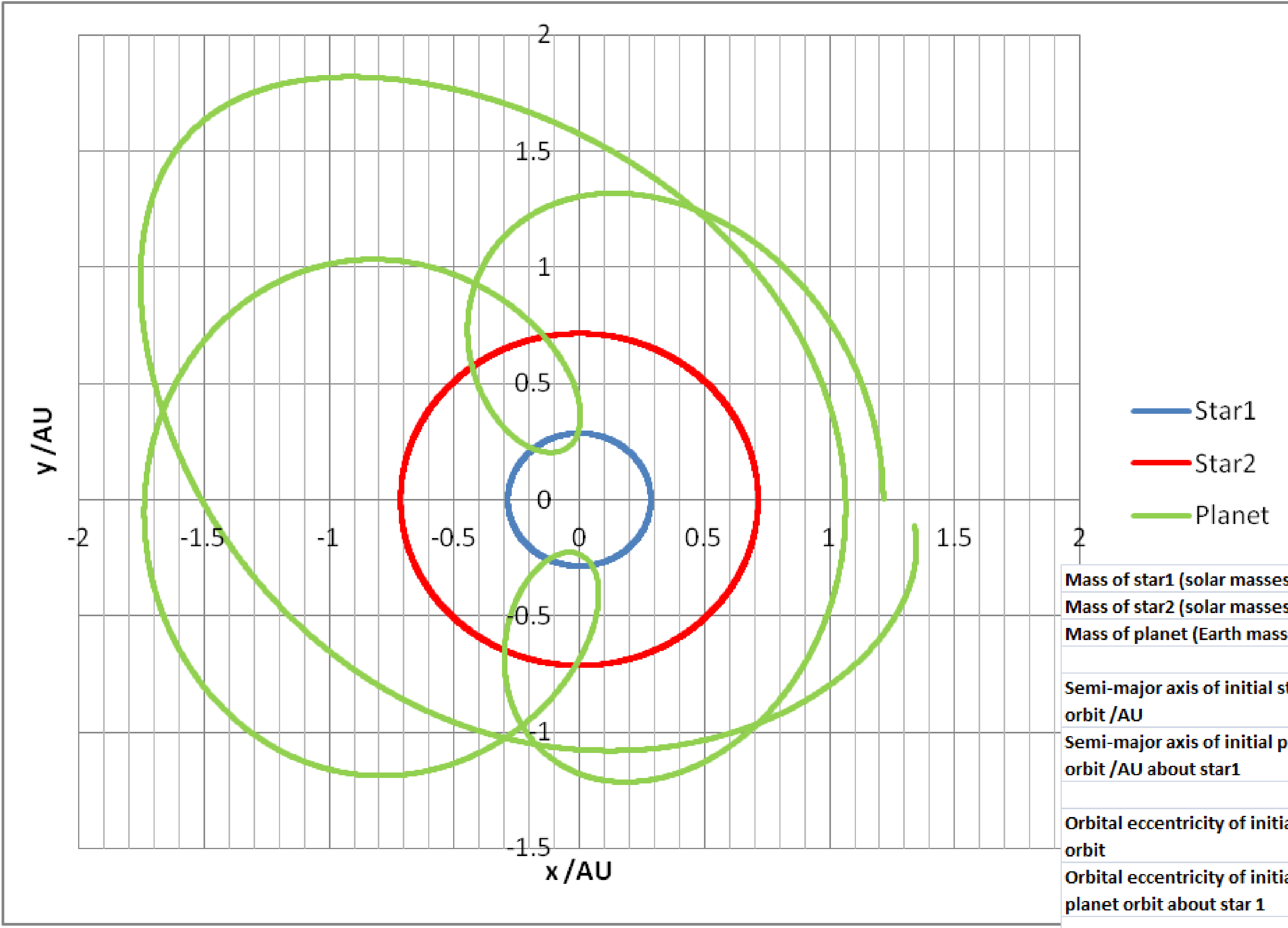
This is a problem with numeric solutions!

timestep = 0.05 years.



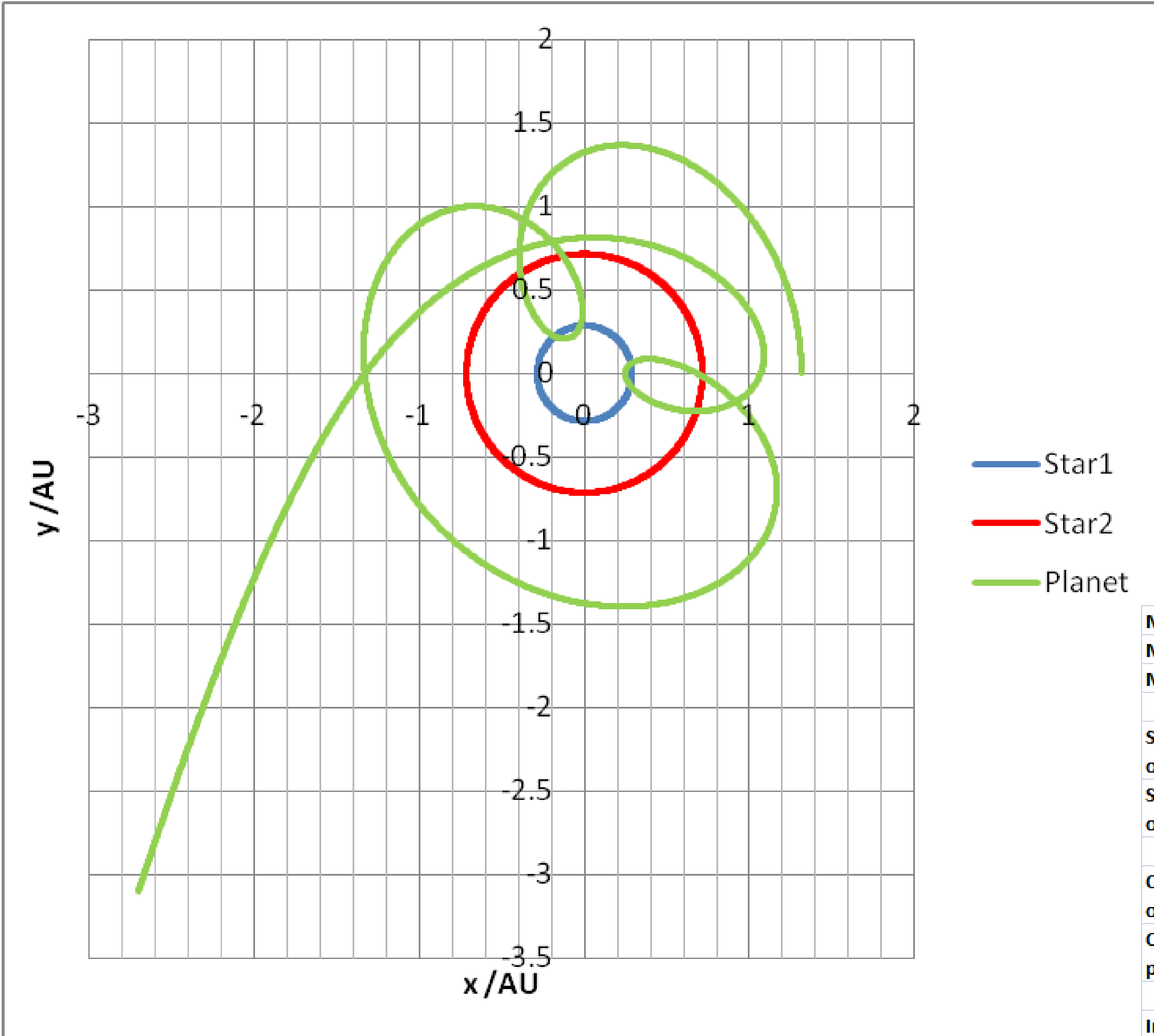
Mass of star1 (solar masses)	5
Mass of star2 (solar masses)	2
Mass of planet (Earth masses)	1
Semi-major axis of initial star orbit /AU	3
Semi-major axis of initial planet orbit /AU about star1	0.4
Orbital eccentricity of initial star orbit	0
Orbital eccentricity of initial planet orbit about star 1	0
Initial polar angle /degrees of initial star orbit	0
Initial polar angle /degrees of initial planet orbit about star 1	0
timestep /years	0.0001

Using the Verlet method to 'solve' a three-body problem i.e. the motion of a planet within the mutual orbits of a binary star system



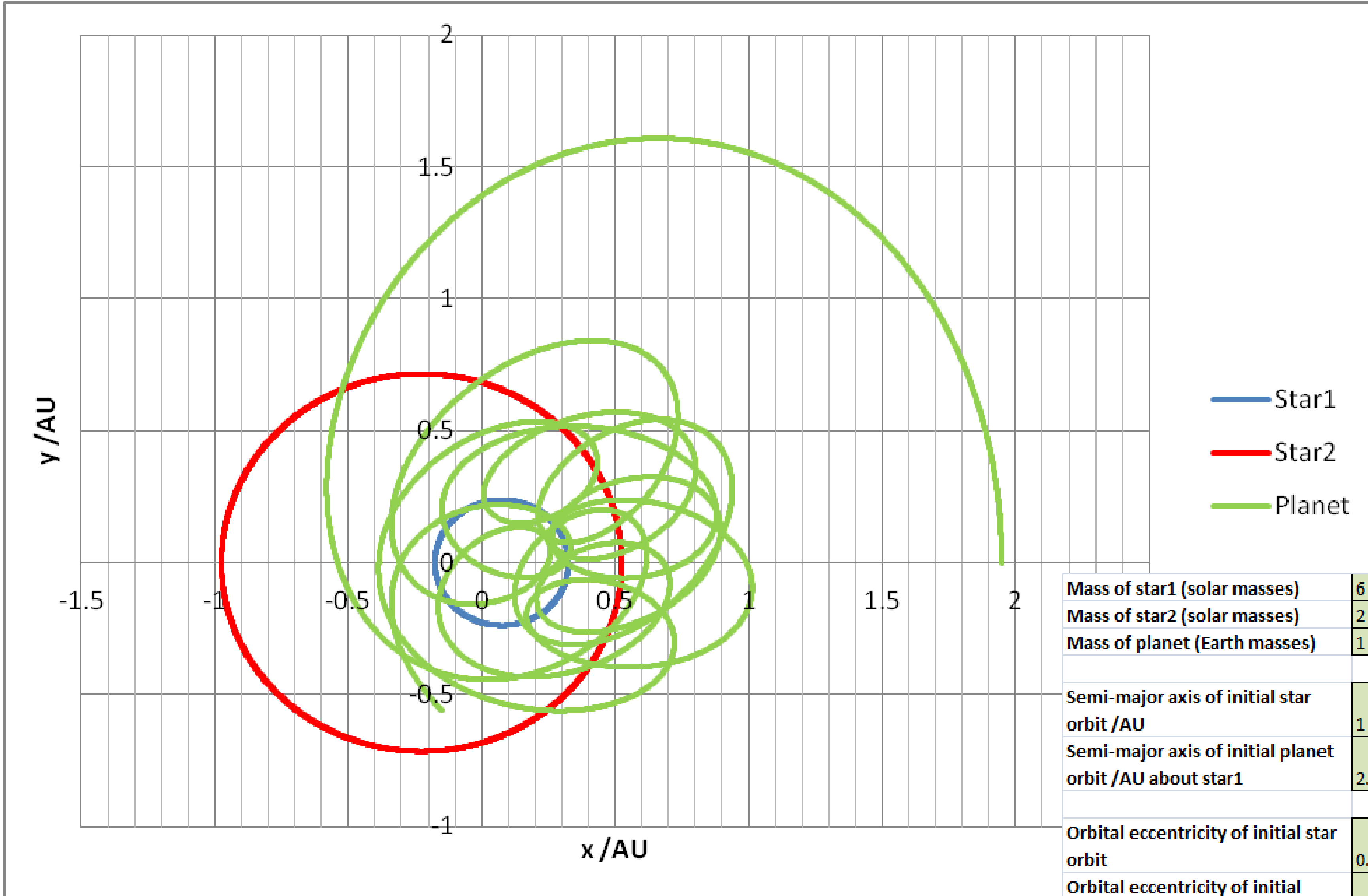
Mass of star1 (solar masses)	5
Mass of star2 (solar masses)	2
Mass of planet (Earth masses)	1
Semi-major axis of initial star orbit /AU	1
Semi-major axis of initial planet orbit /AU about star1	1.5
Orbital eccentricity of initial star orbit	0
Orbital eccentricity of initial planet orbit about star 1	0
Initial polar angle /degrees of initial star orbit	0
Initial polar angle /degrees of initial planet orbit about star 1	0
timestep /years	0.0001

This time the planet starts outside both the orbits of the binary star system



Mass of star1 (solar masses)	5
Mass of star2 (solar masses)	2
Mass of planet (Earth masses)	1
Semi-major axis of initial star orbit /AU	1
Semi-major axis of initial planet orbit /AU about star1	1.6
Orbital eccentricity of initial star orbit	0
Orbital eccentricity of initial planet orbit about star 1	0
Initial polar angle /degrees of initial star orbit	0
Initial polar angle /degrees of initial planet orbit about star 1	0
timestep /years	0.000072

Orbits don't have to be bound .... This one is *unstable*. After a few orbits, the planet is ejected from the system!



Mass of star1 (solar masses)	6
Mass of star2 (solar masses)	2
Mass of planet (Earth masses)	1
Semi-major axis of initial star orbit /AU	1
Semi-major axis of initial planet orbit /AU about star1	2.12345
Orbital eccentricity of initial star orbit	0.3
Orbital eccentricity of initial planet orbit about star 1	0
Initial polar angle /degrees of initial star orbit	0
Initial polar angle /degrees of initial planet orbit about star 1	0
timestep /years	0.000072

Complicated orbits can result if the planet is close to both stars!